Notes on "Differential Rents and the Distribution of Earnings"

from Sattinger, *Oxford Economic Papers* 1979, 31(1)

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- This is a version of an hedonic model.
- It features 1-1 matches.
- Assume that we can rank workers and firms by a skill scale: ℓ is amount of labor skill, c is amount of capital owned by firm.
- $F(\ell, c)$ is output. Assume a common production technology. One worker – one firm match $F_\ell > 0$, $F_c > 0$, $F_{\ell\ell} < 0$, $F_{cc} < 0$, no need to make scale restrictions.

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- Can be increasing returns to scale technologies.
- Homogeneous output of firms, identical technologies.
- Let $G(\ell)$ be cdf of ℓ in population. Let $K(c)$ be cdf of c in population. Assume both monotone strictly increasing, density has positive support – no mass points.
- Let $W(\ell)$ be wage for worker of type ℓ.
- Let $\pi(c)$ denote "profit" for a firm of type c.

- Assume $\dfrac{\partial^2 F}{\partial \ell \partial c} > 0$ (opposite sign produces negative sorting).

- Assume wage function exists.

- This is something to be proved.

- Firm indexed by c.

- Profit maximization requires that

$$\max_{\ell}(F(\ell, c) - W(\ell))$$

$$\text{FOC: } \frac{\partial F}{\partial \ell} = W'(\ell) \qquad \text{SOC: } \frac{\partial^2 F}{\partial \ell^2} - W''(\ell) < 0$$

- Defines demand for worker of type ℓ for firm type c.

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- Differentiate FOC totally with respect to ℓ:

$$W''(\ell) - \frac{\partial^2 F(\ell, c)}{\partial \ell^2} - \frac{\partial^2 F}{\partial \ell \partial c}\frac{dc}{d\ell} = 0$$

$$\underbrace{\left(W''(\ell) - \frac{\partial^2 F(\ell, c)}{\partial \ell^2}\right)}_{>0,\text{ from SOC}} = \underbrace{\left(\frac{\partial^2 F}{\partial \ell \partial c}\right)}_{+}\frac{dc}{d\ell} \qquad (1)$$

- $\therefore \dfrac{dc}{d\ell} > 0$ ("best firms match with best workers")

- Opposite true if we have $\dfrac{\partial^2 F}{\partial \ell \partial c} < 0$ ($dc/dl < 0$).

- Retain $\dfrac{\partial^2 F}{\partial \ell \partial c} > 0$ for specificity.

- Profits residually determined:

$$\pi(c) = F(\ell(c), c) - W(\ell(c)).$$

- Observe that the roles of ℓ and c can be reversed (labor hires capital) and labor incomes could be residually determined.

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- The continuum hypothesis for skills $\implies$ local returns to scale

$$dF = F_\ell d\ell + F_c dc$$

- $\therefore$ we get product exhaustion locally.
- Residual claimant gets marginal product, no matter who is claimant.
- Now suppose number of workers (N_ℓ).
- Number of capitalists (N_c).

- Let W_R be the reserve price of workers (what they could get not working in the sector being studied). Let π_R be reserve price of capitalist. Let ℓ^* be the least productive worker (employed). We need $W(\ell^*) \geq W_R$.

- If all capital employed, and $c \in [\underline{c}, \overline{c}]$, ℓ^* works with
  $$\underset{\text{least productive capitalist}}{\underline{c},} \qquad \text{assuming that } \pi(\underline{c}) \geq \pi_R.$$

- How to establish that decentralized wage setting is optimal and a wage function exists?
- Solve Social Planner's Problem.
$$\frac{\partial^2 F(\ell, c)}{\partial \ell \partial c} > 0 \Rightarrow$$

maximize total output by matching the best with the best.

Proof: trivial based on proof by contradiction

Take a discrete example

$$\text{two workers} \quad \ell_1 < \ell_2$$
$$\text{two firms} \quad c_1 < c_2$$

From complementarity (or supermodularity)

$$F(\ell_2, c_2) + F(\ell_1, c_1) > F(\ell_2, c_1) + F(\ell_1, c_2)$$

because

$$F(\ell_2, c_2) - F(\ell_1, c_2) > F(\ell_2, c_1) - F(\ell_1, c_1)$$

due to

$$\frac{\partial^2 F(\ell, c)}{\partial \ell \, \partial c} > 0.$$

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- Using the fact that the best matches with the best, sort top-down.
- Assume densities "continuous" (absolutely continuous).

$$N_\ell \int_{\ell(c)}^\infty g(\ell)\, d\ell = N_c \int_c^\infty k(c)\, dc$$

$$N_\ell \left(1 - G(\ell(c))\right) = N_c \left(1 - K(c)\right)$$

$$\left(1 - G(\ell(c))\right) = \left(\frac{N_c}{N_\ell}\right) \left(1 - K(c)\right)$$

$$G^{-1}\left[1 - \left(\frac{N_c}{N_\ell}\right) \left(1 - K(c)\right)\right] = \ell(c)$$

- This defines the optimal sorting function.

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Use survivor function:

$$S(x) = \Pr[X \geq x]$$

$$S_G(\ell) = 1 - G(\ell)$$
$$S_K(c) = 1 - K(c)$$

$$S_G(\ell(c)) = \left(\frac{N_c}{N_\ell}\right) S_K(c)$$
$$\ell(c) = S_G^{-1}\left(\frac{N_c}{N_\ell} S_K(c)\right)$$

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- Defines a relationship:

$$\ell = \varphi(c) \quad \text{(most productive match with each order)}$$

This function has an inverse from strictly decreasing survivor function assumption (density has no mass points or holes).

- Feasibility requires, using $\varphi^{-1}(\ell) = c$, that the lowest quality capitalist cover his/her reserve income outside the sector

$$\pi(\underline{c}) = F(\ell(\underline{c}), \underline{c}) - W(\ell^*) \geq \pi_R.$$

- If not satisfied we have unemployed capital.
- Jack up $c^* > \underline{c}$ until constraint satisfied.

- From the allocation derived from the social planner's problem, we can derive the hedonic equation (instead of assuming it).

- The slope of the wage function is given by FOC (using $c = \varphi^{-1}|l$)

$$W'(\ell) = \frac{\partial F}{\partial \ell}(\ell, \varphi^{-1}(\ell))$$

  (the right-hand side determined by the equilibrium sorting).

- This defines the slope of hedonic line with a continuum of labor.

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- Note that if we totally differentiate the right-hand side,

$$W''(\ell) = \underset{<0}{F_{\ell\ell}} + \underset{+}{F_{\ell c}} \underset{+}{\frac{dc}{d\ell}}$$

  $\therefore$ SOC satisfied, because $W''(\ell) - F_{\ell\ell} \geq 0$ as required.
- The marginal wage at minimum quality ℓ^* satisfies

$$W'(\ell_*) = \frac{\partial F}{\partial \ell}(\ell^*, \varphi^{-1}(\ell^*)).$$

- Competitive labor market forces $W(\ell_*) = W_R$.
- You cannot pay any less than reserve wage.
- If you pay more, all workers from the "reserve" will want to work in the sector being studied and hence it forces wages down.

$$W(\ell) = \int_{\ell^*}^{\ell} \frac{\partial F}{\partial x}(x, \varphi^{-1}(x))dx + W_R.$$

"hedonic function"

- Similarly

$$\pi(c) = \int_{c_*}^{c} \frac{dF}{dz}(\varphi(z), z)dz + \pi_R.$$

(Reserve value of capital is nonnegative; $\pi_R \geq 0$.)

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- Under our assumptions (more workers than firms and unemployed worker, $N_c > N_\ell$), rents are assigned to firms.
- Density of earnings is obtained from inverting wage function

  $$w(\ell) = \eta(\ell) \qquad \eta^{-1}(w) = \ell \text{ (exists under our assumptions)}$$

- Density of earnings is

  $$g(\eta^{-1}(w))\frac{d\eta^{-1}(w)}{dw}$$

  Density of profits obtained in a similar way.

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Cobb Douglas Example

- $F(\ell, c) = \ell^\alpha c^\beta$, $\alpha > 0$, $\beta > 0$.
- Assume Pareto distribution of endowments:

$$g(\ell) = j\ell^{-\gamma} \qquad \gamma > 2, \qquad \ell \geq 1$$
$$k(c) = hc^{-\sigma} \qquad \sigma > 2, \qquad c \geq 1.$$

- This ensures finite variances. Obviously $F_{\ell c} > 0$.
- The higher γ, the more equal is the distribution of ℓ.
- The higher σ, the more equal is the distribution of c.

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- Equilibrium:

$$N_c \int_{c(\ell)}^{\infty} h x^{-\sigma} dx = N_\ell \int_\ell^{\infty} j \eta^{-\gamma} d\eta$$

$$c(\ell) = \left[\frac{N_\ell j}{N_c h} \frac{(\sigma - 1)}{(\gamma - 1)} \right]^{\frac{1}{1-\sigma}} (\ell)^{\frac{1-\gamma}{1-\sigma}}.$$

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- FOC (for wages) $\alpha \ell^{\alpha-1} c^\beta = W'(\ell)$.
- Substitute for $c(\ell)$ to reach

$$\therefore \ W'(\ell) = \alpha \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{\beta}{1-\sigma}} \ell^P$$

$$P = \frac{(\alpha - 1)(1 - \sigma) + \beta(1 - \gamma)}{1 - \sigma} \gtrless 0$$

$$W(\ell) = \underbrace{\left[\frac{\alpha \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{\beta}{1-\sigma}}}{\left(\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma} \right)} \right]}_{g_1} \cdot (\ell)^{\left(\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{(1-\sigma)} \right)} + k_1,$$

and where k_1 is a constant of integration, determined by
$W_R : W(\ell^*) \geq W_R$.

- Obviously $W(\ell) \uparrow$ as $\ell \uparrow$. Convexity or concavity in labor quality hinges on whether

$$P \lessgtr 0$$
$$P = (\alpha - 1) + \beta \frac{(1 - \gamma)}{1 - \sigma}.$$

- If $\alpha + \beta = 1$ (CRS)

$$P = \beta \left[-1 + \frac{1 - \gamma}{1 - \sigma} \right]$$
$$= \beta \left[\frac{\sigma - \gamma}{1 - \sigma} \right] = \beta \left[\frac{\gamma - \sigma}{\sigma - 1} \right]$$

- Convexity or concavity of wage function depends on P.

- If $\gamma > \sigma$, $W(\ell)$ is convex in ℓ. (More firms out in tail than workers – workers get scarcity payment).

- Firms less equally distributed (more "productive" firms out in tail).

- If $\beta \uparrow$ (from CRS) reinforces effect (Renders capital relatively more productive).

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- If $\gamma = \sigma$ and $\beta + \alpha > 1$ (β big enough), $P > 0$ and hence produces convexity.

- Increasing returns to scale gives rise to convexity (**scale of productivity of resources effect**).

- Profits can be written as

$$\pi(c) = \ell^\alpha c^\beta - w(\ell)$$

- From the equilibrium matching condition we obtain

$$\ell = g_0(c)^{\frac{1-\sigma}{1-\gamma}} \qquad g_0 = \left[\frac{N_c h(\gamma - 1)}{N_\ell j(\sigma - 1)}\right]^{\frac{1}{1-\gamma}}$$

$$\pi(c) = \left[g_0(c)^{\frac{(1-\sigma)}{(1-\gamma)}}\right]^\alpha c^\beta - g_1\left(g_0(c)^{\frac{(1-\sigma)}{(1-\gamma)}}\right)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}} - k_1$$

$$\frac{\alpha(1-\sigma)}{1-\gamma} + \beta = \frac{\alpha(1-\sigma) + \beta(1-\gamma)}{1-\gamma}$$

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$$\pi(c) = \left[g_0^\alpha - g_1(g_0)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}} \right] \cdot c^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}} - k_1$$

- For positive marginal productivity of capital, this requires that

$$\alpha + \frac{\beta(\gamma-1)}{\sigma-1} > \left[\frac{N_c h(\gamma-1)}{N_\ell j(\sigma-1)} \right]^{\frac{\gamma(\beta-1)}{(\sigma-1)(\gamma-1)}}$$

- Otherwise, coefficient on $c^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}}$ is negative.

$$\pi(c) = ac^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}} - k_2$$

$$a = (g_0)^\alpha - g_1(g_0)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}} > 0$$

(True if $N_c \gg N_\ell$, for example.)

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- ∴ convexity of $\pi(c)$ is determined by sign of

$$\frac{\alpha(1-\sigma) + \beta(1-\gamma)}{1-\gamma} - 1$$

$$= \frac{\alpha(1-\sigma) + (\beta-1)(1-\gamma) - 1 + \gamma}{1-\gamma}$$

$$= \frac{(\gamma-1)(\beta-1) + (\sigma-1)\alpha}{\gamma-1}$$

$$= (\beta-1) + \left(\frac{\sigma-1}{\gamma-1}\right)\alpha.$$

- Observe if $\alpha + \beta > 1$ then both $\pi(c)$ and $W(\ell)$ can be convex in their arguments. With CRS one must be concave, the other convex.

- Linearity arises when we have $\gamma = \sigma$ and $\alpha + \beta = 1$.

- γ big relative to σ (scarcity of labor at top firms (high c firms)).
- α, β big – scale effects – we get convexity at top of distribution.
- Suppose we invoke full employment conditions for capital:

$$N_\ell > N_c \qquad \pi(1) \geq \pi_R$$

- We need to determine the constants for the wage equation.
- Minimum quality labor earns its opportunity cost outside of the sector.
- Rents accrue to other workers.

At lowest level of employment, we have (from matching function $c(\ell)$)

$$1 = \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{1}{1-\sigma}} (\ell^*)^{\frac{1-\gamma}{1-\sigma}}$$

$$\therefore \ \ell^* = \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{1}{\gamma - 1}}$$

$$W(\ell^*) = W_R$$

$$\therefore k_1 =$$

$$W_R - \frac{\alpha(1 - \sigma)}{\alpha(1 - \sigma) + \beta(1 - \gamma)} \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{\beta}{1-\sigma}} (\ell^*)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}}.$$

$\pi(c)$ defined residually. (Need to check $\pi(1) > \pi_R$).

- Pigou's Problem: Why doesn't the distribution of earnings resemble the distribution of ability?
- Distribution of earnings: (generated from distribution of endowments by the pricing function).
- Look at distribution of translated earnings (translated around the constant k_1).

$$(W(\ell) - k_1) \sim (W - k_1)^{-\left[1 + \frac{(\gamma-1)(\sigma-1)}{\alpha(\sigma-1)+\beta(\gamma-1)}\right]}$$

Distribution of raw skills $\sim \ell^{-\gamma}$.

- Higher γ is associated with more equality in the distribution of labor skills.

- One way to measure the market-induced change in inequality is the change in the wage distribution from γ.
- Example:

$$1 + \frac{(\gamma - 1)(\sigma - 1)}{\alpha(\sigma - 1) + \beta(\gamma - 1)} < \gamma$$

(wage inequality $>$ inequality in ℓ)

- For this to happen,

$$\frac{1}{\alpha + \beta \dfrac{(\gamma - 1)}{(\sigma - 1)}} < 1$$

- The higher $\alpha + \beta$, the more unequal the distribution of wages.
- Higher $\gamma > \sigma$ (capital more unequally distributed) the greater the wage inequality.

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- If $\gamma = \sigma$, $\alpha + \beta = 1$, no induced change in inequality.
- If $\gamma = \sigma$, $\alpha + \beta > 1$, *more* inequality in wages than skills.
- If $\sigma \ll \gamma$, then more inequality in wages than skills (Demand for top talent).
- It is not "superstars" but "superfirms".

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- The wage equation is an hedonic function.
- Hedonic Functions (Tinbergen, 1951, 1956; Rosen, 1974). What can you estimate when you regress W on ℓ? Obviously we can estimate k_1,

$$\frac{\alpha(\sigma - 1) + \beta(\gamma - 1)}{(\sigma - 1)}$$

  and slope coefficient (g_1).
- Do not recover any single parameter of interest. We get lowest ℓ in market and from distribution of ℓ and c, we can get γ, σ, h (if c fully employed).
- If we assume $\alpha + \beta = 1$ (CRS) and we observe distributions of the factors, we get σ, γ and hence α, β.

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- If we know ℓ^*, we can get j.
- If we know N_ℓ and N_c, we can identify γ, σ but α, β are unknown.
- $\alpha + \beta$ is known.
- CRS $\Rightarrow \alpha$, β known.

Identify the Technology

- Idea (Rosen, 1974). Two-stage estimation procedure. Assume perfect data.
- Assume $\alpha \neq 1$.
- No error term in model, no omitted variables.
- Use FOC for firm,

$$\ln \alpha + (\alpha - 1) \ln \ell + \beta \ln c = \ln W'(\ell)$$

i.e.,

$$\ln \ell = -\frac{\ln \alpha}{\alpha - 1} + \frac{\ln W'(\ell)}{\alpha - 1} - \frac{\beta \ln c}{\alpha - 1}.$$

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- Apparently, we can regress $\ln \ell$ on $\ln W'(\ell)$.
- Notice however that from the sorting condition,

$$\ln \ell = \ln g_0 + \left(\frac{\sigma - 1}{\gamma - 1} \right) \ln c.$$

- We get no independent variation. $\ln W'(\ell)$ is redundant.
- Alternatively, $\ln W'(\ell)$ and $\ln c$ are perfectly collinear.

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- More general principle:

$$\text{FOC:} \quad \frac{\partial^2 F}{\partial \ell^2} d\ell + \frac{\partial^2 F}{\partial \ell \partial c} dc = dW'(\ell)$$

$$d\ell = \frac{1}{\left(\dfrac{\partial^2 F}{\partial \ell^2}\right)} d[W'(\ell)] - \frac{\dfrac{\partial^2 F}{\partial \ell \partial c}}{\dfrac{\partial^2 F}{\partial \ell^2}} dc.$$

- Functional dependence between c and $W'(\ell)$ does not necessarily imply linear dependence.

- ∴ we might be able to identify the model.
- Need shifter in regression.
- Functional dependence $\not\Rightarrow$ linear independence

$$y = \alpha_0 + \alpha_1 X + \alpha_2 X^2.$$

- Obviously X and X^2 only dependent but not linearly dependent.
- We return to this in a bit.

Pareto Distribution
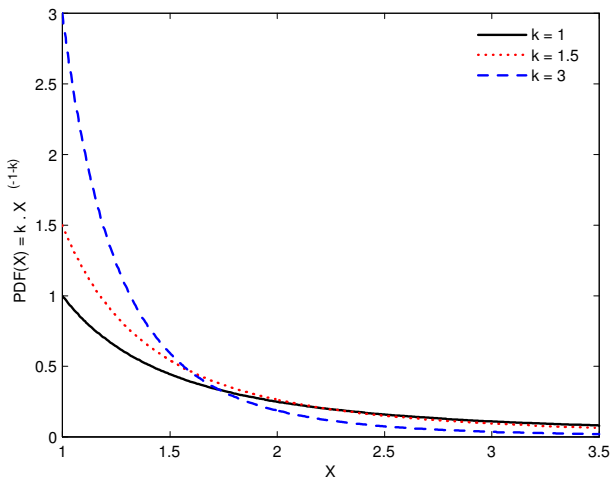
$$X \sim \mathsf{Pareto}(k) \to f_X(x) = k \cdot x^{-(1+k)}$$

Pareto Distribution
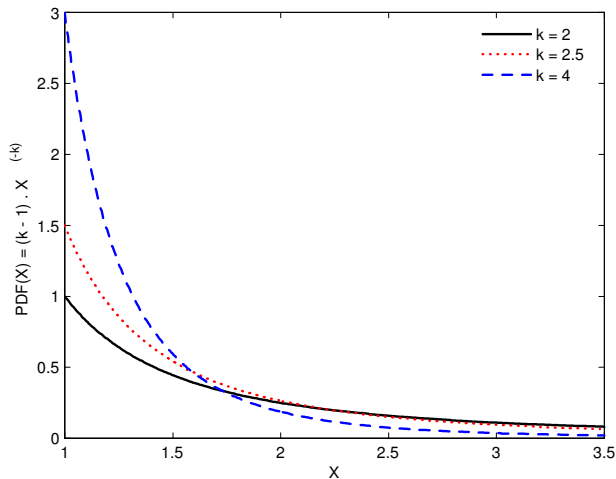
$$X \sim \mathsf{Pareto}(k) \rightarrow f_X(x) = (k-1) \cdot x^{-k}$$

Pareto Distribution
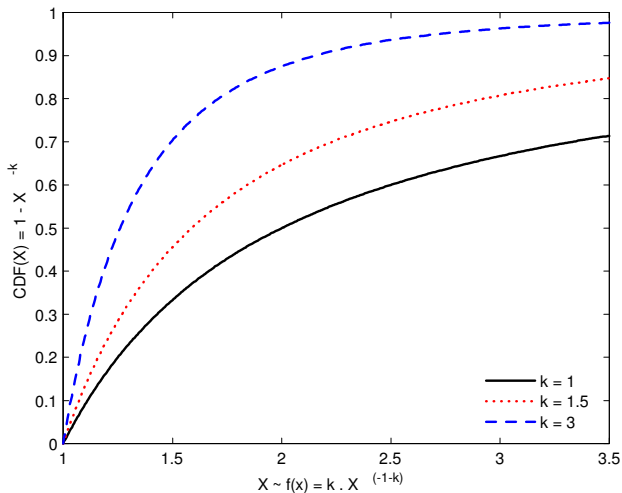
$$X \sim \text{Pareto}(k) \rightarrow F_X(x) = 1 - x^{-k}$$

Pareto Distribution
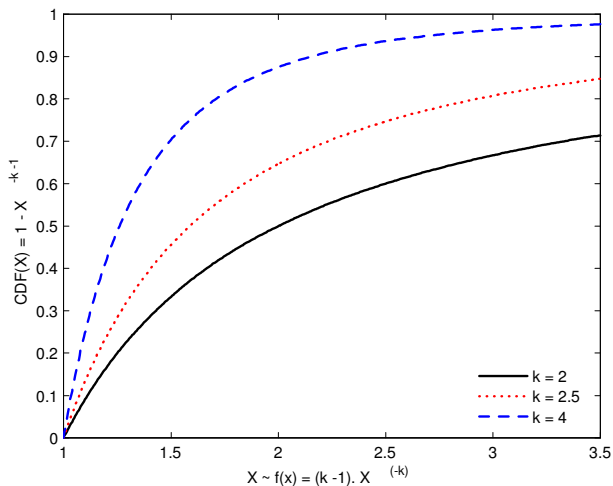
$$X \sim \text{Pareto}(k) \rightarrow F_X(x) = 1 - x^{-(k+1)}$$

Ability Distributions
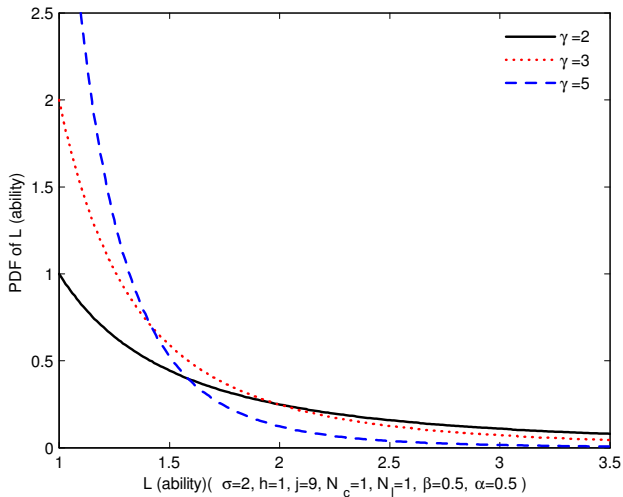
Pareto Percentiles
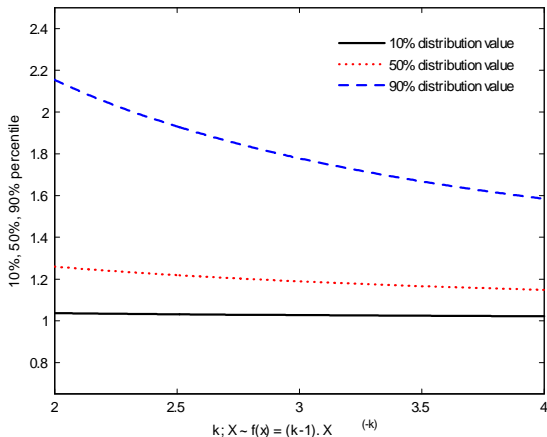
Pareto 10%, 50%, 90% Percentiles for $k \in [2, 4]$

Capital/ability relation
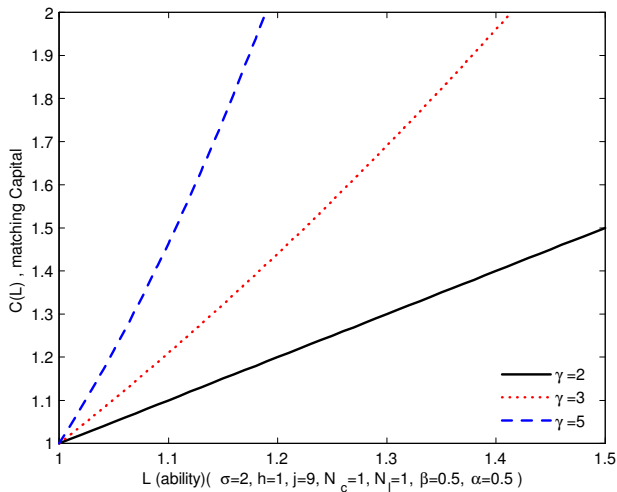
Wage derivative with respect to ability $\dfrac{\partial W(L)}{\partial L}$
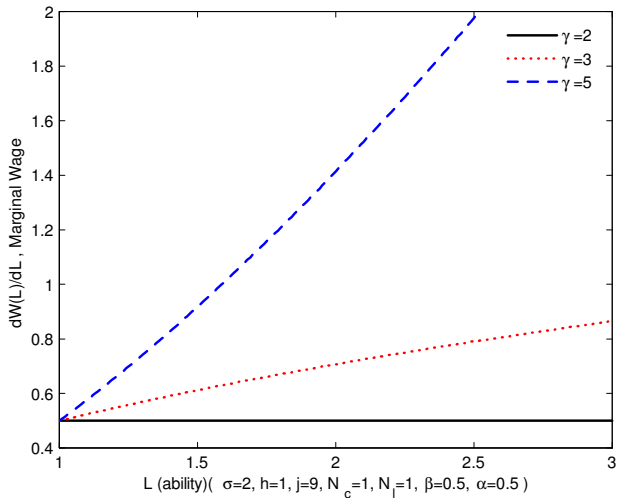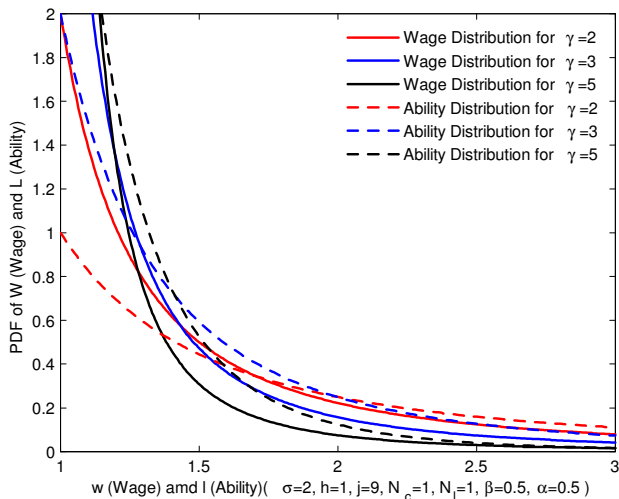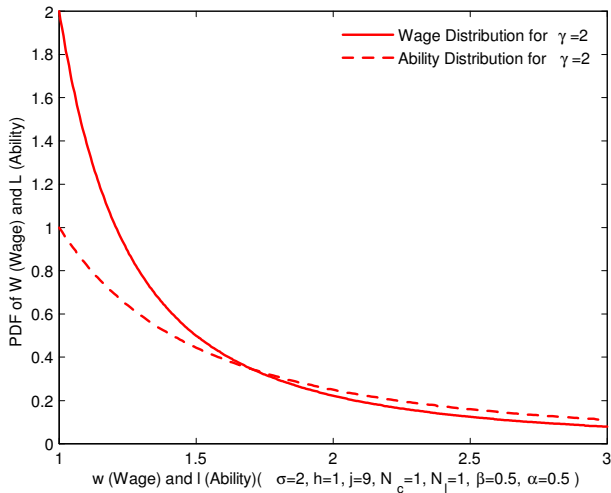
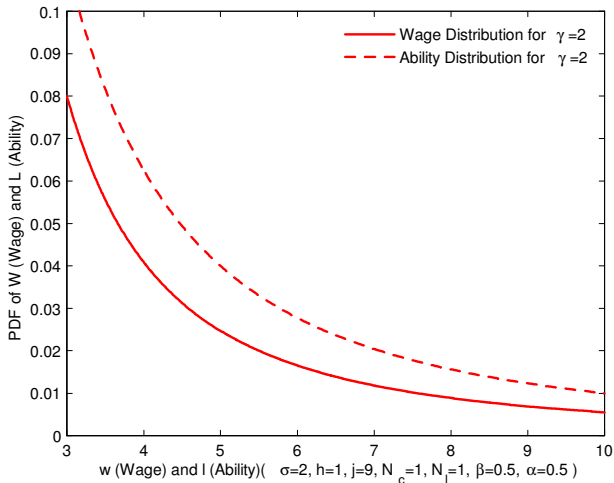# Wage as a function of ability
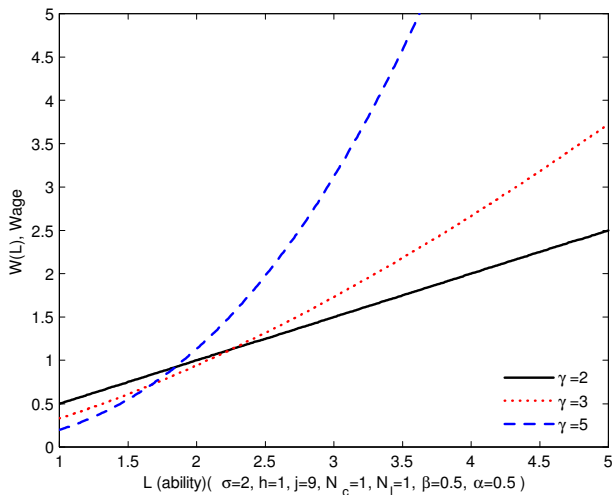
Wage distribution
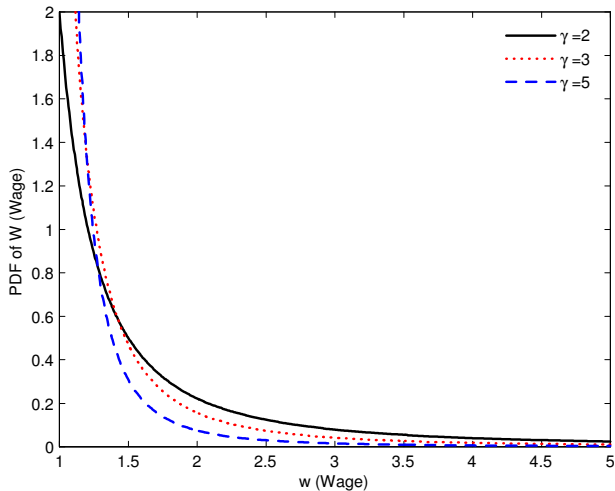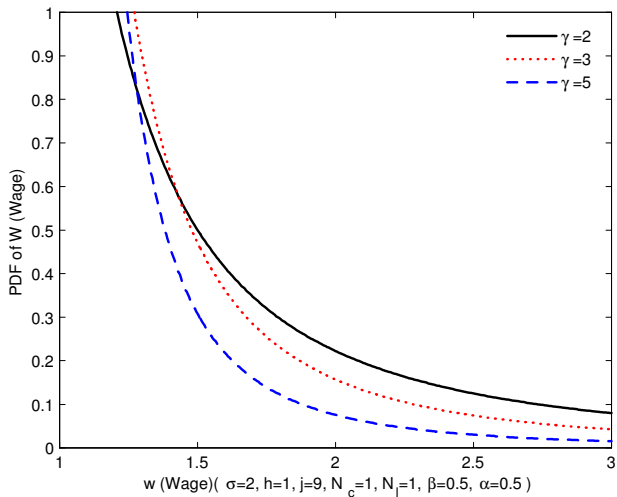
Wage distribution

Wage and ability distribution
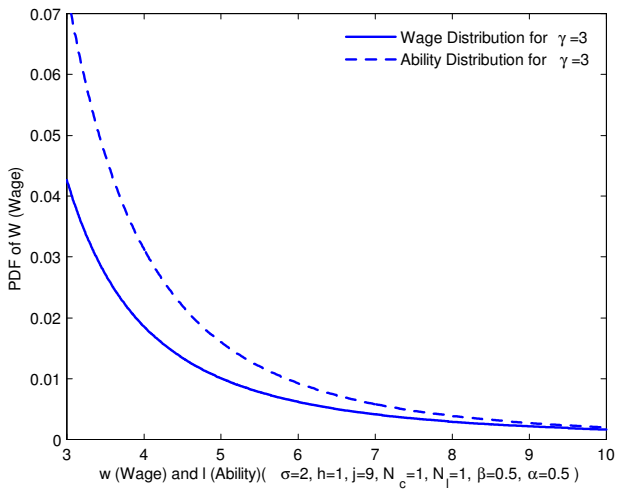

Figure showing PDF of W (Wage) and L (Ability) versus w (Wage) and l (Ability). Legend: Wage Distribution for $\gamma = 2$ (solid line); Ability Distribution for $\gamma = 2$ (dashed line). X-axis label: w (Wage) and l (Ability)($\sigma = 2$, h=1, j=9, N_c=1, N_l=1, β=0.5, α=0.5). Y-axis label: PDF of W (Wage) and L (Ability).

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Wage and ability distribution

Figure legend:
- Wage Distribution for $\gamma = 2$ (solid line)
- Ability Distribution for $\gamma = 2$ (dashed line)

Y-axis: PDF of W (Wage) and L (Ability)

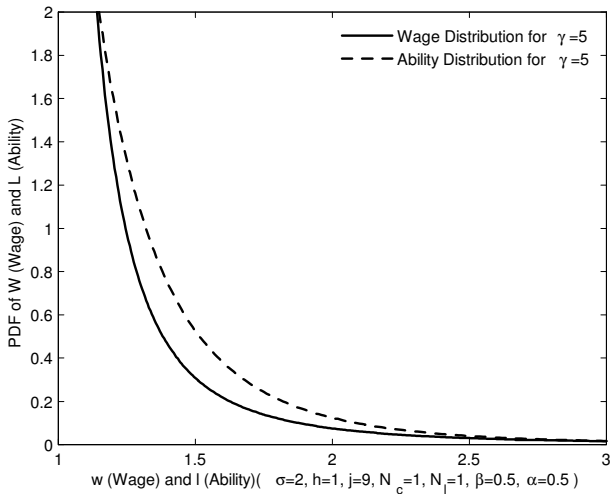X-axis: w (Wage) and l (Ability)($\sigma = 2$, h=1, j=9, $N_c = 1$, $N_l = 1$, $\beta = 0.5$, $\alpha = 0.5$)

Wage and ability distribution

Wage and ability distribution

Wage Distribution for $\gamma = 5$

Ability Distribution for $\gamma = 5$

PDF of W (Wage) and L (Ability)

w (Wage) and l (Ability)($\sigma = 2$, h=1, j=9, $N_c = 1$, $N_l = 1$, $\beta = 0.5$, $\alpha = 0.5$)
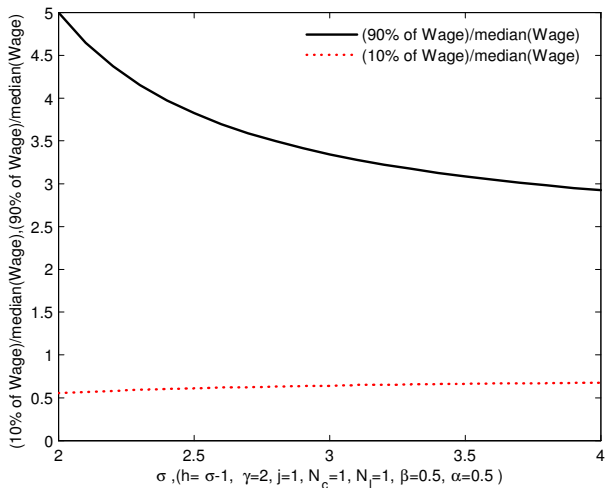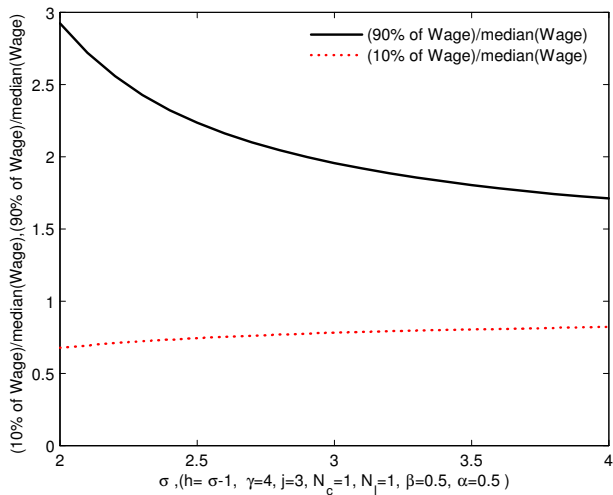
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Wage and ability distribution
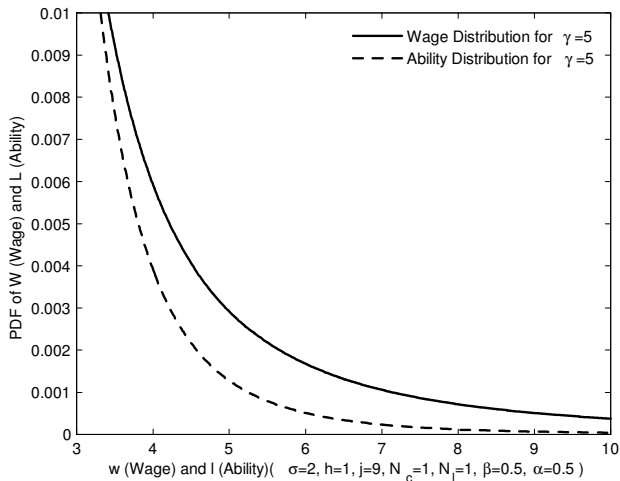
Wage percentile ratios

Wage percentile ratios