Some Generalized Roy Math

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Review Generalized Roy Model

$$
\begin{aligned}
Y_1 &= \mu_1(X) + U_1 \\
Y_0 &= \mu_0(X) + U_0 \\
C &= \mu_C(Z) + U_C \\
\text{Net Benefit: } I &= Y_1 - Y_0 - C \\
I &= \underbrace{\mu_1(X) - \mu_0(X) - \mu_C(Z)}_{\mu_D(Z)} + \underbrace{U_1 - U_0 - U_C}_{-V}
\end{aligned}
$$

$$
\begin{aligned}
(U_0, U_1, U_C) &\perp\!\!\!\perp (X, Z) \\
E(U_0, U_1, U_C) &= (0, 0, 0) \\
V &\perp\!\!\!\perp (X, Z)
\end{aligned}
$$

- Assume Normally Distributed Errors.
- Assume Z contains X but may contain other variables (exclusions)

$$Y = DY_1 + (1 - D)Y_0 \text{ observed } Y \text{ (switching regression model)}$$
$$D = 1(I \geq 0) = 1(\mu_D(Z) \geq V)$$

- Assume $V \sim N(0, \sigma_V^2)$

- "Propensity Score:" a.k.a. probability of choosing $D = 1$

$$\Pr(D = 1 \mid Z = z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$E(Y \mid D = 1, X = x, Z = z) = \mu_1(X) + \underbrace{E(U_1 \mid \mu_D(z) \geq V)}_{K_1(P(z))}$$

because $(X, Z) \perp\!\!\!\perp (U_1, V)$.

- Under normality

$$E\left(U_1 \middle| \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = \frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

- Terms defined below

- Why?

$$U_1 = \text{Cov}\left(U_1, \frac{V}{\sigma_V}\right)\frac{V}{\sigma_V} + \varepsilon_1$$

$$\varepsilon_1 \perp\!\!\!\perp V$$

$$E\left(\frac{V}{\sigma_V} \mid \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = \frac{\int\limits_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} t \frac{1}{\sqrt{2}\pi} e^{\frac{-t^2}{2}} dt}{\int\limits_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} \frac{1}{\sqrt{2}\pi} e^{\frac{-t^2}{2}} dt} = \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$= \frac{\frac{-1}{\sqrt{2}\pi} e^{\left(-\frac{1}{2}\right)\left(\frac{\mu_D(z)}{\sigma_V}\right)^2}}{\Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)} = \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = \frac{-\phi\left(\frac{\mu_D(z)}{\sigma_V}\right)}{\Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)}$$

- Notice

$$\lim_{\mu_D(z)\to\infty} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = 0$$

$$\lim_{\mu_D(z)\to-\infty} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = -\infty$$

- Propensity score:

$$P(z) = \Pr(D = 1 \mid Z = z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\therefore \left(\frac{\mu_D(z)}{\sigma_V}\right) = \Phi^{-1}\left(\Pr(D = 1 \mid Z = z)\right)$$

- Thus we can replace $\frac{\mu_D(z)}{\sigma_V}$ with a known function of $P(z)$
- As $P(Z) \to 1$, selection bias term goes to zero.

- Notice that because $(X, Z) \perp\!\!\!\perp (U, V)$, Z enters the model (conditional on X) only through $P(Z)$: Index Sufficiency.
- It holds true for the LATE model as it does here.
- We can apply our material on the Roy model to the Generalized Roy model.

- Put all of these results together to obtain

$$E\left(Y \mid D=1, X=x, Z=z\right) = \mu_1(x) + \left(\frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right) \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$= E\left(Y_1 \mid D=1, X=x, Z=z\right) = \mu_1(x) + \left(\frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right) \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\tilde{\lambda}(z) = E\left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} < \frac{\mu_D(z)}{\sigma_V}\right) < 0$$

$$\lambda(z) = E\left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} \geq \frac{\mu_D(z)}{\sigma_V}\right) > 0$$

$$E\left(Y \mid D=0, X=x, Z=z\right) = \mu_0(x) + \left(\frac{\mathrm{Cov}(U_0, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right) \lambda\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\mathrm{Var}\left(\frac{V}{\sigma_V}\right) = 1$$

$$\frac{V}{\sigma_V} = -\frac{(U_1 - U_0 - U_C)}{\sigma_V}$$

$$\operatorname{Cov}\left(U_1, \frac{V}{\sigma_V}\right) = -\operatorname{Cov}\left(U_1, \frac{V}{\sigma_V}\right) + \operatorname{Cov}\left(U_0, \frac{V}{\sigma_V}\right) + \operatorname{Cov}\left(U_C, \frac{V}{\sigma_V}\right)$$

In Roy model case ($U_C = 0$),

$$\begin{aligned}\operatorname{Cov}\left(U_1, \frac{V}{\sigma_V}\right) &= -\operatorname{Cov}\left(U_1, \frac{U_1 - U_0}{\sigma_V}\right) \\ &= \frac{\operatorname{Cov}(U_1 - U_0, U_1)}{\sqrt{\operatorname{Var}(U_1 - U_0)}}\end{aligned}$$

- We can identify $\mu_1(x), \mu_0(x)$
- From Discrete Choice model, identify

$$\frac{\mu_D(z)}{\sigma_V} = \frac{\mu_1(x) - \mu_0(x) - \mu_C(z)}{\sigma_V}$$

- If we have a regressor in X that does not affect $\mu_C(z)$ (say regressor x_j, so $\frac{\partial \mu_C(z)}{\partial x_j} = 0$), we can identify σ_V and $\mu_C(z)$.
- $\therefore$ We can identify the net benefit function and the cost function up to scale.
- $\therefore$ We can compute *ex ante* subjective net gains.

- Method generalizes:
  Don't need normality

$$E\left(Y \mid D=1, X=x, Z=z\right)=\mu_1(x)+\overbrace{K_1(P(z))}^{\text{control function}}$$
$$E\left(Y \mid D=0, X=x, Z=z\right)=\mu_0(x)+\underbrace{K_0(P(z))}_{\text{control function}}$$

$$\lim_{P(z)\to 1} E\left(Y \mid D=1, X=x, Z=z\right)=\mu_1(x)$$

$$\lim_{P(z)\to 0} E\left(Y \mid D=0, X=x, Z=z\right)=\mu_0(x)$$

- Identification in limit sets ("large support" conditions)
- e.g., samples where everyone participates or no one participates

- If this condition satisfied, we can identify ATE:

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- ATE is defined in a limit set. **This is true for any model with selection on unobservables** (IV; selection models; LATE)

- What about treatment on the treated?

$$E(Y_1 - Y_0 \mid D = 1, X = x, Z = z)$$

a. From the data, we observe

$$E(Y_1 \mid D = 1, X = x, Z = z)$$

b. Can also create it from the model
c. $E(Y_0 \mid D = 1, X = x, Z = z)$ is a counterfactual

   We know

$$E(Y_0 \mid D = 0, X = x, Z = z) = \mu_0(x) + \mathrm{Cov}\left(U_0, \frac{V}{\sigma_V}\right) \lambda\left(\frac{\mu_D(Z)}{\sigma_V}\right)$$

(this is data)

- We seek

$$E(Y_0 \mid D = 1, X = x, Z = z) = \mu_0(x) + \mathrm{Cov}\left(U_0, \frac{V}{\sigma_V}\right) \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

But under normality, we know $\mathrm{Cov}\left(U_0, \frac{V}{\sigma_V}\right)$

We know $\frac{\mu_D(Z)}{\sigma_V}$

$\tilde{\lambda}(\cdot)$ is a known function.

Can form $\tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$ and can construct counterfactual.

- More generally, without normality (but with $(X, Z) \perp\!\!\!\perp (U, V)$)

$$E(Y_1 \mid D = 1, X, Z) = E(Y \mid D = 1, X = x, Z = z) = \mu_1(x) + K_1(P(z))$$
$$E(Y_0 \mid D = 0, X, Z) = E(Y \mid D = 0, X = x, Z = z) = \mu_0(x) + \tilde{K}_0(P(z))$$

$$\begin{aligned} \text{where } K_1(P(z)) &= E(U_1 \mid D = 1, X = x, Z = z) \\ &= E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V}\right) \\ \tilde{K}_1(P(z)) &= E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V}\right) \\ \tilde{K}_0(P(z)) &= E\left(U_0 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V}\right) \end{aligned}$$

- Use the transformation

$$\frac{F_V}{\sigma_V}\left(\frac{\mu_D(z)}{\sigma_V}\right) = P(z)$$

$$\frac{F_V}{\sigma_V}\left(\frac{V}{\sigma_V}\right) = U_D \qquad \text{(a uniform random variable)}$$

$$D = 1\left(\frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = 1\left(P(z) \geq U_D\right)$$

$$K_1(P(z)) = E(U_1 \mid P(z) > U_D)$$

$$K_1(P(z))P(z) + \tilde{K}_1(P(z))(1 - P(z)) = 0$$

$$\therefore \text{ we can construct } \tilde{K}_1(P(z))$$

$$\therefore \text{ we can form the counterfactual.}$$

- Symmetrically

$$\tilde{K}_0(P(z)) = E(U_0 \mid P(z) \leq U_D)$$
$$K_0(P(z)) = E(U_0 \mid P(z) > U_D)$$
$$(1 - P(z))\tilde{K}_0(P(z)) + P(z)K_0(P(z)) = 0$$

- $\therefore$ If we have "identification in limit sets" we can construct

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- We can construct TT

$$\begin{aligned} &E(Y_1 - Y_0 \mid D = 1, X = x, Z = z) = \\ &\quad = \underbrace{[\mu_1(x) + K_1(P(z))]}_{\text{factual}} - \underbrace{[\mu_0(x) + K_0(P(z))]}_{\text{counterfactual}} \end{aligned}$$

- We form $\mu_1(x) + K_1(P(z))$ from data
- We get $\mu_0(x)$ from limit set $P(z) \to 0$ identifies $\mu_0(x)$
- We can form $K_0(P(z)) = -\tilde{K}_0(P(z))\frac{P(z)}{1-P(z)}$
- $\therefore$ Can construct the desired counterfactual mean.

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- Notice how we can get EOTM: Effect of Treatment for Person at the Margin of Going into the Program

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z)$$

- Under normality we have (as a result of independence and normality)

$$\begin{aligned}
&E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) \\
&= \mu_1(x) - \mu_0(x) + E\left(U_1 - U_0 \mid \frac{\mu_D(z)}{\sigma_V} = \frac{V}{\sigma_V}, X = x, Z = z\right) \\
&= \mu_1(x) - \mu_0(x) + \mathrm{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right)\frac{\mu_D(z)}{\sigma_V}
\end{aligned}$$

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In the Generalized Roy Model where $U_C = 0$ but $\mu_C(z) \neq 0$

$$
\begin{aligned}
&= \mu_1(x) - \mu_0(x) - \sigma_V \left(\frac{\mu_D(z)}{\sigma_V} \right) \\
&= \mu_1(x) - \mu_0(x) - \mu_D(z) \\
&= \mu_C(z)
\end{aligned}
$$

(marginal gain = marginal cost)

- MTE: Marginal Treatment Effect for a person for whom $U = u$

$$E(Y_1 - Y_0 \mid V = v, X = x, Z = z) =$$
$$= \mu_1(x) - \mu_0(x) + \mathrm{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) v$$

- EOTM picks $v = \frac{\mu_D(z)}{\sigma_V}$
- Notice we can use the result that

$$\frac{\mu_D(z)}{\sigma_V} = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(P(z))$$
$$V = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(U_D)$$

- EOTM:

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) =$$
$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(P(z))$$

- MTE:

$$E(Y_1 - Y_0 \mid V = v, X = x, Z = z) =$$
$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(U_D)$$

A useful fact:

$$\begin{aligned} P(z) &= \Pr(D = 1 \mid Z = z) \\ &= \Pr(\mu_D(z) \geq V) \\ &= \Pr\left(\frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) \end{aligned}$$

$$\begin{aligned} P(z) &= F_{\frac{V}{\sigma_V}}\left(\frac{\mu_D(z)}{\sigma_V}\right) \\ U_D &= F_{\frac{V}{\sigma_V}}\left(\frac{V}{\sigma_V}\right); \quad \text{Uniform}(0,1) \end{aligned}$$

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$$
\begin{aligned}
P(z) &= \Pr\left(F_{\frac{V}{\sigma_V}}\left(\frac{\mu_D(z)}{\sigma_V}\right)\right) \geq F_{\frac{V}{\sigma_V}}\left(\frac{V}{\sigma_V}\right) \\
&= \Pr\left(P(z) \geq U_D\right)
\end{aligned}
$$

$P(z)$ is the $p(z)^{\text{th}}$ quantile of U_D.

Recall

$$
\begin{aligned}
Y &= DY_1 + (1-D)Y_0 \\
&= Y_0 + D(Y_1 - Y_0)
\end{aligned}
$$

Keep X implicit (condition on $X = x$)

$$
\begin{aligned}
E(Y \mid Z = z) &= E(Y_0) + \underbrace{E(Y_1 - Y_0 \mid D = 1, Z = z)P(z)}_{\text{from law of iterated expectations}} \\
&= E(Y_0) + E(Y_1 - Y_0 \mid P(z) \geq U_D)P(z)
\end{aligned}
$$

$\therefore$ It depends on Z only through $P(Z)$.

$$
E(Y \mid Z = z') = E(Y_0) + E(Y_1 - Y_0 \mid P(z') \geq U_D)P(z')
$$

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- What is $E(Y_1 - Y_0 \mid P(z) \geq U_D)$?
- Let the joint density of $(Y_1 - Y_0, U_D)$ be

$$f_{Y_1-Y_0,U_D}(y_1 - y_0, u_D).$$

- It does not depend on Z. It will, in general, depend on X.

$$E(Y_1 - Y_0 \mid P(z) \geq U_D)$$
$$= \frac{\int\limits_{-\infty}^{\infty} \int\limits_{0}^{P(z)} (y_1 - y_0) f_{y_1-y_0,u_D}(y_1 - y_0, u_D)\, du_D\, d(y_1 - y_0)}{\Pr(P(z) \geq U_D)}$$

- Now recall that

$$U_D = F_{\left(\frac{V}{\sigma_V}\right)}\left(\frac{V}{\sigma_V}\right).$$

- U_D is a quantile of the V/σ_V distribution.

- By construction, U_D is Uniform$(0,1)$ (this is the definition of a quantile).
- $\therefore f_{U_D}(u_D) = 1$.
- Also, $\Pr(P(z) \geq U_D) = P(z)$.
- Notice, by law of conditional probability,

$$f_{Y_1-Y_0,U_D}(y_1 - y_0, u_D) = f_{Y_1-Y_0,U_D}(y_1 - y_0 \mid U_D = u_D) \underbrace{f_{U_D}(u_D)}_{=1}.$$

$$E(Y_1 - Y_0 \mid P(z) \geq U_D)$$

$$= \frac{\int\limits_0^{P(z)} \int\limits_{-\infty}^{\infty} (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D)\, d(y_1 - y_0)\, du_D}{P(z)}$$

$$E(Y_1 - Y_0 \mid P(z) \geq U_D)$$

$$= \frac{\int\limits_0^{P(z)} \int\limits_{-\infty}^{\infty} (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0 \mid U_D = u_D)\, d(y_1 - y_0)\, du_D}{P(z)}$$

$$= \frac{\int\limits_0^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D)\, du_D}{P(z)}$$

$$\therefore E(Y \mid Z = z) = E(Y_0) + \int_0^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

$$\frac{\partial E(Y \mid Z = z)}{\partial P(z)} = \underbrace{E(Y_1 - Y_0 \mid U_D = P(z))}_{\text{EOTM or marginal gains}}$$

$$E(Y \mid Z = z') = E(Y_0) + \int_0^{P(z')} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

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- Suppose $P(z) > P(z')$

$$\therefore E(Y \mid Z = z) - E(Y \mid Z = z') =$$

$$= \int_{P(z')}^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

$$= E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z')) \Pr(P(z) \geq U_D \geq P(z'))$$

- Notice

$$\Pr(P(z) \geq U_D \geq P(z')) = \int_{P(z')}^{P(z)} du_D$$

$$= P(z) - P(z')$$

$$E(Y \mid Z = z) - E(Y \mid Z = z')$$

$$= \underbrace{E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z'))}_{\text{LATE}}(P(z) - P(z'))$$

- $$\frac{E(Y \mid Z = z) - E(Y \mid Z = z')}{P(z) - P(z')} = \text{LATE}(z, z')$$

$$= \frac{\int_{P(z')}^{P(z)} \text{MTE}(u_D) du_D}{P(z) - P(z')}$$

(Keep X implicit)

$$
\begin{aligned}
E\left(Y_p\right) &= \int_0^1 E(Y_p \mid P_p(Z_p) = t)\, dF_{P_p}(t) \\
&= \int_0^1 \Big[\int_0^1 [\mathbf{1}_{[0,t]}(u_D) E(Y_{1,p} \mid U_D = u_D) \\
&\qquad + \mathbf{1}_{(t,1]}(u_D) E(Y_{0,p} \mid U_D = u_D)]\, du \Big]\ dF_{P_p}(t) \\
&= \int_0^1 \Big[\int_0^1 [\mathbf{1}_{[u_D,1]}(t) E(Y_{1,p} \mid U_D = u_D) \\
&\qquad + \mathbf{1}_{(0,u_D]}(t) E(Y_{0,p} \mid U_D = u_D)]\, dF_{P_p}(t) \Big]\ du_D \\
&= \int_0^1 \big[(1 - F_{P_p}(u_D)) E(Y_{1,p} \mid U_D = u_D) \\
&\qquad + F_{P_p|X}(u_D) E(Y_{0,p} \mid U_D = u_D) \big]\ du_D.
\end{aligned}
$$

- This derivation involves changing the order of integration.
- Note that from finiteness of the mean,

$$E\left|\mathbf{1}_{[0,t]}(u_D)E(Y_{1,p} \mid U_D = u_D) + \mathbf{1}_{(t,1]}(u_D)E(Y_{0,p} \mid U_D = u_D)\right| \\ \leq E(|Y_1| + |Y_0|) < \infty,$$

$\therefore$ the change in the order of integration is valid by Fubini's theorem.

- Comparing policy p to policy p',

$$E\left(Y_p\right) - E\left(Y_{p'}\right) = \int_0^1 \underbrace{E(Y_1 - Y_0 \mid U_D = u_D)}_{\text{MTE}(u_D)} \left(F_{P_{p'}}(u_D) - F_{P_p}(u_D)\right) du_D,$$

which gives the required weights.
- Policies shift the distribution of $P(Z)$.
- They keep the distribution of Y_1 and Y_0 unchanged.