Sheshinski Specification

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Basic Sheshinski Specification

- $\beta = 1$, $\alpha = 1$ in $\dot{H} = AH^{\beta} I^{\alpha} - \sigma H$

$$\dot{H} = AIH - \sigma H$$

$$\mathcal{H} : \quad e^{-rt} R(1 - I)H + \mu(AIH - \sigma H)$$

- Bang-Bang: $I = 1$ if

$$\mu(t)AH \geq e^{-rt} RH$$
$$\mu(t)e^{rt} \geq \frac{R}{A}$$

- Let $g(t) = \mu(t)e^{rt}$.

$$\dot{g} = -R + (R - Ag)I + (\sigma + r)g$$
$$g(T) = 0$$

- Transversality: $\mu(T)H(T) = 0$, i.e., $g(T)H(T) = 0$.
- Observe if $g(0) > \frac{R}{A}$, $I(0) = 1$.
- When $I = 1$,

$$\dot{g} = (\sigma + r - A)g$$

- If $\sigma + r - A > 0$, i.e., $\sigma + r > A$, so $g \uparrow$ and $I = 1$ ever after.
- Violates the transversality condition.
- Nothing bounds the policy.
- $\sigma + r < A$ implies $g \downarrow$.
- Therefore, after g falls to $\frac{R}{A}$, $I = 0$. Then

$$\dot{g} = -R + (\sigma + r)g.$$

- Now with $(\sigma + r)g < R$, if the agent doesn't ever invest again:

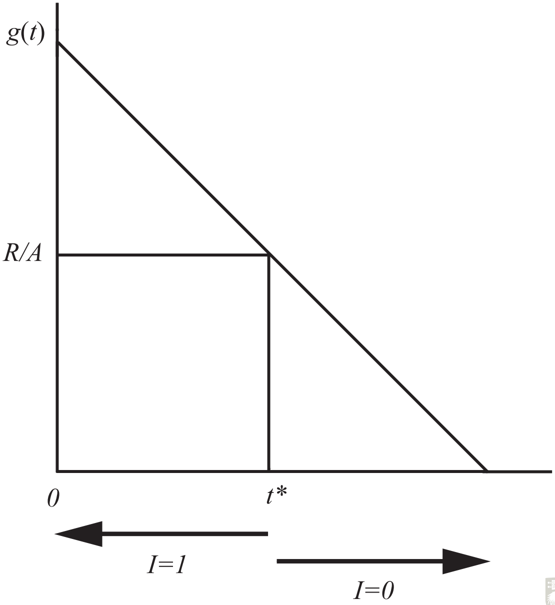
$$g(t) = R \int_t^T e^{(\sigma+r)(t-\tau)} d\tau$$

$$= \frac{R}{\sigma + r}(1 - e^{(\sigma+r)(t-T)}) \leq \frac{R}{\sigma + r}$$

- If invest in future at $\hat{t} > t$

$$\dot{g} = (\sigma + r - A/g) \downarrow$$

- $\therefore g(t)$ is declining everywhere.
- Thus we never invest again in the future.
- Graphically displaying the rule we obtain:


g(t)
R/A
0
t*
I=1
I=0

- At switching age t^*,

$$\frac{R}{A} = \frac{1}{\sigma + r}\left(1 - e^{(\sigma+r)(t^* - T)}\right)$$

$$t^* = T + \frac{1}{\sigma + r} - \frac{1}{A}$$

t^* is schooling.

- $T \uparrow \Rightarrow t^* \uparrow$
- $\sigma, r \uparrow \Rightarrow t^* \downarrow$
- $A \uparrow \Rightarrow t^* \uparrow$
- Initial endowments don't affect schooling.

- For $t \in [0, t^*]$,

$$\frac{\dot{H}}{H} = (A - \sigma)t + \varphi, \qquad H(0) = H_0$$

$$H(t) = e^{(A-\sigma)t} H(0).$$

- Human capital at schooling age t^* is

$$H(t^*) = H(0) e^{(A-\sigma)} \left(T + \frac{1}{\sigma + r} - \frac{1}{A} \right).$$

- Coefficient on schooling: Mincer's "r" is $(A - \sigma)$

$$\begin{aligned} Y(t^*) &= RH(0)H(t^*) \\ \ln Y(t^*) &= \ln RH(0) + (A - \sigma)\underset{\substack{\uparrow \\ \text{years of school}}}{t^*} \end{aligned}$$

Interior Sheshinski Specification

- Now consider $0 < \alpha < 1$:

$$\dot{H} = AI^{\alpha}H - \sigma H$$

$$g(t) = \mu e^{rt}$$

$$\mathcal{H} = e^{-rt}R(1 - I)H + \mu(AI^{\alpha}H - \sigma H)$$

- Therefore, if $g(t) \geq \dfrac{R}{A}$, person invests, full time $I = 1$.
- We get Sheshinski-like policy:

$$\dot{g} = (\sigma + r - A)g$$

- Need $(\sigma + r - A) < 0$ to satisfy optimality of investment ($g(T) = 0$).

Interior Solution Case

- We have

$$RH = \alpha g(t) A l^{\alpha-1} H$$

$$\dot{g} = -R(1-l) - gAl^{\alpha} + (\sigma + r) g$$

- Now

$$g(t) = \int_t^T e^{-(\sigma+r)(t-\tau)} [(R) \underbrace{(1-l)}_{\substack{\text{cash} \\ \text{flow}}} + \underbrace{gAl^{\alpha}}_{\substack{\text{future} \\ \text{productivity}}}] \, d\tau$$

- I is obtained from the first order condition:

$$I = \left[\frac{R}{\alpha g(t)A}\right]^{\frac{1}{\alpha-1}} = \left[\frac{\alpha g(t)A}{R}\right]^{\frac{1}{1-\alpha}}$$

$$\begin{aligned}
\dot{g} &= -R\left(1 - \left(\frac{\alpha A}{R}\right)^{\frac{1}{1-\alpha}} g(t)^{\frac{1}{1-\alpha}}\right) \\
&\quad -gA\left(\frac{\alpha A}{R}\right)^{\frac{\alpha}{1-\alpha}} g^{\frac{\alpha}{1-\alpha}} + (\sigma + r)g \\
&= -R + (g)^{\frac{1}{1-\alpha}}\varphi + (\sigma + r)g
\end{aligned}$$

$$\dot{g} = -R + (g)^{\frac{1}{1-\alpha}}\varphi + (\sigma + r)g,$$

where

$$\begin{aligned}\varphi &= R\left(\frac{\alpha A}{R}\right)^{\frac{1}{1-\alpha}} - A\left(\frac{\alpha A}{R}\right)^{\frac{\alpha}{1-\alpha}} \\ &= (A)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R}\right)^{\frac{\alpha}{1-\alpha}}(\alpha - 1) < 0.\end{aligned}$$

When $\sigma + r = 0$, $\dot{g} < 0$ for sure.

- Note: Solution does not depend on initial conditions.
- Case $\alpha = \frac{1}{2}$ produces Riccati equation:

$$\dot{g} = -R + g^2\varphi + (\sigma + r)g$$

- Solution: Let

$$g^2\varphi + (\sigma + r)g - R = 0$$
$$(g - r_+)(g - r_-) = 0$$

- r_+ and r_- are roots of equation (may be complex). Then, we can easily solve.

- Suppose $r_+ \neq r_-$ (distinct roots)

$$\frac{g(t) - r_+}{g(t) - r_-} = c\, e^{\varphi(r_+ - r_-)t}$$

- Transversality $\Rightarrow g(T) = 0$. Therefore,

$$\frac{r_+}{r_-} = ce^{\varphi(r_+ - r_-)T}$$

$$c = \left(\frac{r_+}{r_-}\right) e^{-\varphi(r_+ - r_-)T}$$

- For $r_+ = r_- = r_0 \neq 0$ because $(\sigma + r) > 0$, $R > 0$;

$$g(t) - r_0 = \frac{1}{c - \varphi t}$$

$$g(t) = r_0 + \frac{1}{c - \varphi t}$$

$$g(T) = 0 \Rightarrow c = \varphi T - y\frac{1}{r_0}$$

- Complex case is of economic interest.

$$r_{\pm} = \frac{-(\sigma + r) \pm \sqrt{(\sigma + r)^2 + 4\varphi R}}{2\varphi}$$

for $\alpha = \frac{1}{2}$, $\varphi = -(A)^2 R^{-1} \frac{1}{4}$.

- Therefore:

$$(\sigma + r)^2 - \frac{4R}{4}(A^2)R^{-1}$$

$$(\sigma + r)^2 - A^2, \text{ but } < 0 \text{ from transversality}$$

$$\begin{aligned} r_{\pm} &= \frac{-(\sigma + r) \pm \sqrt{(\sigma + r)^2 - A^2}}{-\frac{1}{2}A^2 R^{-1}} \\ &= \frac{+2R(\sigma + r)}{A^2} \mp \frac{2R\sqrt{(\sigma + r)^2 - A^2}}{A^2}. \end{aligned}$$

- Now solution is very simple.

$$(g(t) - r_+) = \left(\frac{r_+}{r_-}\right) e^{\varphi(r_+ - r_-)(t-T)}(g(t) - r_-)$$

$$g(t)\left[1 - \frac{r_+}{r_-} e^{\varphi(r_+ - r_-)(t-T)}\right] = r_+(1 - e^{\varphi(r_+ - r_-)(t-T)})$$

$$g(t) = r_+ \frac{1 - e^{\varphi(r_+ - r_-)(t-T)}}{1 - \frac{r_+}{r_-} e^{\varphi(r_+ - r_-)(t-T)}}.$$

- Now,

$$r_+ = a + bi, \qquad r_+ - r_- = (2bi), \qquad r_- = a - bi$$

- Set $\theta = \varphi(2b)(t - T)$ (in radians)

$$\begin{aligned} g(t) &= r_+ \frac{(1 - e^{i\theta})}{1 - \frac{r_+}{r_-} e^{i\theta}} \\ &= (r_+ r_-) \frac{(1 - e^{i\theta})}{(r_- - r_+ e^{i\theta})} \end{aligned}$$

- $r_+ r_- = a^2 + b^2$. Now multiply by $e^{-i\theta/2}$,

$$g(t) = (r_+ r_-) \frac{(e^{-i\theta/2} - e^{i\theta/2})}{(r_- e^{-i\theta/2} - r_+ e^{i\theta/2})}$$

Using $\cos(-x) = \cos x \quad \sin(-x) = -\sin x$,

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
g(t) &= (r_+ r_-)\left[\frac{\cos(\theta/2) - i\sin\theta/2 - \cos(\theta/2) - i\sin\theta/2}{-2ai\sin\theta/2 - 2b_i\cos\theta/2}\right] \\
&= (r_+ r_-)\left[\frac{\sin\theta/2}{a\sin\theta/2 + b\cos\theta/2}\right] \\
&= \left(\frac{r_+ r_-}{a}\right)\left[\frac{1}{1 + \frac{b}{a}\cot\theta/2}\right]
\end{aligned}
$$

Therefore,

$$
g(t) = \frac{(a^2 + b^2)}{a}\left[\frac{1}{1 + \frac{b}{a}\cot\varphi b(t - T)}\right]
$$

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$$a = \frac{2(\sigma + r)R}{A^2} \qquad b = \frac{2R}{A^2}(A^2 - (\sigma + r)^2)^{1/2}$$

$$\varphi b = -\frac{1}{2}(A^2 - (\sigma^2 + r^2))^{1/2}$$

$$\frac{b}{a} = \frac{2R(A^2 - (\sigma^2 + r^2))^{1/2}/A^2}{2\frac{(\sigma + r)R}{A^2}} = \frac{[A^2 - (\sigma^2 + r^2)]^{1/2}}{\sigma + r}$$

When $\sigma + r = 0$,

$$r_{\pm} = \pm\frac{\sqrt{4\varphi R}}{2\varphi} = \pm\sqrt{\frac{R}{\varphi}} = \pm\sqrt{\frac{4R}{-A^2 R^{-1}}} = \left(\frac{2R}{A}\right) i$$

$$\varphi b = \left[-(A)^2 \frac{R^{-1}}{4}\right]\left[\frac{2R}{A^2}A\right] = -\frac{A}{2}$$

$$\begin{aligned} g(t) &= \left(\frac{4R^2}{A^2}\right) \frac{\tan}{\frac{2R}{A^2}A}(\theta/2) \\ &= \left(\frac{2R}{A}\right) \tan\left[\frac{\theta}{2}\right] \\ &= \left(\frac{2R}{A}\right) \tan\left(-\frac{A}{2}(t-T)\right) \\ &= \left(\frac{2R}{A}\right) \tan\left(\frac{A}{2}(T-t)\right) \end{aligned}$$

- From definition of θ, we obtain

$$g(t) = \left(\frac{2R}{A}\right) \tan\left(\frac{A}{2}(T-t)\right)$$

Modified Sheshinski Specification (More Interesting)

$$\dot{H} = AI - \sigma$$

$$\mathcal{H} = e^{-rt}R(1 - I)H + \mu(t)(AI - \sigma H)$$

- $I = 1$ if $\mu A \geq e^{-rt}R$
- $I = 0$ otherwise

$$g(t) = \mu(t)e^{rt}$$
$$\dot{g} = -R(1-I) + g(\sigma + r)$$
$$g(t) = R\int_t^T e^{+(\sigma+r)(t-\tau)}(1-I)\,d\tau$$
$$g \geq \frac{R}{A}H, \qquad I = 1$$

- When $I = 1$, $\dot{g} = g(\sigma + r) > 0$ and g$\uparrow$
- Intuition: as $t\uparrow$ agent is getting nearer the payoff period.
- While the agent invests he/she gets no return.

- First take case when $\sigma = 0$

$$\dot{g} = -R(1-I) + rg$$

- For $t = 0$, if $g(t) \geq \frac{R}{A}H(t)$; $I = 1$; $\dot{H} = A$,

$$H(t) = At + H(0)$$

- Let $\hat{t}$ be the age of the first interior solution.
- At $\hat{t}$, $g(\hat{t}) = \frac{R}{A}H(\hat{t})$,

$$g(0)e^{r\hat{t}} = \frac{R}{A}[A\hat{t} + H(0)]$$

- Observe that
$$g(t) \leq R \int_t^T e^{+(\sigma+r)(t-\tau)}\, d\tau$$
(i.e. set $I(\tau) = 0$).
- Therefore, $g(t) \leq \frac{R}{\sigma+r}\left(1 - e^{+(\sigma+r)(t-\tau)}\right) \leq \frac{R}{\sigma+r}$
- Therefore, $\dot{g} < 0$ (after the period of investment)
- Thus at most one period of specialization and it comes at the beginning of life if at all. Will not arise if $g(0) < \frac{R}{A}$, i.e. $A < r$ precludes this (return by investment < return by saving in lending market.
- This is a model of schooling.

- Therefore, t^* is solution from

$$\frac{R}{r}\left(1-e^{r(t^*-T)}\right) = \frac{R}{A}(At^*+H_0)$$

$$\left(1-e^{r(t^*-T)}\right) = \frac{r}{A}(At^*+H_0)$$

- The higher H_0, the lower t^*.
- Need $r < A$ for feasibility.
- Human capital stock at end of school:

$$H = At^* + H_0$$

$$Y(t^*) = R(At^* + H_0)$$

- Take case where $\sigma > 0$. Now, by the previous logic, $g \leq \frac{R}{\sigma + r}$.
- Therefore, $\dot{g} < 0$.
- Now investment pattern *may* be more complex.
- Suppose $g(0) \geq \frac{R}{A} H(0)$. Then $I(0) = 1$.

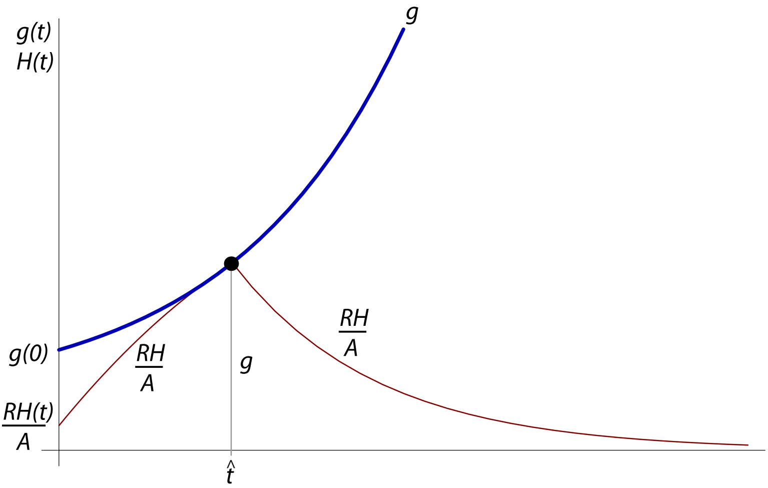
$$\frac{\dot{g}}{g} = (\sigma + r)$$
$$\dot{H} = A - \sigma H$$

$$\begin{aligned} H(t) &= A \int_0^t e^{-\sigma(t-\tau)} \, d\tau + H(0) e^{-\sigma t} \\ &= \frac{A}{\sigma}(1 - e^{-\sigma t}) + H(0) e^{-\sigma t} \\ &= \frac{A}{\sigma} + \left[H(0) - \frac{A}{\sigma} \right] e^{-\sigma t} \end{aligned}$$

- $$
\begin{aligned}
g(0)e^{(\sigma+r)\hat{t}} &= \frac{R}{A}\left(\frac{A}{\sigma}(1-e^{-\sigma\hat{t}}) + H(0)e^{-\sigma\hat{t}}\right) \\
&= R\left(\frac{H(0)}{A} - \frac{1}{\sigma}\right)e^{-\sigma\hat{t}} + \frac{R}{\sigma}
\end{aligned}
$$

- To ensure $\dot{H} > 0$ at $t = 0$, need $A - \sigma H(0) > 0 \Rightarrow A > \sigma H(0) \Rightarrow \frac{1}{\sigma} > \frac{H(0)}{A}$.

For intersection to occur, we have:



- $$g(0) \geq \frac{R}{A} H(0)$$
  $$H(t) = \frac{A}{\sigma} + \left[H(0) - \frac{A}{\sigma} \right] e^{-\sigma t}$$
- t_1 is the first point where $g(t_1) = \frac{R}{A} H(t_1)$
- $\dot{g}(t) = (\sigma + r) g$ so $g(t_1) = e^{(\sigma + r) t_1} g(0)$.
- Then,
  $$\frac{R}{\sigma} + \frac{R}{A} \left[H(0) - \frac{A}{\sigma} \right] e^{-\sigma t_1} = g(0) e^{(\sigma + r) t_1}.$$

- Then at t_1, $I = 0$,

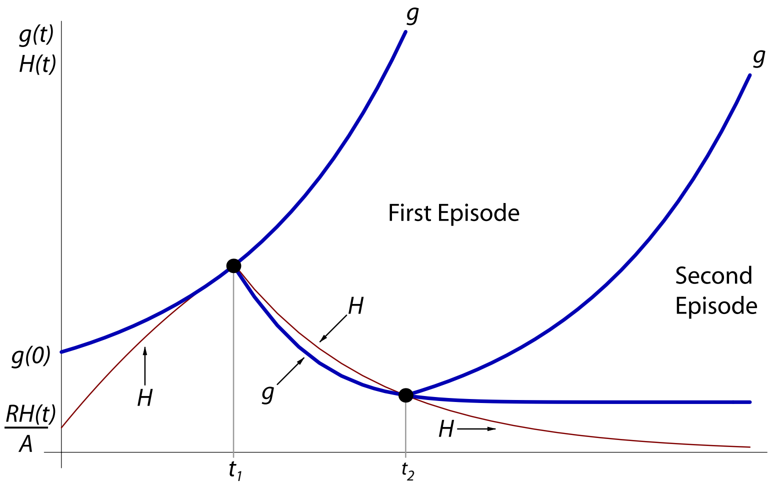
$$\begin{aligned}
\dot{g} &= -R + (\sigma + r)g \\
H(t) &= H(t_1)e^{-\sigma(t-t_1)} \qquad t_1 < t < t_2 \\
g(t) &= \frac{R}{\sigma + r}\left(1 - e^{+(\sigma+r)(t-t_2)}\right) + g(t_2)e^{(\sigma+r)(t-t_2)}
\end{aligned}$$

- At t_2, we have that

$$\begin{aligned}
\frac{RH(t_2)}{A} &= RH(t_1)e^{-\sigma(t_2-t_1)} \\
&= g(t_2) = \int_{t_2}^{T} e^{-(\sigma+r)(t_2-\tau)}(1 - I(\tau))\, d\tau
\end{aligned}$$

- Then person bangs in at $I = 1$ and, possibly a sequence of intervals of specialization.
- $t_2 < t < t_3$; etc.

One possible trajectory



- We could also have one shot indefinitely (but last shots are "short").
- Observe:

$$g(t) = R \int_{t_1}^{t_2} e^{(\sigma+r)(t-\tau)}\, d\tau + \cdots + \int_{t_3}^{t_4} e^{(\sigma+r)(t-\tau)}\, d\tau + \cdots$$

- For $t < t_1$, $t \uparrow$, $g \uparrow$ can happen.
- For this to occur:
  - In a neighborhood of t_1:

$$\dot{g}(t_1) < \left.\frac{R\dot{H}(t)}{A}\right|_{t=t_1}$$

    (demand price less than opportunity cost).
  - The curves must cross. Otherwise, we get failure of transversality.

- Whether or not such investment activity occurs depends on initial $H(0)$ and other parameters.
- Thus, at time t_1, for this to arise, we need:

$$\dot{g}\Big|_{t=t_1} < \frac{R\dot{H}(t)}{A}\Bigg|_{t=t_1}.$$

- g is continuous at t_1 (but not necessarily differentiable and, in our case, definitely not).

- At t_1, $g(t_1) = \frac{R}{A}H(t_1)$

$$\dot{g} = -R + (\sigma + r)g \qquad \text{(from right)}$$

$$\frac{R}{A}\dot{H}(t_1) = -\sigma\frac{R}{A}H(t_1) = -\sigma g(t_1)$$

- Therefore, we need:

$$-R + (\sigma + r)g(t_1) < -\sigma g(t_1) = \left.\frac{R\dot{H}(t)}{A}\right|_{t=t_1}$$

- However, this is not guaranteed by $\frac{R}{\sigma+r} > g$. We need a tighter bound.

- For specialization to occur at 0, we need:

$$g(0) \geq \frac{R}{A} H(0),$$

but we need the slope of $\left.\frac{RH(t)}{A}\right|_{t=0}$ to exceed $\left.\dot{g}\right|_{t=0}$ (otherwise, g curve and $\frac{R}{A}H(t)$ curves do not intersect).

- For the required condition we need (using expression for $RH(t)$ in a neighborhood of $t = 0$):

$$R\left(1 - \frac{\sigma H(0)}{A}\right) > g(0)(\sigma + r)$$

- Sufficient condition:

$$\left(1 - \frac{\sigma H(0)}{A}\right) \geq 1 - e^{(\sigma + r)T}$$

(but this is way too strong)

- Necessary condition:

$$\frac{\sigma H(0)}{A} < 1$$

(otherwise, never pays to specialize)

- Therefore, if $H(0)$ is too high, agent never specializes.
- At $g(0)$, we must have:

$$\frac{R}{\sigma + r}\left(1 - \frac{\sigma H(0)}{A}\right) > g(0) > \frac{RH(0)}{A}.$$

If $H(0)$ big enough, cannot happen.

Observe that:

$$g(t) = R \int_t^T e^{-(\sigma+r)(t-\tau)} (1 - I(\tau))\, d\tau$$

Recall that I switches between 0 and 1. Therefore:

- For $0 < t < t_1$ (person invests),

$$g(t) = \frac{R}{\sigma+r} e^{(\sigma+r)t} \sum_{k \geq 1} (-1)^{k+1} e^{-(\sigma+r)t_k}$$

- For $t_1 < t < t_2$ (person does not invest),

$$g(t) = \frac{R}{\sigma+r} \left[1 - e^{(\sigma+r)(t-t_2)}\right] + \frac{R}{\sigma+r} e^{(\sigma+r)t} \sum_{k \geq 3} (-1)^{k+1} e^{-(\sigma+r)t_k}$$

- For $t_2 < t < t_3$ (etc.),

$$g(t) = \frac{R}{\sigma+r} e^{(\sigma+r)t} \sum_{k \geq 3} (-1)^{k+1} e^{-(\sigma+r)t_k}$$

- Cannot prove that $g(t_3) < g(t_1)$ for all policies.
- Person may build up stock of human capital over the lifetime.