The Race Between Demand and Supply: Tinbergen's Pioneering Studies of Earnings Inequality

James J. Heckman

Econ 350, Winter 2021
This draft, February 17, 2021 4:35pm

THE UNIVERSITY OF CHICAGO

Tinbergen's Hedonic Model (1956)
Applied and Extended in Rosen (JPE, 1974)

- A model for heterogenous firms and workers
- One worker: one firm match
- In the scalar case, an example is hours supplied

- Consider the supply of attributes to the market.
- Define $U(c, z, x, \varepsilon, A)$ as the preferences of workers where x and ε represent observed (by the econometrician) and unobserved characteristics of workers that vary across persons, A represents preference parameters common across persons and c is consumption where $c = P(z) + R$.
- Higher values of z lead to lower values of U.
- For ease of exposition, assume $R = 0$.

- Given $P(z)$, a twice continuously differentiable price function, and assuming the utility function is twice differentiable, one obtains the following first and second order conditions for a maximum

$$U_c\left(c, z, x, \varepsilon, A\right) P_z\left(z\right) + U_z\left(c, z, x, \varepsilon, A\right) = 0 \tag{1}$$

$$U_{zz'} + U_c P_{zz'} + 2 P_z U_{cz'} + P_z U_{cc}\left(P_z\right)' \text{ is negative definite.} \tag{2}$$

THE UNIVERSITY OF CHICAGO

- These conditions characterize optimal job attribute choices for each worker.
- For each location z in attribute space, they characterize the set of workers who choose that location.
- Workers are heterogenous and differ in their willingness to supply attribute z to the market.

- Assume that the production function is twice differentiable. Profits are

$$\Pi\left(z, y, \eta, B, P\left(z\right)\right) = \Gamma(z, y, \eta, B) - P(z)$$

- The first and second order conditions for a maximum are

$$\Gamma_z\left(z,y,\eta,B\right)-P_z\left(z\right)=0 \tag{3}$$

$$\Gamma_{zz'}-P_{zz'} \text{ is negative definite.} \tag{4}$$

- Throughout I follow the classical hedonic literature and assume the regular case in which the second order conditions hold as strict inequalities, $\Gamma_{z\eta'}$ is positive definite, and $P_z U_{c\varepsilon'} + U_{z\varepsilon'}$ is positive definite.
- These conditions guarantee positive sorting on unobservables in the sense that in equilibrium $\frac{\partial \eta}{\partial z} > 0$ and $\frac{\partial \varepsilon}{dz} > 0$.
- Firms are heterogeneous.
- Tinbergen anticipated modern developments on sorting in the labor market.

- Heterogeneous workers differ in their preference attributes vectors x and ε.
- Heterogeneous firms differ in their productivity vectors y and η.
- Let the densities of x and ε be f_x and f_ε and let x be independent of ε.
- x and ε have supports $\mathcal{X}$ and $\mathcal{E}$ respectively.
- The densities of y and η are f_y and f_η. y is assumed independent of η and y and η have supports $\mathcal{Y}$ and $\mathcal{H}$ respectively.

- Assume that x, ε, y, and η are absolutely continuous random variables.
- We focus on the case in which $\dim(\varepsilon) = \dim(\eta) = \dim(z)$.
- For this case, there is no bunching in equilibrium.
- That is, in equilibrium every bundle of characteristics has population measure zero of demanders or suppliers.

- Given the previous assumptions, a local implicit function theorem applies and one can invert the first order conditions (FOC) (1) and (3) to obtain ε and η as functions of z and x and y, respectively. Inverting the FOC (1) for the worker one obtains

$$\varepsilon = \varepsilon\left(z, P_z, P(z), x, A\right).$$

- Similarly, inverting the FOC (3) for the firm one obtains

$$\eta = \eta\left(z, P_z, y, B\right).$$

- Using these relationships, use f_x and f_ε to find the density of z supplied given $P(z)$, and use f_y and f_η to find the density of z demanded given $P(z)$.

- The **Supply Density** is:

$$\int_{\mathcal{X}} f_\varepsilon\left(\varepsilon\left(z, P_z, P(z), x, A\right)\right) \det\left[\frac{d\varepsilon\left(z, P_z, P(z), x, A\right)}{dz}\right] f_x\left(x\right) dx.$$

- where the term in square brackets is the Jacobian matrix with respect to vector z (i.e., its effect on all arguments of ε that depend on z).
- This is the density of the amenity supplied as a function of the price function, preference parameters A, and the densities of x and ε.

- The **Demand Density** is:

$$\int_{\mathcal{Y}} f_{\eta}\left(\eta\left(z, P_{z}, y, B\right)\right) \det\left[\frac{d\eta\left(z, P_{z}, y, B\right)}{dz}\right] f_{y}\left(y\right) dy.$$

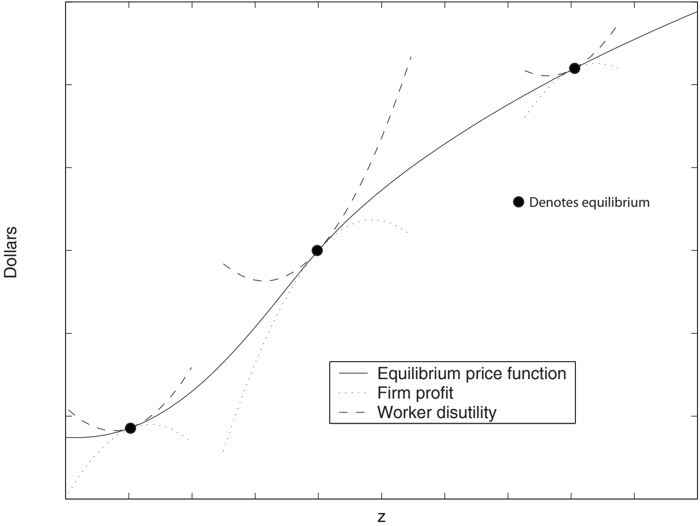
- Again, the term in square brackets is the Jacobian matrix with respect to vector z.
- This is the density of demand for a given price function, vector of technology parameters B, and pair of densities of y and η.
- From the second order conditions (2) and (4), respectively, the Jacobian terms are both positive.

- **It was Tinbergen's great insight that equilibrium in hedonic markets requires that supply and demand be equated at each point of the support of** z.
- Equilibrium prices $P(z)$ must satisfy the following second order partial differential equation

$$\int_{\mathcal{X}} f_{\varepsilon}\left(\varepsilon\left(z, P_z, P(z), x, A\right)\right) \det\left[\frac{d\varepsilon\left(z, P_z, P(z), x, A\right)}{dz}\right] f_x(x)\, dx = \tag{5}$$

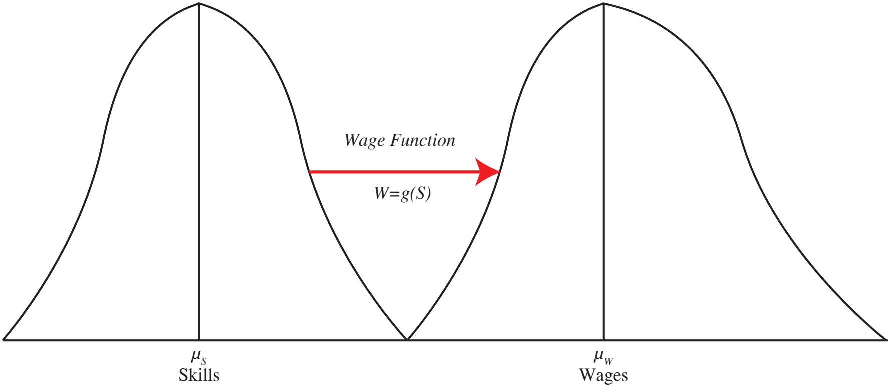
$$\int_{\mathcal{Y}} f_{\eta}\left(\eta\left(z, P_z, y, B\right)\right) \det\left[\frac{d\eta\left(z, P_z, y, B\right)}{dz}\right] f_y(y)\, dy.$$

Figure 1: Optimal job choice for three worker-firm pairs



Source: Ekeland et al. (2004)

Figure 2: Density of Skills versus Density of Wages



Source: Stallings (2020)

- Similar function for demand distribution
- $g(S)$ determined by equilibrium
- Demand (at S) $=$ supply (at S) for all S in the market

- When z is scalar and utility is quasi-linear so that $U(c, z, x, \varepsilon, A) = c - V(z, x, \varepsilon, A)$, $\frac{d\varepsilon}{dz} = \frac{P_{zz} - V_{zz}}{V_{z\varepsilon}}$ and $\frac{d\eta}{dz} = \frac{P_{zz} - \Gamma_{zz}}{\Gamma_{z\eta}}$.
- Since $V_{z\varepsilon} < 0$ and $\Gamma_{z\eta} > 0$, one can substitute these expressions into (5) to obtain an explicit expression for P_{zz}, the second derivative of the pricing functional:

$$P_{zz} = \frac{\int\limits_{\mathcal{X}} f_\varepsilon f_x \left(\frac{V_{zz}}{-V_{z\varepsilon}} \right) dx + \int\limits_{\mathcal{Y}} f_\eta f_y \left(\frac{\Gamma_{zz}}{\Gamma_{z\eta}} \right) dy}{\int\limits_{\mathcal{Y}} \frac{f_\eta f_y}{\Gamma_{z\eta}} dy + \int\limits_{\mathcal{X}} \frac{f_\varepsilon f_x}{-V_{z\varepsilon}} dx} \tag{6}$$

- The arguments of the functions have been suppressed for ease of exposition.

Tinbergen's Specific Version of The Hedonic Model

- Assume that preferences are quadratic in z and linear in c, unearned income $R = 0$, and that individual heterogeneity (x, ε) only affects utility through the single index $\theta = \mu_\theta(x) + \varepsilon$ where $\dim(\theta) = \dim(z)$.
- Workers maximize

$$U(c, z, \theta, A) = P(z) + \theta' z - \frac{1}{2} z' A z.$$

- The conditions determining a worker's maximum are

$$P_z + \theta - Az = 0$$

where $P_{zz'} - A$ is negative definite.
- On the firm side, assume that the production function is quadratic in z and that firm heterogeneity only affects profits through the single index $\nu = \mu_\nu(y) + \eta$ where $\dim(\nu) = \dim(z)$.

- Profits are

$$\Pi\left(z,\nu,B,P\left(z\right)\right)=\nu' z-\frac{1}{2}z' Bz-P\left(z\right)$$

- The conditions determining a firm's optimum are

$$\nu - Bz - P_z = 0$$

where $-(B + P_{zz'})$ is negative definite.

- The distributions of θ and ν in the population are normal.
- The distribution of θ is $\theta \sim N(\mu_\theta, \Sigma_\theta)$, and the distribution of ν is $\nu \sim N(\mu_\nu, \Sigma_\nu)$.

- An arbitrary price function induces a density of demand and a density of supply at every location z.
- The equilibrium price function can be found by equating these densities at every point z and thereby solving the differential equation (5) .
- However, in the linear-quadratic-normal case one can correctly guess that the price function that solves 5 is quadratic in z

$$P(z) = \pi_0 + \pi_1' z + \frac{1}{2} z' \pi_2 z$$

- Then find the coefficients (π_0, π_1, π_2) that satisfy the equilibrium equation.
- Assuming that the price function is quadratic, the first order condition for a worker is

$$\pi_1 + \pi_2 z + \theta - Az = 0. \tag{7}$$

- For a firm, it is

$$\nu - Bz - \pi_1 - \pi_2 z = 0. \tag{8}$$

- The second order conditions require that both $A - \pi_2$ and $B + \pi_2$ are positive definite. Thus we may solve for z from (7) to obtain

$$z = (A - \pi_2)^{-1}(\theta + \pi_1) \tag{9}$$

and from (8) to obtain

$$z = (B + \pi_2)^{-1}(\nu - \pi_1). \tag{10}$$

- These equations define mappings from workers θ and firms ν to job types z.
- These mappings determine the density of supply and demand at every bundle of characteristics that appears in the market or attributes and the types of workers and firms at every location.
- Equilibrium is characterized by a vector π_1 and a matrix π_2 that equate demand and supply at all z.
- However, since both θ and ν are normally distributed, this only requires equating the mean and variance of supply and demand.

- The mean supply $E^S(z)$ is obtained from (9):

$$\textbf{(Average Supply)} \qquad E^S(z) = (A - \pi_2)^{-1} E(\theta + \pi_1)$$

- The mean demand is obtained from (10):

$$\textbf{(Average Demand)} \qquad E^D(z) = (B + \pi_2)^{-1} E(\nu - \pi_1) \ .$$

THE UNIVERSITY OF CHICAGO

- Since $\mu_\theta = E(\theta)$ and $\mu_\nu = E(\nu)$, the condition $E^S(z) = E^D(z)$ implies that

$$\textbf{(Equality of means)} \quad (A - \pi_2)^{-1}(\mu_\theta + \pi_1) = (B + \pi_2)^{-1}(\mu_\nu - \pi_1) \ .$$

- Rearranging terms, we obtain an explicit expression for π_1 in terms of $A, B, \mu_\theta, \mu_\nu$ and π_2 :

$$\pi_1 = [(A - \pi_2)^{-1} + (B + \pi_2)^{-1}]^{-1}[-(A - \pi_2)^{-1}\mu_\theta + (B + \pi_2)^{-1}\mu_\nu].$$

- To determine π_2, compute the variances of supply and demand from (9) and (10) respectively to obtain:

$$\Sigma_z^S = (A - \pi_2)^{-1} \Sigma_\theta \left[(A - \pi_2)^{-1} \right]'$$

$$\Sigma_z^D = (B + \pi_2)^{-1} \Sigma_\nu \left[(B + \pi_2)^{-1} \right]'$$

- Σ_z^S is the variance of supply and Σ_z^D is the variance of demand.
- From equality of variances of the demand and supply distributions we obtain an implicit equation for π_2 :

$$\textbf{(Equality of variances)} (A - \pi_2)^{-1} \Sigma_\theta \left[(A - \pi_2)^{-1} \right]' = (B + \pi_2)^{-1} \Sigma_\nu \left[(B + \pi_2)^{-1} \right]' .$$

- Once one has have solved for π_1 and π_2, (9) and (10) also define the equilibrium matching function linking the characteristics of suppliers (θ) to those of demanders (ν).
- Substituting out for z, this function is

$$(A - \pi_2)^{-1}(\theta + \pi_1) = (B + \pi_2)^{-1}(\nu - \pi_1)$$

- The equilibrium relationship between θ and ν is

$$\theta = (A - \pi_2)\,(B + \pi_2)^{-1}(\nu - \pi_1) - \pi_1. \tag{11}$$

- Because of sorting, equilibrium worker and firm characteristics are related.

- In the special case where Σ_θ, Σ_ν, A, and B are diagonal, π_2 is diagonal.
- Effectively, this is a scalar case where each attribute is priced separately.
- In the scalar case, equality of variances implies that $(A - \pi_2)^2 \Sigma_\nu = (B + \pi_2)^2 \Sigma_\theta$.
- The second order conditions imply that $A - \pi_2 > 0$ and $B + \pi_2 > 0$.

- Defining $\sigma_\theta = (\Sigma_\theta)^{\frac{1}{2}}$ and $\sigma_\nu = (\Sigma_\nu)^{\frac{1}{2}}$, this implies that

$$\pi_2 = \frac{A\sigma_\nu - B\sigma_\theta}{\sigma_\theta + \sigma_\nu}.$$

$$\pi_1 = \frac{-\mu_\theta\sigma_\nu + \mu_\nu\sigma_\theta}{\sigma_\theta + \sigma_\nu}.$$

- π_2, the curvature of the price function, is a weighted average of the curvatures of workers' and firms' preference and technology functions.
- π_1 is a weighted average of the means of worker and firm distributions of heterogeneity.
- In both expressions, the weights depend on the relative variances of worker and firm heterogeneity.
- If workers are much more heterogeneous than firms $\sigma_\theta >> \sigma_\nu$, π_2 will approximately equal B, the curvature of firms' technology.
- If $\sigma_\theta = \sigma_\nu$ and $A = B$, $\pi_2 = 0$ is a solution and the equilibrium price function is linear in z.

THE UNIVERSITY OF CHICAGO

- If $\sigma_\theta = \sigma_\nu$, but $A \neq B$, then $\pi_2 = \dfrac{A - B}{2}$.
- In the polar cases when $\sigma_\theta = 0$ or $\sigma_\nu = 0$ there is effectively only one type of consumer or one type of firm respectively.
- If $\sigma_\theta = 0$ and $\sigma_\nu > 0$, then $\pi_2 = A$ and $\pi_1 = -\mu_\theta$.
- Then prices reveal the parameters of consumer preferences.

- If $\sigma_\nu = 0$ and $\sigma_\theta > 0$, $\pi_2 = B$ and $\pi_1 = \mu_\nu$.
- Only in these two polar cases do prices directly reveal consumer preferences or firm productivities respectively.
- Similar results hold when z, θ, and ν are vectors.
- As previously noted, much of the applied literature still ignores these fundamental properties of hedonic equilibria.