What Accounts for the Racial Gap in Time Allocation and Intergenerational Transmission of Human Capital?

George-Levi Gayle
Washington University

Limor Golan
Washington University

Mehmet A. Soytas
Ozyegin University

James J. Heckman
University of Chicago

Econ 350, Winter 2021
This draft, February 10, 2021 1:18pm

- Model of parental time allocation and investment in the human capital of young children that captures differences in the intergenerational transmission of human capital across racial and socioeconomic groups.
- Builds on previously developed dynastic models that analyze transfers and the intergenerational transmission of human capital.
- In some models, such as Loury (1981) and Becker and Tomes (1986), fertility is exogenous, while in others, such as those of Becker and Barro (1988) and Barro and Becker (1989), fertility is endogenous. We will endogenize fertility.

- The life-cycle model includes individual choices about time allocation decisions, investments in children, and fertility.
- This benchmark model is developed in Gayle, Golan, and Soytas (2014).

- The main goal: to capture the effect of family structure on investment in *children.*
- Our goal: to investigate life cycle IGE.
- Thus, we further extend the basic model to include gender and decisions made by two individuals in married-couple households, marriage, divorce, and assortative mating.
- In this framework, single versus married parenthood is endogenous, which allows us to account for the effect of family structure on children's outcomes and the selection into different types of families.

Basic Setup

- The genderless individuals from each generation $g \in \{0, ...\infty\}$ live for $t = 0, ...T$ periods, where $t = 0$ is the childhood and at period 1 the individual becomes an adult.
- Adults in each generation derive utility from their own consumption, leisure, and from the utility of their **adult** offspring.
- The utility of adult offspring is determined probabilistically by the educational outcome of children, which in turn is determined by parental time and monetary inputs during early childhood, parental characteristics (such as education), and luck.
- Parents make decisions in each period about fertility, labor supply, time spent with children, and monetary transfers.

- The only intergenerational transfers are transfers of human capital, as in Loury (1981).
- Abstract from social investment, assets, and bequests and focus on the trade-offs parents face between personal consumption and leisure and their children's well-being (but will marry with Bayer et al. 2016).
- Assume no borrowing or savings.
- This needs to be relaxed, but is a starting point.
- We will relax.

- Fertility decisions capture the quantity-quality trade-off of children.
- Incorporating life-cycle behavior allows us to model the optimal time spacing of children, an important aspect of the time allocation problem because time input is especially important during early childhood.
- Especially important in engaging with steep life cycle earning profiles.

Choices, technology and budget constraint

- Children only consume and otherwise do nothing. Adults make discrete choices about labor supply, h_t, time spent with children, d_t, and birth, b_t, in every period $t = 1, ... T$.
- They play no active role in learning; they are investment vessels.

- For labor time individuals choose no work, part-time or full-time ($h_t \in (0, 1, 2)$), and for time spent with children individuals choose none, low, and high ($d_t \in (0, 1, 2)$).
- The birth decision is binary ($b_t \in (0, 1)$).

- All the discrete choices can be combined into one set of mutually exclusive discrete choices, represented as k, such that $k \in (0, 1, ...17)$.
- Let I_{kt} be an indicator for a particular choice k at age t; I_{kt} takes the value 1 if the kth choice is chosen at age t and 0 otherwise.

- Indicators defined as follows:

$$
\begin{aligned}
I_{0t} &= I\{h_t = 0\}I\{d_t = 0\}I\{b_t = 0\}, \\
I_{1t} &= I\{h_t = 0\}I\{d_t = 0\}I\{b_t = 1\}, \dots, \\
I_{16t} &= I\{h_t = 1\}I\{d_t = 2\}I\{b_t = 1\}, \\
I_{17t} &= I\{h_t = 2\}I\{d_t = 2\}I\{b_t = 1\}
\end{aligned}
$$

- Since these indicators are mutually exclusive, then $\sum_{k=0}^{17} I_{kt} = 1$.
- Define vector x: includes the time-invariant characteristics of education, skill, and race of the individual.
- Incorporating this vector, define the vector z to include all past discrete choices as well as time-invariant characteristics, such that $z_t = \left(\left\{I_{k1}\right\}_{k=0}^{17}, \ldots, \left\{I_{kt-1}\right\}_{k=0}^{17}, x\right)$.

- These choices allow them to proxy for disutility associated with labor market activities and home hours, and therefore, a proxy for relative utility from leisure associated with the activities.
- They allow for different degrees of utility/disutility associated with different types of activities and their combinations.
- For example, spending the same number of hours working in the labor market or on a combination of home hours and working may imply the same number of hours of leisure, but it can be associated with different levels of utilities.

- Denote the earnings function by $w_t(z_t, h_t)$; it depends on the individual's time-invariant characteristics, choices that affect human capital accumulated with work experience, and the current level of labor supply, h_t.
- The choices and characteristics of parents are mapped onto their offspring's characteristics, x', via a stochastic production function of several variables.
- The offspring's characteristics are affected by their parents' time-invariant characteristics, parents' monetary and time investments, and the presence and timing of siblings.

- These variables are mapped into the child's skill and educational outcome by the function $M(x'|z_{T+1})$, since z_{T+1} includes all parental choices and characteristics and contains information on the choices of time inputs and monetary inputs.
- z_{T+1} also contains information on all birth decisions.
- It captures the number of siblings and their ages.

- Assume there are four mutually exclusive outcomes of offspring characteristics: less than high school, high school, some college, and college.
- $M(x'|z_{T+1})$ is a mapping of parental inputs and characteristics into a probability distribution over these four outcomes.

- Normalize the price of consumption to 1. Raising children requires parental time, d_t, and market expenditure.
- The per-period cost of expenditures from raising a child is denoted by pc_{nt}.
- Per-period budget constraint is given by

$$w_t \geq c_t + pc_{nt} \tag{1}$$

- To simplify the presentation of the model:
    - the price of consumption is normalized to 1,
    - and we assume that pc_{nt} is proportional to an individual's current wages and the number of children,
- But we allow this proportion to depend on state variables.
- This assumption allows us to capture the differential expenditures on children made by individuals with different incomes and characteristics.
- Practically this allows us to observe differences in social norms of child-rearing among different socioeconomic classes.

- Explicitly:

$$pc_{nt} = \alpha_{Nc}(z_t)(N_t + b_t)w_t(x, h_t) \tag{2}$$

- Incorporating the assumption that individuals cannot borrow or save and equation (2), the budget constraint becomes

$$w_t(x, h_t) = c_t + \alpha_{Nc}(z_t)(N_t + b_t)w_t(x, h_t). \tag{3}$$

Preferences

- Adults from each generation have the same utility function.
- But we can stop with 3 generations.
- An individual receives utility from discrete choices and from consumption of a composite good, c_t.
- The utility from consumption and leisure is assumed to be additively separable because the discrete choice, I_{kt}, is a proxy for leisure, and is additively separable from consumption.

- The utility from I_{kt} is further decomposed into two additive components:
  - a systematic component, denoted by $u_{1kt}(z_t)$,
  - and an idiosyncratic component, denoted by ε_{kt}.
- The systematic component associated with each discrete choice k represents an individual's net instantaneous utility associated with the disutility from market work, the disutility/utility from parental time investment, and the disutility/utility from birth.

- The idiosyncratic component is standard in empirical discrete choice models; it represents preference shocks associated with each discrete choice k that are transitory in nature.

- To capture this feature of ε_{kt} we assume that the vector $(\varepsilon_{0t}, .., \varepsilon_{17t})$ is independent and identically distributed across the population and time, and is drawn from a population with a common distribution function, $F_{\varepsilon}(\varepsilon_{0t}, .., \varepsilon_{17t})$.

- The distribution function is assumed to be absolutely continuous with respect to the Lebesgue measure and has a continuously differentiable density.

- The per-period utility from the composite consumption good:

$$u_{2t}(c_t, z_t).$$

- Assume that $u_{2t}(c_t, z_t)$ is concave in c, that is, $\partial u_{2t}(c_t, z_t)/\partial c_t > 0$ and $\partial^2 u_{2t}(c_t, z_t)/\partial c_t^2 < 0$.

- Implicit in this specification is intertemporally separable utility in the consumption good, but not for the discrete choices, because $u_{2t}$is a function of z_t, which is itself a function of past discrete choices but is not a function of the lagged values of c_t.
- Altruistic preferences are introduced under the same assumption as the Barro-Becker model: Parents obtain utility from their adult offsprings expected lifetime utility.

- Two separable discount factors capture the altruistic component of the model.
- The first, β, is the standard rate of time preference parameter, and the second, $\lambda N^{1-\nu}$, is the intergenerational discount factor, where N is the number of offspring an individual has over his lifetime.
- Here $\lambda (0 < \lambda < 1)$ should be understood as the individual's weighting of his offsprings' utility relative to her own utility.

- If $\lambda = 1$, the individual values his own utility as his children's utility.
- The individual discounts the utility of each additional child by a factor of $1 - \nu$, where $0 < \nu < 1$: diminishing marginal returns from offspring.
- The functional form assumption is similar to the one in Barro and Backer (1988); for further discussion on the functional form assumptions on the discount factor see Alvarez (1999).

- The sequence of optimal choices for both discrete choices and consumption is denoted as I_{kt}^{o} and c_t^{o}, respectively.
- Denote the expected lifetime utility at time $t = 0$ of a person with characteristics x in generation g, excluding the dynastic component, as

$$U_{gT}(x) = E_0\left[\sum_{t=0}^{T} \beta^t [\sum_{k=0}^{17} I_{kt}^{o}\{u_{1kt}(z_t) + \varepsilon_{kt}\} + u_{2t}(c_t^{o}, z_t)]|x\right] \tag{4}$$

- The total discounted expected lifetime utility of an adult in generation g, including the dynastic component is

$$U_g(x) = U_{gT}(x) + \beta^T \lambda E_0 \left[N^{1-\nu} \frac{\sum_{n=1}^{N} U_{g+1,n}(x_n')}{N} \middle| x \right] \tag{5}$$

- $U_{g+1,n}(x'_n)$ is the expected utility of child n ($n = 1, .., N$) with characteristics x'.
- In this model, individuals are altruistic and derive utility from their offsprings utility, subject to discount factors β and $\lambda N^{1-\nu}$.
- This formulation creates links across all generations, and by recursive substitution can be written as a discounted sum of the life-cycle utility, $U_{gT}(x)$, of all generation (for example, the discount rate on grandchildren utility is $\beta^{2T}\lambda^2$).

- Solving for consumption from equation (3) and substituting for consumption in the utility equation, we can rewrite the third component of the per-period utility function, specified as $u_{2kt}(z_t)$, as a function of just z_t :

$$u_{2kt}(z_t) = u_t[w_t(x, h_t) - \alpha_{Nc}(z_t)(N_t + b_t)w_t(x, h_t), z_t] \tag{6}$$

- Note that the discrete choices and fixed characteristics,now map into different levels of utility from consumption.
- Therefore, we can eliminate consumption as a choice and write the systematic contemporary utility associated with each discrete choice k as

$$u_{kt}(z_t) = u_{1kt}(z_t) + u_{2kt}(z_t). \tag{7}$$

- Incorporating the budget constraint manipulation, we can rewrite equation (4) as

$$U_{gT}(x) = E_0\left[\sum_{t=0}^{T}\beta^t\sum_{k=0}^{17} I_{kt}^{o}[u_{kt}(z_t)+\varepsilon_{kt}]|x\right]. \qquad (8)$$

- Thus, this expression is the expected utility at time 0 of the lifetime utility, excluding the dynastic component of an individual in generation g and characteristics x.
- This expression is similar to the standard representation of expected utility in standard life-cycle models of discrete choice.
- Except for age, which changes over the life-cycle, the environment in our model is assumed to be stationary.
- We will relax this – it's crucial – so we use ingredients but *not* stationarity.
- Therefore, we can omit the generation index g in the analysis from equation (8) and write $U_T(x)$ instead.

Optimal discrete choice

- The individual chooses the sequence of alternatives yielding the highest utility by following the decision rule.
- $I(z_t, \varepsilon_t)$, where ε_t is the vector $(\varepsilon_{0t}, ..., \varepsilon_{17t})$.
- The optimal decision rules are given by

$$
\begin{aligned}
& I^o(z_t, \varepsilon_t) = \\
& \arg\max_I E_I \left[\sum_{t=0}^{T} \beta^t \sum_{k=0}^{17} I_{kt}[u_{kt}(z_t) + \varepsilon_{kt}] \right. \\
& \left. + \frac{\beta^T \lambda}{N^\nu} \sum_{n=1}^{N} U_{g+1,n}(x_n') | x \right]
\end{aligned}
$$

- The expectations are taken over the future realizations of z and ε induced by I^o.

- In any period $t < T$, the individual maximization problem can be decomposed into two parts: the utility received at t plus the discounted future utility from behaving optimally in the future.

- Can write the value function of the problem, which represents the expected present discounted value of lifetime utility from following I^o, given z_t and ε_t, as

$$V(z_{t+1}, \varepsilon_{t+1}) = \max_I E_I \left(\left\{ \sum_{t'=t+1}^{T} \beta^{t'-t} \sum_{k=0}^{17} I_{kt'}[u_{kt'}(z_{t'}) + \varepsilon_{kt'}] + \frac{\beta^{T-t'}\lambda}{N^{\nu}} \sum_{n=1}^{N} U_{g+1,n}(x_n') \right\} \Big| z_t, \varepsilon_t\right) \quad (9)$$

- By Bellman's principle of optimality, the value function can be defined recursively as

$$
\begin{aligned}
& V(z_t, \varepsilon_t) = \\
& \max_{I} \left[\sum_{k=0}^{17} I_{kt} \left\{ u_{kt}(z_t) + \varepsilon_{kt} + \right. \right. \\
& \left. \left. \beta E(V(z_{t+1}, \varepsilon_{t+1}) | z_t, I_{kt} = 1) \right] \right\} \\
= & \sum_{k=0}^{17} I_{kt}^{o}(z_t, \varepsilon_t) [u_{kt}(z_t) + \varepsilon_{kt}] + \\
& \beta \sum_{z} \int V(z, \varepsilon) f_{\varepsilon}(\varepsilon) d\varepsilon F(z | z_t, I_{kt}^{o} = 1)
\end{aligned}
$$

- $f_{\varepsilon}(\varepsilon_{t+1})$ is the continuously differentiable density.
- $F_{\varepsilon}(\varepsilon_{0t}, .., \varepsilon_{17t})$, and $F(z_{t+1}|z_t, I_{kt} = 1)$ is a transition function for state variables, which is conditional on choice k.
- In this simple version, the transitions of the state variables are deterministic given the choices of labor market experience, time spent with children, and number of children.

- Next, we further characterize the choice probabilities.
- Define the *ex ante* (or integrated) value function, $V(z_t)$, as the continuation value of being in state z_t before ε_t is observed by the individual.

- $V(z_t)$ is given by integrating $V(z_t, \varepsilon_t)$ over ε_t before it is observed by the agent.
- Define the probability of choice k at age t by $p_k(z_t) = E[I^o_{kt} = 1 | z_t]$.
- The ex ante value function can be written more compactly as

$$
\begin{aligned}
& V(z_t) = \\
& \textstyle\sum_{k=0}^{17} p_k(z_t) \left[u_{kt}(z_t) + E_\varepsilon[\varepsilon_{kt} | I_{kt} = 1, (z_t)] \right. \\
& \left. + \beta \textstyle\sum_z V(z) F(z | z_t, I_{kt} = 1) \right]
\end{aligned}
$$

- In this form, $V(z_t)$ is now a function of the conditional choice probabilities, the expected value of the preference shock, the per-period utility, the transition function, and the ex ante continuation value.
- All components except the conditional probability and the ex ante value function are primitives of the initial decision problem.
- By writing the conditional choice probabilities as a function of only the primitives and the ex ante value function, we can characterize the optimal solution of the problem (i.e., the ex ante value function) as implicitly dependent on only the primitives of the original problem.

- To create such a representation we define the conditional value function, $\upsilon_k(z_t)$, as the present discounted value (net of ε_t) of choosing k and behaving optimally from period $t = 1$ forward:

$$\upsilon_k(z_t) = u_{kt}(z_t) + \beta \sum_z V(z)F(z|z_t, I_{kt} = 1). \qquad (10)$$

- The conditional value function is the key component to the conditional choice probabilities.
- Equation (9) can now be rewritten using the individual's optimal decision rule at t to solve

$$I^{o}(z_t, \varepsilon_t) = \arg\max_{I} \sum_{k=0}^{17} I_{kt}[\upsilon_k(z_t) + \varepsilon_{kt}]. \tag{11}$$

- Therefore, the probability of observing choice k, conditional on z_t, is $p_k(z_t)$ and is found by integrating out ε_t from the decision rule in Equation (11):

$$p_k(z_t) =$$
$$\int I^o(z_t, \varepsilon_t) f_\varepsilon(\varepsilon_t) d\varepsilon_t =$$
$$\int \left[\prod_{k \neq k'} \left\{ 1 \upsilon_k(z_t) - \upsilon_{k'}(z_t) \geq \varepsilon_{kt} - \varepsilon_{tk'} \right\} \right] f_\varepsilon(\varepsilon_t) d\varepsilon_t$$

- Therefore, $p_k(z_t)$ is now entirely a function of the primitives of the model (i.e., $u_{kt}(z_t)$, β, $F(z_{t+1}|z_t, I_{kt} = 1)$, and $f_\varepsilon(\varepsilon_t)$) and the *ex ante* value function.
- Hence substituting equation (51) into equation (47) gives an implicit equation defining the ex ante value function as a function of only the primitives of the model.

Discussion

- Time allocation decisions involve the usual trade-offs of the non-pecuniary costs associated with the combinations of activities (representing different levels of leisure) and current consumption.
- When allocating consumption and leisure over time, reducing the labor supply has dynamic effects since it reduces labor market experience.
- Since there are no savings in the model, the only way parents can increase consumption in the future is by accumulating labor market experience.
- This is similar to Loury (1981).

- In addition, both income when children are young and parental time may affect the outcomes of children.
- These dynamic effects of time allocation on the outcomes of children makes the solution to the labor supply decisions nontrivial, despite the linearity of the per-period utility function.

- In dynastic models of investment (Loury, 1981) wealthier parents invest more in their children. In Becker and Barro (1988) and Barro and Becker (1989), however, there is no correlation between wealth and investment because unlike Loury (1981), fertility is endogenous and wealthier parents adjust their own consumption and increase the number of children, but the investment per child does not change.
- As a result, there is no intergenerational persistence in outcomes.

- Alvarez (1999) shows that relaxing the following three assumptions in the Barro-Becker model can generate persistence in outcomes across generations:
- First, the marginal costs of raising children is increasing instead of constant.
- Second, separability of utility from consumption of parents and children utility.
- Third, investment of past generations does not affect the marginal costs of raising children.

- In our model, persistence is achieved because the first and third assumption are relaxed.
- The cost of investment in children is not constant in our model because the cost of time investment is not linear.
- This nonlinearity is captured in $u_{1kt}(z_t)$ as discussed below.
- Furthermore, the opportunity cost of time in the form of loss of labor market experience and future earnings may not be linear.
- Also, the budget constraints are non-separable across generations.

- The cost of an individual's investment in children in each generation depends on the investment made by previous generation through education, which affect the opportunity cost of time.
- In addition, education affects earnings and we allow for the costs of children to depend on earnings.

- In Barro and Becker (1988), children are a normal good.
- Hence, wealthier individuals have more children.
- This is in contrast to empirical evidence.
- If time allocation is endogenous, however, there are income and substitution effects on fertility decisions.
- More-educated parents have a higher opportunity cost of time, possibly explaining the lower fertility rates of educated women.

- The quantity-quality trade-off (Becker and Lewis, 1973) is captured by the resource constraint. Income and time are limited.
- Our model include the life-cycle.
- Thus, spacing of children is endogenous.
- Since the time available to have children is limited and the opportunity costs of time vary over the life-cycle, our model does not, in general, predict that time with children is independent of parental education.

- Since we focus on early childhood investment, spacing of children affects the quantity-quality trade-off.
- Thus, decisions on timing of having children are affected by several factors:
- First, if income increases with age, the opportunity cost of time increases.
- At the same time, having children later in life implies that the same amount of money can be earned working less.
- Second, there is a limited time during which one can have children.
- Thus, having fewer children allows for longer spacing between children and less quantity-quality trade-offs implied by having the same number of children with shorter spacing.

Model of Households

- This extends the basic framework to include household decisions.
- The model incorporates marriage and assortative mating by allowing the education outcome of the child to affect who they marry.
- Both educational outcomes of children and their marriage market outcomes are determined when children become adults, after all parental investments are made.
- Marriage and divorce are not modeled as choice variables.

- However, they depend stochastically on choices.
- Therefore, forward-looking individuals take into account the effect of their decisions on marriage and divorce probabilities.
- Thus, these variables are endogenous in a predetermined sense.
- Household structure is an important determinant of parental transfers to children.

- However, most dynastic models are written as a single decision-maker problem ignoring marriages.
- In our model, couples can share the costs of raising children and income can be transferred between spouses, whereas a single-parent consumption depends on her own income only.
- This allows us to capture the different costs and trade-offs faced by single and married parents.
- For example, for a married person, an increase in time sent with children and a decrease in labor supply may not reduce consumption if the spouse makes transfers and increases labor supply in response.

- We model the household decision process as a simultaneous move game.
- In our model, the Markov perfect equilibria can be Pareto ranked and we assume there is no other Markov perfect equilibrium that Pareto dominates the equilibrium implemented.
- Thus, our approach to modeling household decisions is similar to the Ligon, Thomas, and Worrall (2002) model of non-cooperative behavior in households in which the equilibrium is constrained Pareto efficient.

- An individual's gender, subscripted as σ, takes the value 1 for a male and 2 for a female: $\sigma = \{1, 2\}$.
- Gender is included in the vector of invariant characteristics x_σ.
- In the extension, only females make birth decisions, so males and females face a different set of choices.
- Let K_σ describe the number of possible combinations of actions available to each gender, so $K_2 = 17$ and $K_1 = 8$.
- All individual variables, preferences, and earnings are indexed by the gender subscript σ.

- We omit the gender subscript when a variable refers to the household (both spouses).
- The state variables are extended to include the gender of the offspring.
- Let the vector ζ_t indicate the gender of a child born at age t, where $\zeta_t = 1$ if the child is a female and $\zeta_t = 0$ otherwise.
- We also define an indicator for marriage: ψ_t.
- It equals 1 if the individual is married and 0 otherwise.

- The vector of state variables is expanded to include the gender of the offspring:

$$z_{t\sigma} = (\{I_{\sigma k1}\}_{k=0}^{K_\sigma}, ..., \{I_{\sigma kt-1}\}_{k=0}^{K_\sigma}, \zeta_0, .., \zeta_{t-1}, \psi_0, ..., \psi_t, x_\sigma).$$

- We denote the household state variables by $z_t = (z_{t\sigma}, z_{t-\sigma})$, where $-\sigma$ refers to the individual's spouse.
- Married individuals and single individuals who live with their children make decisions of labor supply, home hours and birth (females).
- For a single person household $z_t = z_{t\sigma}$.

- We assume that single parents who do not live in the same household with their children choose only labor supply and birth decisions (if female).
- Thus, they do not choose transfers of money and time.
- Instead the transfers are fixed and depend on the parent's characteristics.

- Married individuals and single parents who live with the children invest time and money in the children in the household.
- We make these assumptions to simplify the analysis and because of certain considerations of the data.
- These assumptions are standard in the family economics literature (Browning, Chiappori, and Weiss, 2011) where the noncustodial parents is normally assumed to not have a choice in how much time and resources he or she can spend with and on the child.
- However, the noncustodial parent continues to enjoy the benefit of the child, albeit at a possibly reduced level.

- The idea here is that the family court sets the level of child support and visitation rights, which are strictly adhered to.
- Allowing it to depend on parental characteristics proxies for the discretion that the court normally displaces by taking into account the ability to pay and the desire to spend time with the child, which may vary by education level and other socioeconomic factors.

- The function $w_{\sigma t}(z_{\sigma t}, h_{\sigma t})$ denotes the earnings function; the only difference from the single agent problem is that gender is included in $z_{\sigma t}$ and can thus affect wages.
- The educational outcome of the parents' offspring is mapped from the same parental inputs as the single agent model: income and time investment, number of older and younger siblings, and parental characteristics such as education, race, and labor market skill.
- In the extension gender is also included as a parental characteristic.

- Thus, the production function is still denoted by $M(x'|z_{T+1})$, where z_{T+1} represents the state variables at the end of the parent life-cycle, T.
- For single parents not living with their children, we assume there is no time or monetary input.
- However, the parental fixed characteristics are in the production function, implying that we restrict these parents to be making the same transfers conditional on their fixed characteristics.

- Our justification is similar to the one discussed above in addition to data limitations and tractability considerations; specifically the assumption that income of single parents not in the child's household is not in the production function is made to avoid analyzing a game between ex-spouses, as many times spouses remarry and this requires formulating a game between more than two players.
- Nevertheless, the individual's fixed effect and education are controlled for in the production function, capturing the effect of a permanent part of the individual income.

Household budget constraint

- In the household, the total per-period expenditures cannot exceed the combined income of the individual and the spouse.
- To formulate the individual's problem we describe a sharing rule:
- Let $\tau_\sigma(z_t)$ denote the net transfer to spouse σ.
- By this definition $\tau_{-\sigma}(z_t) = -\tau_\sigma(z_t)$.

- Thus, the budget constraint for the married individual is given by

$$w_{\sigma t} + \tau_{\sigma}(z_t) \geq c_{\sigma t} + \alpha_{\sigma m N c}(z_t)(N_t + b_t) w_t(z_t, h_t) \qquad (12)$$

- $w_t(z_t, h_t) = w_{\sigma t}(z_{\sigma t}, h_{\sigma t}) + w_{-\sigma t}(z_{-\sigma t}, h_{-\sigma t})$ is the total household labor income.

- Each individual's resources are given by his own income plus the net transfer $\tau_\sigma(z_t)$, which depends on the state variables of the household.
- The right-hand side represents expenditures on personal consumption, $c_{\sigma t}$, and on children.
- The individual's share of child care expenditures is represented by the term $\alpha_{\sigma mNc}(z_t)$.
- m subscript denotes the couple's sharing of the cost.
- $\alpha_{\sigma mNc}(z_t) + \alpha_{-\sigma mNc}(z_t) = 1$.

- Total household expenditures cannot exceed the combined income of the parents.
- Married individuals pay for the children living in their household, regardless of the biological relationship, and do not transfer money to any biological children living outside the household.

- There are no transfers between divorced individuals therefore the budget constraint for a single individual is similar to the one in the gender-less model:

$$w_{\sigma t} \geq c_{\sigma t} + \alpha_{\sigma Nc}(z_{\sigma t})(N_t + b_t) w_{\sigma t}(z_{\sigma t}, h_{\sigma t}). \tag{13}$$

- Thus, the monetary cost of and time spent with children depend not only on the parents' characteristics but on the marital status as well.

Timing, information, and strategies

- We assume married couples play a simultaneous move game and the timing and information are as follows:
- At the beginning of each period, both spouses observe all the systematic state variables and the independently distributed taste shocks:
- $\varepsilon_t = (\varepsilon_\sigma, \varepsilon_{-\sigma})$.
- The individual and the spouse choose their actions simultaneously.
- After the decisions are observed, consumption is allocated according to the sharing rule described above.

- In the extension, we define $I_{\sigma kt}$, the kth element of the discrete Markov strategy profile at time t, as a mapping of any possible state variables z_t, ε_t onto $\{0,1\}$, such that $I_{\sigma kt} : [z_t, \varepsilon_t] \Longrightarrow \{0, 1\}$.
- The Markov strategy profile for the individual in period t is defined as $I_{\sigma t} = \left[\{ I_{\sigma k1}(z_1, \varepsilon_1) \}_{k=0}^{K_\sigma}, .., \{ I_{\sigma kT}(z_T, \varepsilon_T) \}_{k=0}^{K_\sigma} \right]$,.
- We can thus write the strategies of both spouses as $I_t = (I_{\sigma t}, I_{-\sigma t})$.

- The sequence of optimal strategies for both discrete choices by $I^o_{\sigma t}$ and $c^o_{\sigma t}$.
- c^o is a mapping of state variables onto the optimal consumption strategy.
- Then write the expected lifetime utility at time $t = 0$ of an individual with characteristics x_σ in generation g, excluding the dynastic component, as

$$U_{\sigma g T}(x) =$$
$$E_0 \left[\sum_{t=0}^{T} \beta^t [\sum_{k \in K_\sigma} I^o_{\sigma kt} \{u_{1\sigma kt}(z_t)\right.$$
$$\left. + \varepsilon_{\sigma kt}\} + u_{2\sigma t}(c^o_{\sigma t}, z_t)] | x_\sigma \right]$$

- In addition to the choices made by the individual, the household's state variables and the spouse's expected choices now affect the individual per-period utility.
- Individuals are not altruistic toward their spouse.
- Therefore each individual's utility depends only on their own consumption and not the spouse's.

- The total discounted expected lifetime utility of an adult in generation g, including the dynastic component, is

$$U_{\sigma g}(x) = U_{\sigma g T}(x) + \beta^{T} \lambda E_0 \left[N_{\sigma}^{-\nu} \sum_{n=1}^{N} U_{\sigma', g+1}(x_n') | x_{\sigma} \right]. \quad (14)$$

- The above formulation allows the expected utility (at age zero) of a child, denoted with subscript σ', to depend on gender and birth order.

- As in the single agent model, we can eliminate the continuous choice in the lifetime utility problem so that households face a purely discrete choice problem.
- As in the single agent problem, we substitute for consumption in $u_{2\sigma}$ as follows:

$$\begin{aligned} u^{(-k)}_{2\sigma kt}(z_t) = u_t[w_{\sigma t}(z_{\sigma t}, h_{\sigma t}) \\ +\tau_\sigma(z_t) - \alpha_{\sigma mNc}(z_t)(N_t + b_t)w_t(z_t, h_t), z_t] \end{aligned} \tag{15}$$

- The subscript σ_k denotes the actions of the individual σ, and the superscript $-k$ denotes the actions of the spouse.
- The spouse's actions affect the household income, and therefore consumption through labor supply choices, and a male's consumption is affected by his wife's birth decisions.

- Note that the share of expenditure on children and net transfers both depend on the household characteristics z_t, so we can write the utility function $u_{\sigma kt}^{(-k)}(z_t) = u_{1\sigma kt}(z_t) + u_{2\sigma kt}^{(-k)}(z_t)$ as a function of state variables.
- Incorporating the budget constraint manipulation, we can rewrite equation (87) as

$$U_{\sigma gT}(x) =$$
$$E_0 \left[\sum_{t=0}^{T} \beta^t \sum_{k=0}^{K_\sigma} I_{\sigma kt}^o \right.$$
$$\left. \left[\sum_{k'=0}^{K_{-\sigma}} \{ I_{-\sigma k' t}^o u_{\sigma kt}^{(-k')}(z_t) \} \psi_t + u_{\sigma kt}(z_t)(1-\psi_t) + \varepsilon_{\sigma kt} \right] \middle| x_\sigma \right] . \tag{16}$$

Optimal strategies

- The strategy at each node of the game (i.e., on and off the equilibrium path) is similar to the decision problem in the single agent model.
- In the single agent model, the individual takes the state variables as given, and in the extension the individual also takes the strategy of the spouse as given.
- The equilibrium strategy is such that given the spouse's strategy and state variables, the individual cannot make a unilateral single deviation that increases his utility.

- Since this is a complete information game this means that at time t the information held by both players includes current state variables– that is, both the random and systematic component, z_t and ε_t.
- Denote a sequence of decision policy functions for player σ at time t' by $\mathbb{I}_{\sigma t}$ from the moment t to T by

$$\mathbb{I}_{\sigma t} = \langle I_{\sigma t}, I_{\sigma t+1}, ..., I_{\sigma T} \rangle = \langle I_{\sigma t}, \mathbb{I}_{\sigma t+1} \rangle \, . \tag{17}$$

- Then at the moment t, after the preference shock for that period is observed by both partners, the expected discounted payoff for partner σ is

$$
\begin{aligned}
& V_{\sigma}(z_t, \varepsilon_t, \mathbb{I}_{\sigma t}, \mathbb{I}_{-\sigma t}) \\
& = E_{z_{t+1}, \varepsilon_{t+1}, \ldots z_T, \varepsilon_T} \\
& \Big(\textstyle\sum_{t'=t+1}^{T} \beta^{t'-t} \sum_{k=0}^{K_{\sigma}} I_{\sigma k t'} \\
& [\textstyle\sum_{k'=0}^{K_{-\sigma}} \{ I_{-\sigma k' t'} u_{\sigma k t'}^{(-k')}(z_{t'}) \} \psi_{t'} + u_{\sigma k t'}(z_{t'})(1 - \psi_{t'}) + \varepsilon_{\sigma k t'}] \\
& + \tfrac{\beta^{T-t'} \lambda}{N_{\sigma}^{\nu}} \textstyle\sum_{n=1}^{N} U_{g+1, \sigma'}(x'_{\sigma n}) | z_t, \varepsilon_t \Big) .
\end{aligned} \tag{18}
$$

THE UNIVERSITY OF CHICAGO

- A pair of policy functions, $\left\langle \mathbb{I}^o_{\sigma t}, \mathbb{I}^o_{-\sigma t} \right\rangle$, provides the Nash equilibrium for a pair of value functions.
- $\langle V_\sigma(.,.,.,.), V_{-\sigma}(.,.,.,.) \rangle$, if, for all possible values of z_t and ε_t, we have

$$\left\{ \begin{array}{c} V_\sigma(z_t, \varepsilon_t, \mathbb{I}^o_{\sigma t}, \mathbb{I}^o_{-\sigma t}) = \max\limits_{\mathbb{I}_{\sigma t}} V_\sigma(z_t, \varepsilon_t, \mathbb{I}_{\sigma t}, \mathbb{I}^o_{-\sigma t}) \\ V_{-\sigma}(z_t, \varepsilon_t, \mathbb{I}^o_{-\sigma t}, \mathbb{I}^o_{\sigma t}) = \max\limits_{\mathbb{I}_{-\sigma t}} V_{-\sigma}(z_t, \varepsilon_t, \mathbb{I}_{-\sigma t}, \mathbb{I}^o_{\sigma t}) \end{array} \right. . \tag{19}$$

- In what follows, we denote the Nash equilibrium discounted payoff as $V_\sigma(z_t, \varepsilon_t) = V_\sigma(z_t, \varepsilon_t, \mathbb{I}^o_{\sigma t}, \mathbb{I}^o_{-\sigma t})$ for $\sigma =$1,2.
- It follows that we can write the expected discounted payoff for partner σ recursively as

$$
\begin{aligned}
& V_\sigma(z_t, \varepsilon_t, \mathbb{I}_{\sigma t}, \mathbb{I}_{-\sigma t}) \\
& = \textstyle\sum_{k=0}^{K_\sigma} \sum_{k'=0}^{K_{-\sigma}} I_{\sigma k t} I_{-\sigma k' t} [u^{(-k')}_{\sigma k t'}(z_t)\psi_t + u_{\sigma k t}(z_t)(1-\psi_t) + \varepsilon_{\sigma k t} \\
& + \beta \textstyle\sum_z \int V_\sigma(z, \varepsilon_{t+1}, \mathbb{I}_{\sigma t+1}, \mathbb{I}_{-\sigma t+1}) f(\varepsilon) d\varepsilon F_{k,k'}(z|z_t)],
\end{aligned}
$$

- Denote the transition for couples as $F_{k,k'}(z_{t+1}|z_t) = F(z_{t+1}|z_t, I_{\sigma kt} I_{-\sigma k' t} = 1)$.
- Therefore, the Nash equilibrium value function is

$$
\begin{aligned}
& V_\sigma(z_t, \varepsilon_t) = \\
& \textstyle\sum_{k=0}^{K_\sigma} \sum_{k'=0}^{K_{-\sigma}} I^o_{\sigma kt} I^o_{-\sigma k' t} \\
& [u^{(-k')}_{\sigma kt'}(z_t)\psi_t + u_{\sigma kt}(z_t)(1-\psi_t) \\
& + \varepsilon_{\sigma kt} + \beta \textstyle\sum_z \int V_\sigma(z, \varepsilon) f(\varepsilon) d\varepsilon F_{k,k'}(z|z_t)]
\end{aligned}
$$

- Since ε_t is unobserved, further define the ex ante (or integrated) Nash equilibrium value function, $V_\sigma(z_t)$, similar to in a manner similar to the single agent case by integrating over ε_t.
- Define the joint probability of choices of $I^o_{\sigma kt} I^o_{-\sigma k' t} = 1$ at age t by $p_{k,k'}(z_t) = E[I^o_{\sigma kt} I^o_{-\sigma k' t} = 1 | z_t]$.
- The expectation of the preference shock conditional on $I^o_{\sigma kt} I^o_{-\sigma k' t} = 1$ and z_t as $e_{\sigma kk'}(z, p) = E_\varepsilon[\varepsilon_{\sigma kt} | I^o_{\sigma kt} I^o_{-\sigma k' t} = 1, z_t]$.

- Then the ex ante value function can be written more compactly as

$$
\begin{aligned}
&V_\sigma(z_t) = \textstyle\sum_{k=0}^{K_\sigma} \sum_{k'=0}^{K_{-\sigma}} p_{k,k'}(z_t) \\
&[\; u_{\sigma kt}^{(-k')}(z_t)\psi_t + u_{\sigma kt}(z_t)(1-\psi_t) + e_{\sigma kk'}(z_t, p_t) \\
&+ \beta \textstyle\sum_z V_\sigma(z) F_{k,k'}(z|z_t) \;]
\end{aligned} \tag{20}
$$

- This is now a function of the joint conditional choice probabilities, the expected value of the preference shock, per-period utility, the transition function, and the ex ante continuation value.
- With the exception of the conditional choice probabilities and the ex ante continuation value, all of the above are primitives of the original decision problem.
- If we can write the conditional choice probabilities as only a function of the primitives and the ex ante value function, then we would have characterized the optimal solution of problem (i.e. the ex ante value function) as the implicit solution of an equation that depends only on the primitives of the original problem.

- The joint household's choice probabilities achieve this; we first define the conditional best response function, $\upsilon_{\sigma kk'}(z_t)$, as the present discounted value (net of ε_t) of choosing k and behaving optimally from period $t = 1$ forward:

$$\begin{aligned} &Couples: \ \upsilon_{\sigma kk'}(z_t) = \\ &u_{\sigma kt}^{(-k')}(z_t) + \beta \textstyle\sum_z V_\sigma(z) F_{k,k'}(z|z_t) \end{aligned} \tag{21}$$

$$\begin{aligned} &Singles: \ \upsilon_{\sigma k}(z_t) = \\ &u_{\sigma kt}(z_t) + \beta \textstyle\sum_z V_\sigma(z) F_k(z|z_t) \end{aligned} \tag{22}$$

- Note that for singles the current-period utility does not depend on any spouse decision; the continuation value $V_\sigma(z)$ is the Nash equilibrium value function since next period there is a chance the person will get married to an individual with characteristics $z_{-\sigma t+1}$.
- We assume that this happens with probability matching function $G(z_{-\sigma t+1} | z_{\sigma t+1})$.
- This function is assumed to be exogenous and embodies the marriage market equilibrium, which is also taken as exogenous.

- However, since ψ_{t+1} is an element $z_{\sigma t+1}$ and the transition function $F_k(z_{t+1}|z_t)$ (or $F_{k,k'}(z_{t+1}|z_t)$ if it is a couple) depends on the current decision hence, marriage is endogenous to the Nash equilibria profile.
- The conditional value function is the key component to the conditional best response probabilities.
- We then restate equation (9), the individual's optimal decision rule at t.
- First condition on the spouse choosing choice k' in period t and both partners following the equilibrium strategies from $t+1$ to T.

- That is, the best response policy function and is defined as

$$I_{\sigma}^{o}(z_t, \varepsilon_t | k') = \arg\max_{I} \sum_{k=0}^{K_{\sigma}} I_{\sigma kt}[\upsilon_{\sigma kk'}(z_t) + \varepsilon_{\sigma kt}] \tag{23}$$

- $I_{\sigma}^{o}(z_t, \varepsilon_t | k') = \left\langle I_{\sigma 0}^{o}(z_t, \varepsilon_t | k'), I_{\sigma 1}^{o}(z_t, \varepsilon_t | k'), ..., I_{\sigma K_{\sigma}}^{o}(z_t, \varepsilon_t | k') \right\rangle$.

THE UNIVERSITY OF CHICAGO

- Therefore the probability of observing choice k made conditional on z_t and the spouse choosing k', $p_{\sigma k}(z_t|k')$, is found by integrating out ε_t from the decision rule in equation (23):

$$p_{\sigma k}(z_t|k') =$$
$$\int I^o_{\sigma k}(z_t, \varepsilon_t|k') f_\varepsilon(\varepsilon_t) d\varepsilon_t =$$
$$\int \left[\prod_{k\neq\widehat{k}}^{K_\sigma} 1\{\upsilon_{\sigma kk'}(z_t) - \upsilon_{\sigma\widehat{k}k'}(z_t) \geq \varepsilon_{kt} - \varepsilon_{\widehat{k}t}\} \; f_\varepsilon(\varepsilon_t) d\varepsilon_t\right.$$

- Therefore, according to the definition of equilibrium, the joint probability $p_{k,k'}(z_t) = p_{\sigma k}(z_t|k')p_{-\sigma k'}(z_t)$ where $p_{-\sigma k'}(z_t) = \sum_{\widehat{k}}^{K_{-\sigma}} p_{-\sigma k'}(z_t|\widehat{k})$.

Equilibrium

- We solve for a Markov Perfect equilibrium of the game, restricting attention to pure strategies equilibria.

Definition 1 (Markov Perfect equilbrium)

A strategy profile $\langle I^o_{\sigma t}, I^o_{-\sigma t} \rangle$ is said to be a Markov Perfect equilibrium if for any $t \leq T$, $\sigma \in \{1, 2\}$, and $(z_t, \varepsilon_t) \in (Z, R^{K_\sigma + K_{-\sigma}})$: (1) $\upsilon_{\sigma k k'}(z_t) + \varepsilon_{\sigma k t} \geq \upsilon_{\sigma \widehat{k} k'}(z_t) + \varepsilon_{\sigma \widehat{k} t}$; (2) all players play Markovian strategies.

- In general, a pure strategy Markovian perfect equilibrium for complete information stochastic games may not exist; however, they impose sufficient conditions on the primitives of the game and show that there exists at least one pure strategies Markov perfect equilibrium.
- To show this results, they use some of the properties and definitions of super modular games on lattice theory.

Link to Appendix

Discussion

- The equilibria in super-modular games can be Pareto ranked.
- The key feature is the presence of strategic complementarities, or positive externalities, which naturally arise in the context of families.
- We are therefore able to show that there exists a Pareto best (and worst) equilibrium.
- In the context of families, it is reasonable to assume that families can coordinate on the best equilibrium.
- The highest-ranked equilibrium is constrained Pareto efficient.

- In this sense, it can be thought of as a result of a contractual agreement on the (constrained) Pareto frontier as in Ligon, Thomas, and Worrall (2002) formulation of a solution to the household problem with limited commitment.
- In contrast to Ligon, Thomas, and Worrall (2002), players live for a finite number of periods and we restrict our strategies to payoff-relevant strategies.
- Therefore, cannot invoke folk theorems and achieve efficient solution (Abreu, 1988; Kocherlakota, 1996).

- They have a super-modular game and public goods that provide the result that the equilibria can be Pareto ranked.
- Since they have no commitment and incomplete asset markets, the constrained efficient equilibrium is not expected to yield the same outcome and provision of public good (investment in children and fertility) that a fully efficient solution would yield.

- Married individuals are affected by the action of a spouse from a different dynasty.
- The income externalities within a household imply that the utility of an individual in generation g depends on the future spouses of one's own children and their children's spouses from different dynasties.
- As shown by Bernheim and Bagwell (*JPE*, 1988), it is possible that within a few generations there will be links between most or all dynasties, in which case, the representation of the problem may be complicated.

- Notice that we circumvent this problem because our formulation of dynasties is anonymous in the sense that it is only the state variables of future generations that affect individual utilities and not their identity.
- Similarly, the spouses of future offspring affect the individual's utility through their state variables and not the identity of the dynasty they come from.

- By stationarity, the valuation function of a person with state variable x (which includes a spouse's characteristics) is the same across generations.
- Ex ante, individuals with different characteristics have a different probability distribution over different "types" of offspring (x').
- This creates different "types" of dynasties, each with a different life-time expected utility, a different expected number of offspring, and a different distribution probabilities over their children's types.

- The trade-offs an individuals makes when married and single are different.
- First, marriage allows for some degree of specialization (not necessarily full) within the household.
- For example, it is possible that in equilibrium one spouse increases the time spent with children and decrease labor supply, but own consumption may not decline if the partner increases labor supply since transfers are proportional to the income.

- In the single agent problem, decreasing the labor supply implies lower consumption.
- A second point is that we assume that women make fertility decisions; in the household framework, this does not mean that men cannot affect fertility decisions.
- For example, it is possible that females' best response to males working longer hours when there are children in the home is to increase fertility.

- In contrast to the model of a single decision maker, there are additional elements in the extended model related to the marriage market and the interactions between spouses within households.
- The cost of investment in a child is an equilibrium outcome: Investment of time by each parent depends on the education of both spouses and the resulting allocation or resources, the degree of specialization in time with children and labor market activities, and how they vary by education level of spouses.

- In the basic model, parental investment affects the education of the children and therefore affects the cost of investment in the children of the offspring.
- Interestingly, in the extended model, parental investment also affects the costs of investment in children and the feasible set of their children through the effect on the marriage market.
- The educational outcomes of a child may change the probability of the child being a single parent, changing the costs of investment directly (recall that the coefficients in the utility function on children depend on marital status).
- It also affects the education of the spouse of the child, taking into account assortative mating.

- Should calibrate and simulate the model to Danish data – Rafeh has code; let's calibrate
- Look at the structural IGE from it – it gives timing and spacing of births (simulate data and Corakify)
- It's a starting point but let's start

Appendix

- A binary relation $\geq$ on a non-empty set is a partial order if it is reflexive, transitive, and anti symmetric.

- A partially ordered set is said to be a lattice if for any two elements the supremum and infimum are elements of the set.
- A two-person game is said to be super-modular if the set of actions for each player σ is a compact lattice, the payoff function is super-modular in $I_{\sigma kt}$ for fixed $I_{-\sigma kt}$, and satisfies increasing differences in $(I_{\sigma kt}, I_{-\sigma kt})$.
- Following Watanabe and Yamashita (2010), if the continuation values in every period and state satisfy the conditions below, the game is super modular and there exists a pure strategies Markov perfect equilibrium.
- Following the convention, we use $\vee$ to denote the supremum of two elements and $\wedge$ to denote the infimum of two elements.

Condition 1 (S)

$\upsilon_{\sigma k k'}(z_t)$ is super-modular in k for any z_t and k' if

$$\upsilon_{\sigma k \vee \widehat{k}, k'}(z_t) + \upsilon_{\sigma k \wedge \widehat{k}, k'}(z_t) \geq \upsilon_{\sigma \widehat{k}, k'}(z_t) + \upsilon_{\sigma k, k'}(z_t)) \quad (24)$$

for all $(\widehat{k}_{\sigma t}, k_{\sigma t})$.

Condition 2 (ID)

$\upsilon_{\sigma k k'}(z_t)$ has increasing differences in $(k_\sigma, k_{-\sigma})$ for any z_t if

$$\upsilon_{\sigma k' k'}(z_t) - \upsilon_{\sigma k k'}(z_t) \geq \upsilon_{\sigma k' k}(z_t) - \upsilon_{\sigma k k}(z_t) \qquad (25)$$

for all $\langle I_{k\sigma t}, I_{-k\sigma t}\rangle$ and $\langle I_{k'\sigma t}, I_{-k'\sigma t}\rangle$ where the outcome of choice that $\langle I_{k'\sigma t} = 1, I_{-k'\sigma t} = 1\rangle$ is greater than or equal to the outcome for the choice $\langle I_{k\sigma t} = 1, I_{-k\sigma t} = 1\rangle$ for both σ and $-\sigma$.

- In order to apply these conditions we need some natural ordering of our set of choices.
- This is satisfied in our application as each choices has natural ordering, e.g. working or spending full time at home is greater that working or spending part time at home.
- Watanabe and Yamashita (2010) provide sufficient conditions on the stochastic transitions functions and the per period utility for existence of a pure strategies Markov perfect equilibrium.
- These conditions impose restrictions on the functional forms of the per- period utility sharing rules, wage functions, value of children, and the return investment in children in our model.

Return to main text