An Equilibrium Model of Child Maltreatment

Hideo Akabayashi. (2006).

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1. Introduction

- The purpose of this paper is to propose a dynamic equilibrium model of a child's human capital formation and the parents' style of interactions with the child and thereby explain complicated phenomena in modern families, such as child maltreatment (abuse).

- The principal-agent framework is used to describe a family consisting of an altruistic parent and a growing child.
- The key assumptions are
  1. a child's human capital develops through his or her own effort under parental influence and interventions,
  2. a child's rate of time preference is a decreasing function of the human capital,
  3. the parent cannot directly observe the child's human capital and the parent's observation errors can be reduced by spending additional time with the child, and
  4. the parent updates her beliefs regarding the child's human capital level using available information.

- The dynamic equilibrium process of the parent's beliefs about the child's human capital indicates that the parental beliefs may diverge.
- It is then suggested that the parent with a high initial expectation about the child's ability tends to maintain an unreasonably high expectation about the child's behavior, which leads to a persistently negatively biased assessment of the child's effort.
- The parent's optimal interactions with the child tend to be punitive rather than positive, thereby providing an explanation of child maltreatment.

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3. The Model

3.1. Law of motion of a child's human capital

- Family consists of one parent and one child.
- Denote the child's human capital at the beginning of period t by h_t, where $t = 1, \ldots, T + 1$.
- $T + 1$ is the period when a child becomes independent of the parent and starts relying only on the value of his or her own human capital accumulated over the previous periods.
- Child's initial human capital or 'potential ability': h_1.
- A child's human capital at subsequent periods is assumed to be determined by the child's human capital level in the immediate past, the level of effort, the parent's time spent with the child, and the family environment including the parent's human capital level.

- Parent and the child know that *the law of motion of human capital* is described by the following linear process:

$$h_{\tau+1} = \underbrace{(1-\delta)}_{\text{depreciation}} h_\tau + \varphi \underbrace{s_\tau H^\gamma}_{\substack{\text{parental human}\\ \text{capital}}} + \phi \underbrace{a_\tau}_{\substack{\text{child}\\ \text{investment}}} \quad \text{for } \tau = 1, \ldots, T, \tag{1}$$

- H is the parent's human capital (assumed to be positive and constant over time),
- $s_\tau \in (0,1]$ is her (normalized) time spent with a child, and
- a_τ is the child's effort level.

- Assume that δ is strictly positive and less than 1 so that the first term represents the depreciation of human capital.
- The second term represents the parent's investment in the child's human capital (or 'education'), which is a function of the time spent by the parent and her human capital level.
- The third term represents the child's own investment in human capital (or 'learning by own effort').
- φ and ϕ are presumably positive marginal effects of 'education' and 'effort' on human capital, respectively.
- We assume that ϕ is less than $1 - \delta$ so that the effect of effort is smaller than the effect of past human capital.

- $h_{T+1} = (1-\delta)^{T-t+1} h_t + \sum_{\tau=t}^{T} (1-\delta)^{T-\tau} (\varphi s_\tau H^\gamma + \phi a_\tau)$, for arbitrary $t < T+1$.
- Therefore, the human capital level when a child becomes independent of the parent (period $T+1$) can be expressed as a function of the human capital at an arbitrary period t and inputs of the child's effort as well as the parent's time spent at and after t.

3.2. Observation equation of the child's behavior

- While the child knows his or her own h_t, the parent cannot directly observe her child's human capital or effort.
- Can observe the child's performance at period τ, y_τ.
- The child's performance is determined by his human capital, effort, and a random shock in that period, according to the following linear *observation equation*:

$$y_\tau = h_\tau + a_\tau + \nu_\tau \quad \text{for } \tau = 0, 1, \ldots, T, \tag{2}$$

- v_τ is a random variable distributed as $\mathrm{N}(0, \sigma^2_{\nu\tau})$ for all τ.

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- The random shock, ν_τ, includes shocks to the child's performance as well as the parent's measurement error.
- The first two terms indicate that a more mature child with more effort tends to behave better.
- The nature of the third stochastic term depends on the duration of the parent's observation.
- Assume that $\sigma^2_{\nu t}$ is decreasing in s_τ, the parent's time spent with the child, because spending more time with a child would presumably reduce the parent's measurement error in the observations.
- The error term cannot be eliminated even if the parent spends the maximum possible time with the child, because the child may still make unintended mistakes.
- We assume that $\sigma^2_{\nu\tau} \equiv K/s_\tau$, where K represents the parent's monitoring ability, possibly correlated with her human capital level.

- The information structure specified as follows.
- The parent's *information set* at t is defined as the set of all information available at period t, denoted by $I_t \equiv \{y_t, y_{t-1}, \ldots, y_1\}$.
- Denote the parent's subjective expectation and mean squared forecasting error of h_t based on I_s by $\hat{h}_{t|s}$ and $\sigma^2_{ht|s}$, respectively.
- Define $\hat{h}_t \equiv \hat{h}_{t|t-1}$ and $\sigma^2_{ht} \equiv \sigma^2_{ht|t-1}$.
- The best one-step-ahead predictor and mean-squared error, respectively.
- Let the pair $(\hat{h}_t, \sigma^2_{ht})$ denote the parent's *belief at period* t.
- We assume that the parent has a prior belief $(\hat{h}_1, \sigma^2_{h1})$ at the time of the child's birth.

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3.3. Parent's incentive schedule

- Parent considers a child's happiness as her own happiness (altruism) and can create and transfer 'services' to the child.
- The child derives utility from these services.
- d_τ: the amount of services created and transferred at period τ.
- Services are assumed to consist of two components: the time spent with the parent (s_τ), and parent's interactions (kiss, hug, spank, etc.).
- The first component is directly productive since it appears in (1), while the second component is assumed to have only psychological effects.
- At each period, the parent sets the time spent with the child and promises a schedule based on which she interacts with the child.

- Since the parent cares about the child's future human capital, her actual choice of interaction depends on the promised schedule and her estimate of the child's effort given the available observations, $\mathrm{E}[a_\tau | I_\tau]$.
- After interacting with the child, the parent revises her belief regarding the child's human capital.
- More specifically, we consider only the following linear *incentive schedule*, by which the parent produces the argument of the child's utility function measured in hours multiplied by a measure of the parent's human capital

$$d_\tau = (s_\tau + b_\tau \mathrm{E}[a_\tau | I_\tau]) H^\gamma \quad \text{for } \tau = 0, 1, \ldots, T. \tag{3}$$

Among the components of d_τ,

- $s_\tau H$ is the service created by spending time with the child
- $b_\tau \mathrm{E}[a_\tau | I_\tau] H$ is the service from the parent's interactions contingent on the new observation, y_τ
- b_τ is called the *slope of incentive*

- Parent's marginal change of interactions with the child measured in the equivalent unit of time when her estimate of the child's effort changes.
- Assume that the child's 'effective' incentive that is created is a multiple of H, the parent's human capital level.
- Therefore, it is reasonable to term $b_\tau \mathrm{E}[a_\tau | I_\tau]$ the *parent's observed interaction*.
- Since b_τ is shown to be positive in equilibrium, the observed interaction is large ('praise') when the parent observes good performance and forms a high estimate of the child's effort, and it reduces ('punishment') when poor performance is observed.

- Given this structure, a set of the two variables from period t onward, $\{s_\tau, b_\tau\}_{\tau=t}^{T}$, completely defines the parent's *plan of parenting at period t*.
- The assumptions that the parent can choose some part of the child's utility and that the parent cares about the child create a connection between the parent and the child.

- From (2), we have
  $\mathrm{E}[a_\tau | I_\tau] = \mathrm{E}[y_\tau - h_\tau - \nu_\tau | I_\tau] = y_\tau - \hat{h}_\tau = h_\tau + a_\tau + \nu_\tau - \hat{h}_\tau;$
- (3) is then rewritten as

$$d_\tau = (s_\tau + b_\tau(h_\tau + a_\tau + \nu_\tau - \hat{h}_\tau))H^\gamma \quad \text{for } \tau = 0, 1, \ldots, T. \quad (4)$$

observed

guess

- Therefore, given a series of the current and past observations (I_τ), the parent's incentive provision is based on the difference between the behavior observed today and the best estimate of the child's human capital.
- Clearly, given an observation y_τ at period τ, if the parent had a high expectation of the child's human capital level (high $\hat{h}_\tau$) at the beginning of that period, she tends to have a low estimate of the child's effort (low $\mathrm{E}[a_\tau | I_\tau]$) and tends to 'punish' him or her (low d_τ), and vice versa.

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3.4. Preferences

- Assume that the rate of time preference is a decreasing function of human capital (Becker and Mulligan, 1997).
- Denote the child's rate of time preference and the parent's rate of time preference by $\rho_{ct}(\equiv \rho(h_t))$ and $\rho_p(\equiv \rho(H))$, respectively.
- Assume that $\lim_{h\to\infty}(\mathrm{d}/\mathrm{d}h)(1/(1+\rho_{ct}))=0$.
- Also there is a value of h, h^c, such that $1/(1+\rho_{ct})$ is concave in $h_t \in (h^c, \infty)$.
- These are natural assumptions since the discount factor is bounded from above.

- The assumption on the limit may be restated as 'the discount factor tends to be inelastic with respect to human capital as the level of human capital increases,' like many characteristics that tend to be fixed as a child becomes an adult.
- For instance, $(1/(1+\rho_{ct})) = 1/(1+\exp(-\eta h_t))$ with $\eta > 0$ satisfies these requirements for $h_t > 0$, which will be used later.

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- A child is assumed to be myopic in three ways.
- First, the child's rate of time preference is generally greater than the parent's rate, because the child is less mature (as measured by the child's level of human capital).
- Second, although the child knows about the law of motion of his or her own human capital, that future tastes might change with the evolution of human capital is not known to the child, and therefore, the child considers the current rate of time preference as given in deciding the future effort allocation plan.

- Finally, due to the lack of knowledge of changing preferences, the child does not know how his or her choice today may influence the parent's future parenting choices through her improved knowledge of the child's preferences.
- Clearly, the child's decision might be time-inconsistent, and the child might regret and revise the plan.
- In contrast, the parent is less myopic in the sense of having a lower rate of time preference (ρ_p) and has the knowledge that the child's rate of time preference changes as the child grows.

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- Further, we assume that effort is painful to the child and provide disutility $-v(a_\tau)$, where $v(\cdot)$ is a positive, increasing and convex function.
- In particular, we assume $v(a) = (a - \underline{a})^2/2\psi$, where $\underline{a}$ is an individual fixed characteristic representing the child's least painful level of effort.
- $\underline{a}$ is the *child's natural level of effort*.
- $1/\psi$ determines the child's marginal disutility of effort.
- The child's one-period utility is determined by the sum of this disutility of effort and the incentive schedule provided by the parent, namely $d_\tau - v(a_\tau)$.

- Assume that both parent and child have exponential preferences toward risk in their life-cycle utility; the parent maximizes the expected value of $U(\cdot) \equiv -[\exp\{-R(\cdot)\}]$ while the child maximizes the expected value of $u(\cdot) \equiv -[\exp\{-r(\cdot)\}]$, where $(\cdot)$ takes each agent's sum of utility over the life cycle as its argument and R and r are the parameters governing attitudes toward risk.

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4. Optimal Interactions and Equilibrium

4.1. Child's decision problem

- A series of decisions in one period.
- Given a belief about a child's human capital $(\hat{h}_t, \sigma^2_{ht})$ at the beginning of period t, the parent decides upon a plan of parenting $\{s_\tau, b_\tau\}_{\tau=t}^{T}$.
- Given this, the child chooses a plan of efforts $\{a_\tau\}_{\tau=t}^{T}$.
- Next, the child's performance is observed according to (2).
- The parent determines the amount of the services to be provided to the child via (4) and revises her belief.
- Finally, the child's human capital develops according to (1).

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- Consider the child's problem at period t.
- The child's optimization problem is

$$\text{Max } \mathrm{E}\left[u\left(\sum_{\tau=t}^{T}\left(\frac{1}{1+\rho_{ct}}\right)^{\tau-t}(d_\tau - v(a_\tau)) + \left(\frac{1}{1+\rho_{ct}}\right)^{T-t+1} B\cdot h_{T+1}\right)\middle| I_{t-1}\right]$$

subject to (1), (4) for $\tau = t, \ldots, T,$ given $\{s_\tau, b_\tau\}_{\tau=t}^{T}$ and h_t,

- $B \cdot h_{T+1}$ (B is a positive constant) determines the child's utility from his or her own human capital at $T+1$.

$$d_\tau = (s_\tau + b_\tau(h_\tau + a_\tau + \nu_\tau - \hat{h}_\tau))H^\gamma \quad \text{for } \tau = 0, 1, \ldots, T. \; (4)$$

$$h_{\tau+1} = \underbrace{(1-\delta)}_{\text{depreciation}} h_\tau + \varphi \underbrace{s_\tau H^\gamma}_{\text{parental human capital}} + \phi \underbrace{a_\tau}_{\text{child investment}} \quad \text{for } \tau = 1, \ldots, T, \qquad (1)$$

- First-order condition for a_t is

$$b_t H^{\gamma} - (a_t - \underline{a})/\psi + \left(\frac{1}{1+\rho_{ct}}\right)^{T-t+1} (1-\delta)^{T-t} B\phi = 0. \quad (5)$$

- While deciding the level of effort using (5), the child compares the immediate marginal pain (the second term) with the sum of the immediate marginal return from the parent's love (the first term) and the subjectively discounted future return from his or her human capital stock upon becoming an adult at $T+1$ (the third term).
- Notice that the child's decision regarding today's effort is independent of his or her future decisions or future human capital levels, due to additivity and the child's myopia over changing preferences.

- His or her planned future efforts as of today might differ from efforts actually chosen in the future, because the rate of time preference changes and the way in which it will change is unknown today.
- Furthermore, the child might make a wrong guess about the parent's future actions.
- This inconsistency does not pose a problem in interpreting the child's decision *today* because it depends only on his or her human capital and parental incentives *today*.

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- By defining the child's *subjective marginal return to investment* at age t, $D_t(h_t) \equiv (1/(1+\rho_{ct}))^{T-t+1}(1-\delta)^{T-t}B$, (5) is solved for the optimal effort at t in response to the parental incentive:

$$a^* = \underline{a} + \psi b H^\gamma + \phi\psi D_t(h). \tag{6}$$

- The child's natural level of effort ($\underline{a}$) and the parental incentive (the second term) have positive effects on his effort.
- Since $D_t(h)$ is increasing in h and in t independently, an older or more mature child tends to make more effort.

- The reason that age has an independent effect is that the need for effort becomes more apparent as the child becomes older (finite-horizon effects).
- A larger level of effort would be chosen if the child's marginal disutility of effort $(1/\psi)$ is smaller, or if effort is more productive in the accumulation of human capital (larger ϕ).
- Notice that a^*, which is known to the child, is an unobservable stochastic variable to the parent even though the parent controls the slope of incentive (b), because the uncertainty regarding h still remains.
- Thus, the child's choice of effort partly depends upon his 'maturity,' regarding which the parent can update her belief from past observations.

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- From (6), the observation equation (2) can be rewritten as $y^* = h + \phi\psi D_t(h) + \underline{a} + \psi bH^\gamma + \nu$.
- Given the parent's choice of the 'effective incentive slope,' bH, the child's observed performance is positively correlated with his human capital level for at least two reasons.
- The first term represents the exogenous effect of h on the child's performance.
- The second term represents the endogenous effect of human capital on the child's performance, because the choice of effort depends on the child's rate of time preference which, in turn, depends on the child's human capital.

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4.2. Parent's decision problem

- At period t, the parent chooses a plan of parenting. $\{s_\tau, b_\tau\}_{\tau=t}^{T}$, to maximize her expected utility from family consumption and the child's happiness, given the child's response function.
- Let π be the wage rate of one efficiency unit of the parent's human capital, and let α describe the parent's degree of altruism toward the child, both of which are assumed to be time-invariant.
- Assuming the parent has one unit of time to spend either with the child or working, she will spend $1 - s_t$ units of time working in the market.

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The parent's problem is to

$$\text{Max } \mathrm{E}\left[U\left(\sum_{\tau=t}^{T} \left(\frac{1}{1+\rho_p} \right)^{\tau-t} [c_\tau + \alpha(d_\tau - v(a_\tau^*))] + \alpha \left(\frac{1}{1+\rho_p} \right)^{T-t+1} Bh_{T+1} \right) \middle| I_{t-1} \right]$$

subject to $c_\tau = \pi(1 - s_\tau)H^\gamma$, (1), (4), and (6) for $\tau = t, \ldots, T$.

- Notice that, for the same reason as stated before, the parent's decision today is independent of the parent's (or the child's) future decisions.
- Therefore, we can focus on the choice of the 'current' parenting plan, $\{s_t, b_t\}$, and the child's current response, a_t^*.
- Choosing a large s_t is costly because it can be achieved only if the parent spends less time at work.

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- Although there is no explicit market cost in choosing a large b_t, a risk averse, altruistic parent has a reason to avoid this, because she would prefer less variability in her interactions with the child.
- This is clearly seen from the following first-order condition for b_t (with time subscript being suppressed again).

$$bH^{\gamma} = \frac{1}{R\alpha(\sigma_h^2 + K/s)}(\underline{a} + b\psi H^{\gamma} + \psi\phi D_t(H))$$
$$= \frac{1}{R\alpha(\sigma_h^2 + K/s)}(\hat{a}^* + \psi\phi[D_t(H) - D_t(\hat{h})]), \tag{7a}$$

- where $D_t(H) \equiv (1/(1+\rho_p))^{T-t+1}(1-\delta)^{T-t}B$ defines the parent's *subjective marginal return to investment* and
- $\hat{a}^*$ is the parent's estimated effort of the child based on the estimate of human capital.

- Thus, the parent needs to set a steeper incentive slope if she wants to induce greater effort ($\hat{a}^*$) or if she estimates a larger difference in her own and the child's subjective marginal return to investment, other things being equal.
- As long as H is larger than $\hat{h}$ and $\underline{a}$ is positive, the optimal slope must be positive.
- The first-order condition for s_t is

$$\pi = \alpha + \alpha\varphi D_t(H) + \frac{R\alpha^2 H^\gamma K b^2}{2s^2}. \tag{7b}$$

happiness H.C Info.

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- Left-hand side is the opportunity cost of being with the child.
- Right-hand side is the marginal return to time spent with the child and consists of the following three components:
  - The first term is the immediate marginal happiness derived from being with the child,
  - The second term is the future marginal return to increasing the child's human capital, and
  - The last term is the return to improved information about the child's behavior.
- The last term appears because being with the child makes it easier for the parent to monitor the child, which reduces the risk of punishing a good child.

- To clarify this point further, rearrange (7b) to obtain a proportional relationship between s and b as follows:

$$s = \left(\frac{2H^{\gamma}}{R\alpha^2 K} [\pi - \alpha - \alpha\varphi D_t(H)] \right)^{-1/2} bH^{\gamma} \equiv Q_t^{-1} bH^{\gamma}. \quad \text{(7b')}$$

- An endogenous complementarity between the time spent with the child and the parent's interactions.
- This is because, if the parent spends more time with the child, she can observe the child's behavior with less error and can set stricter criteria for judging the child's performance.

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- It is also easy to see the following implications for the parent's substitution between 'time spent' and 'interactions.'
- First, if the wage rate (π) is higher, she tends to shift away from time spent with the child and toward a stricter discipline.
- Second, the reverse shift would occur if the parent is more altruistic (larger α).
- Finally, if K or R is larger, she also tends to spend more time with the child for reducing the risk of making a mistake in judging the child's effort.

4.3. Equilibrium system equation and the parent's expectation process

- The parent's and the child's optimal actions are determined jointly by (7a) and (7b').
- Assuming that the second order condition is always satisfied ($R\alpha\sigma_h^2 - \psi > 0$) and that an interior solution can be achieved, we have the following reduced form solutions for the slope of incentive, bH^γ, and parental time with the child, s:

$$b^* H^\gamma = \frac{1}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t), \tag{8a}$$

$$s^* = \frac{1}{Q_t(R\alpha\sigma_h^2 - \psi)}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t). \tag{8b}$$

- Applying (8a) to (6), we obtain the reduced form solution for the child's effort,

$$a^* = \underline{a} + \phi\psi D_t(h) + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t). \tag{8c}$$

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- Thus, better information about the child's human capital (smaller σ_h^2) has a positive effect on each of the following: the slope of incentive, the time spent with the child, and the child's effort.
- Care must be taken in interpreting the result that the parent's choice variables are independent of the estimated *level* of the child's human capital ($\hat{h}$).
- This is because of our additivity assumption on the human capital production function, and if we allow any complementarity between the inputs of the human capital production, $\hat{h}$ should affect the parent's choice.

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- Notice also that b may not be monotonically related to the parent's human capital, H; although a more educated parent needs less severe interactions on the left-hand side of (8a), such a parent will tend to have a higher $D_t(H)$; which would make her choose a larger b.
- A more risk-averse (larger R) parent, being afraid of punishing a good child, tends to choose a smaller b and the induced effort tends to be small.
- Although the parent is tempted to shift from 'intervention' to "being with the child," she finally chooses a smaller s, because the return to reducing observation error reduces when a less strict intervention plan is chosen.
- In this way, the endogenous complementarity of the parent's interactions (b) and the time spent with the child (s) tend to generate another force in the choice of the plan of parenting.

- Due to this complementarity, some of the comparative statics results are now ambiguous.
- For example, a more altruistic (larger α) parent would be willing to stay with the child longer (larger s), thereby reducing the observation error; hence the improved accuracy in the observations would allow the parent to opt for a stricter discipline (larger b).
- At the same time, the parent tends to shift the parenting plan away from the use of intervention.
- It turns out that the effects of the degree of altruism are ambiguous, depending on the relative strengths of these two opposite forces.
- Among the child's other characteristics, a larger ψ (less marginal disutility of effort) and a larger $\underline{a}$ (natural level of effort) have positive effects on b, s, and a.

- Substituting (8a)–(8c) into (1) and (2) yields a state-space representation of the dynamic equilibrium system.
- The *equilibrium law of motion* (system equation) defined as

$$h' = (1-\delta)h + \phi a^* + \varphi s^* H^\gamma = F(h) + G(\sigma_h^2), \tag{9a}$$

$$h' \equiv h_{t+1},$$

$$F(h) \equiv (1-\delta)h + \phi^2 \psi D_t(h),$$

and

$$G(\sigma_h^2) \equiv \phi \left[\underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi} (\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \right] + \varphi s^* H^\gamma.$$

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- The equilibrium law of motion of human capital has two major components: $F(\cdot)$ is related to the child's current human capital, including its endogenous effect on effort.
- $G(\cdot)$ is related to the parent's background (H), the level of uncertainty regarding the child's human capital (σ_h^2), and preference parameters.
- Both components are time-dependent functions due to the finite horizon nature of the model.

- The *equilibrium observation equation*:

$$y = h + a^* + \nu = A(h) + C(\sigma_h^2) + \nu, \tag{9b}$$

- where

$$A(h) \equiv h + \phi\psi D_t(h) \text{ and } C(\sigma_h^2) \equiv \underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t).$$

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- The equilibrium observation equation also consists of two major components:
- $A(\cdot)$ is the contribution of the current human capital, including its endogenous effect on the child's effort (the second term in the definition of $A(\cdot)$).
- $C(\cdot)$ is a function of the other factors that affect the child's effort.
- Both functions are time-dependent for the same reason as stated before.
- Considering this nonlinear equilibrium system (9a)–(9b), assume that the parent forms and updates the expectation about the child's human capital with linear approximation as follows.

- Given a belief $(\hat{h}, \sigma_h^2)$ at the beginning of period t and a new observation on behavior, y, the parent first updates the belief of the child's human capital today from $\hat{h}(\equiv \hat{h}_{t|t-1})$ to $\hat{h}^u(\equiv \hat{h}_{t|t})$ using the Bayesian updating rule.
- She then uses the optimal recursive projection formula (Kalman filter) to construct the one-step-ahead projection of the child's human capital and its error variance $(\hat{h}', (\sigma_h^2)')$ that becomes her belief at the beginning of the next period.
- Call the stochastic process of the parent's belief, $\{\hat{h}_t, \sigma_{ht}^2\}$, constructed in this way, the *parent's expectation process*.

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- To construct the expectation process, assume that the parent uses the following algorithm.
- First, she uses $\hat{h}$ as the first guess to linearly approximate $A(\cdot)$ and updates it using a new observation y.
- The updating rule is essentially an average of the previous belief and the information obtained from the new observation weighted by the degree of uncertainty, written as follows:

$$\hat{h}^u = \hat{h} + \frac{A'(\hat{h})\sigma_{ht}^2}{A'(\hat{h})^2\sigma_h^2 + \sigma_\nu^2}(y - A(\hat{h}) - C(\sigma_h^2)), \qquad (10)$$

- $A'(\hat{h}) = 1 + \phi\psi D_t'(\hat{h})$ is the first-order Taylor coefficient from (9b).

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- She then uses the updated value as the second approximation, obtains a better approximation of $A(\cdot)$, and updates it by applying (10) again.
- After iterating on this procedure, the parent reaches an estimate $\hat{h}^u$ and uses it with the linear approximation of (9a) to estimate the human capital at the beginning of the next period.

- Thus the parent's expectation process is

$$\hat{h}' = F(\hat{h}^u) + G(\sigma_h^2), \tag{11a}$$

$$(\sigma_h^2)' = \Phi \cdot \sigma_h^2, \tag{11b}$$

$$\Phi \equiv \frac{F'(\hat{h}^u)^2 \sigma_\nu^2}{A'(\hat{h}^u)^2 \sigma_h^2 + \sigma_\nu^2}$$

$F'(\hat{h}^u) \equiv (1-\delta) + \phi^2 \psi D_t'(\hat{h}^u)$: first-order Taylor coefficient from (9a).

- The time-dependent coefficient, Φ; characterizes the stability of the parent's expectation process.
- Since this is a finite-horizon problem and the rate of time preference is endogenous, the process is state-dependent.
- There is no stationary state and we cannot expect the error variance to converge mechanically as predicted by the theory of the time-invariant Kalman filter.

However, if Φ is less than 1, the parent's belief converges in the sense stated in the following lemma:

Lemma 1

The error variance σ_h^2 of the parent's estimate of the child's human capital monotonically decreases over time if $\Phi < 1$ is satisfied.

Proof.

Obvious from (11a)–(11b). □

- From the definition of Φ, it is clear that a sufficient condition for Φ to be less than 1 is that $F'(\hat{h}^u)$ is less than 1.
- Since $D_t(\hat{h}^u)$ is concave in $\hat{h}^u$ due to the assumption regarding the discount factor function, the condition in the lemma is satisfied if the parent's expectation is kept adequately high to make $D_t'(\hat{h}^u)$ sufficiently small.
- Let us define h_1^* and h_2^* to be the two levels of human capital that solve $F'(h) = 1$.

- If $\hat{h}^u$ is strictly higher than h_2^*, the child is perceived to be sufficiently mature and therefore the child's rate of time preference is insensitive to the change in his or her level of human capital.
- Then, the lemma states that, starting from any initial σ_h^2, the parent's uncertainty decreases over time.
- Clearly, such an h_2^* must be larger if ψ or ϕ is larger or if δ is smaller.
- However, if $F'(\hat{h}^u)$ is not less than 1, Φ can be greater than 1.
- In fact, we can show that the following proposition holds:

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Proposition 1

Suppose that $\rho_{ct} = \exp(-\eta h_t)$ *with* $\eta > 0$ *and that there exists a level of h for which* $F'(h) > 1$. *Then, (i) there exists a combination of parameter values such that there exists a level of* $\overline{\sigma}^2$ *and the associated range of* $\hat{h}^u$, $R(\overline{\sigma}^2) = (h_L, h_H)$, *such that for any* $\sigma_h^2 > \overline{\sigma}^2$ *(equivalently,* σ_ν^2 *is larger than* $\sigma_h^2(1-(1-\delta-\phi)^2)/\phi^2$*) and any* $\hat{h}^u \in R(\overline{\sigma}^2)$, Φ *is greater than one, (ii) given an updated belief* $(\hat{h}^u, \sigma_h^2)$ *and the parameter values that satisfy the condition described in (i), the parent's expectation process becomes divergent, and (iii)* h_L *is decreasing and* h_H *is increasing in* $\overline{\sigma}^2$.

Proof.

See Appendix E. □

- Proposition 1 indicates that the necessary condition for the parent's belief no be divergent is that the parent-child pair satisfies $\sigma_h^2 > \min(\overline{\sigma}^2) \equiv \overline{\sigma}_m^2 > 0$.
- Although this condition may not look intuitive, it is equivalent to $\sigma_\nu^2 > \sigma_h^2(1-(1-\delta-\phi)^2)/\phi^2$ (see Appendix E), which implies that the uncertainty due to the observation error is larger (by a certain factor) than the uncertainty regarding the child's current human capital.
- Recalling that the actual observation is the sum of these components, it is no surprise that, in such a situation, an additional observation will not improve the knowledge about the child.

Link to Appendix E

5. Interpretation

5.1. Interpreting the stable and unstable expectation processes

- We have seen that the nonlinear equilibrium system equations (11a)–(11b) are state-dependent, due to the endogenous development of the child's rate of time preference and the finite horizon.
- A child's human capital develops endogenously as the child matures, because it is enhanced by the child's effort, which depends on the child's rate of time preference (see (9a) and the definition of $F(\cdot)$).
- Therefore, despite the regression-to-the-mean nature of the *given* law of motion of human capital (1), the *equilibrium* law of motion of human capital may not exhibit regression to the mean.

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- The same reasoning applies to the parent's expectation process.
- When the parent updates her belief about the child's maturity upwards, she also revises the expectation of the child's effort because she thinks, *"the kid seems to get smarter, so he must be more responsible."*
- The expectation may self-generate a higher expectation of the child's development with increased uncertainty due to the endogenous system coefficient, Φ, in (11a)–(11b).
- Φ can still be less than 1 if the revision of the expectation of the effort is sufficiently small.

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- This is likely, as implied by the lemma and Proposition 1,
  1. if the child is sufficiently grown up and the child's behavior is not greatly affected by a small change in his or her level of human capital, or
  2. if the parent has adequately good initial knowledge about the child and therefore spends a long time with the child in order to maintain low observation uncertainty (and low uncertainty from the unobservable effort).

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- When the conditions described in Proposition 1 are satisfied, the equilibrium development process of human capital becomes endogenously explosive, at least locally, and so does the associated parental expectation process.
- While providing the full statistical characteristics of this locally explosive process is beyond the scope of this paper, we will discuss two typical cases in which different initial conditions may generate dramatically different equilibrium paths.
- The two cases are then visualized with a numerical simulation in order to enhance the understanding of qualitative discussions on the stability of the equilibrium path and the emergence of child maltreatment – a persistent negatively biased belief and interaction toward the child.

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(a) *Case of converging belief – normal family.*

- Fig. 1 illustrates the phase diagram of the parent's expectation process based on Proposition 1.

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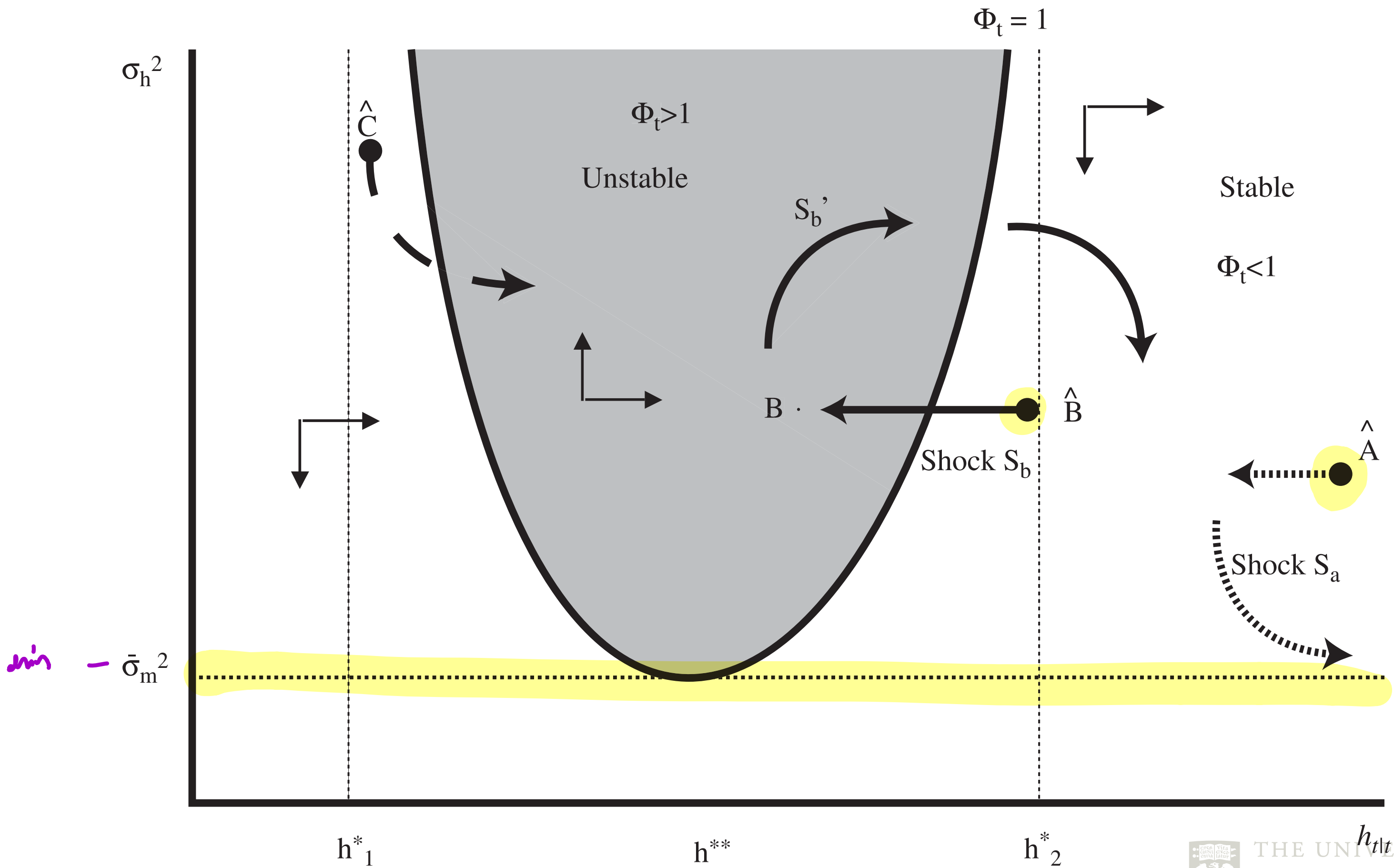
Figure 1: Dynamics of the parent's expectation process



- Figure treats only the domain where the child is in the middle of development and the parent estimates a low level of $\hat{h}^u$.
- Therefore, in the figure, $\hat{h}^u$ is increasing over time except for perturbations by random shocks.
- We will focus on what occurs around the curve $\Phi = 1$, since it provides us with the most important and interesting interpretations in characterizing the dynamics.

- Suppose that a parent is very sure that the child has a high level of human capital with a small error variance, as shown by $\hat{\mathbf{A}}$ in Fig. 1.
- The parent believes that the child has 'grown-up' characteristics, and therefore $D_t(\hat{h}^u)$ is insensitive to $\hat{h}^u$ and the effect of uncertainty about human capital (the second term in (9a)) is small.
- Since Φ is likely to remain less than 1 even with some shocks to the parent's observations (the arrow $\mathbf{S}_a$ in Fig. 1), the parent's belief is likely to be stable and to converge monotonically.

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- In particular, if the initial uncertainty is less than $\overline{\sigma}_m^2$, the process would never diverge due to any shock.
- The parent's knowledge about the child improves as each new observation becomes available, and the expectation tends to become unbiased after many observations.
- As in (8a), the slope of incentive, b, is increasing over time as the parent collects more information about the child.
- The child's induced effort is also increasing over time and approaches the first-best level as the child matures and the parent becomes confident about the child's human capital.

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- We next show that the probability of 'punishment' decreases over time if the parent's belief converges.
- First, we construct *the equilibrium distribution of the parent's interaction* from the distribution of the parent's observed interactions, $b_t \mathrm{E}[a_t | I_t]$, defined in Section 3.3.
- Second, we define a statistic of the parent's observed negative interactions that measures the probability of 'non-punishment.'
- Finally, we examine how this statistic changes when the parent's belief is converging.

- The parent's observed interaction in equilibrium, e_t^*, is defined and evaluated as follows:

$$
\begin{aligned}
e_t^* &\equiv b_t^* \mathrm{E}[a_t^* | I_t] = b^*(h - \hat{h} + a^* + \nu) = b^*[h - \hat{h} + \nu + \underline{a} + \psi b^* H^\gamma + \psi\phi D_t(h)] \\
&\cong b_t^*[h - \hat{h} + \nu + \underline{a} + \psi b^* H^\gamma + \psi\phi D_t(\hat{h}) + (h - \hat{h})\psi\phi D_t'(\hat{h})] \\
&= b^*[(h - \hat{h})(1 + \psi\phi D'(\hat{h})) + \nu + \underline{a} + \psi b^* H^\gamma + \psi\phi D_t(\hat{h})],
\end{aligned}
$$

- (6) is used to obtain the first line $D_t(h)$ is linearly approximated to derive the second line from the first.

- Since ν and h are Gaussian and uncorrelated with each other, when the parent has a belief $(\hat{h}, \sigma_h^2)$ at the beginning of the period and if this belief is unbiased, the equilibrium distribution of the parent's interaction, e_t^*, is defined as $\mathrm{N}(\mu_e, \sigma_e^2)$, where

$$\mu_e = b^*(\underline{a} + \psi b^* H^\gamma + \psi\phi D_t(\hat{h})),$$
$$\sigma_e^2 = b^{*2}[(1 + \psi\phi D_t(\hat{h}))^2\sigma_h^2 + \sigma_\nu^2]. \quad (12)$$

- The expected value of the parent's interaction is higher when the parent expects a high effort level due to a high expectation of the child's human capital (large $\hat{h}$) or chooses strict discipline (large slope of incentive, b^*).
- This also has a scale effect on both the expected value and the standard error.
- Suppose that we recognize that parental behavior becomes 'punitive' when the parent's interaction falls below a certain threshold level.
- Since e_t^* is Gaussian, the probability that the parent does not become punitive is described by a 'non-punishment' statistic, $t_e \equiv \mu_e / \sigma_e$.

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- When t_e is large, it is unlikely that the parent's interaction falls below the threshold level.
- Using (12) and (7b'), this is evaluated as

$$t_e = \mu_e/\sigma_e = \frac{b^*(\underline{a} + \psi b^* H^\gamma + \psi\phi D_t(\hat{h}))}{b^*[(1+\psi\phi D_t'(\hat{h}))^2\sigma_h^2 + \sigma_\nu^2]^{1/2}} = \frac{(\underline{a} + \psi b^* H^\gamma + \psi\phi D_t(\hat{h}))}{[(A'(\hat{h}))^2\sigma_h^2 + (K/b^* H^\gamma Q_t)]^{1/2}}. \tag{13}$$

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- Since b^* is decreasing in σ_h^2 from (8a), it is found that $\partial t_e / \partial(\sigma_h^2) < 0$.
- This is because, as the parent becomes more certain about the child's human capital, she spends more time with the child and chooses stricter discipline to induce a greater effort that makes a low level of interaction less likely.
- Thus, we have proved the following proposition:

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Proposition 2

When the parent's belief is unbiased, as the uncertainty about the child becomes smaller, t_e becomes larger and the parent employs low level of interactions less often.

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- Therefore, when the parent's belief converges over time, the probability of parental interaction below a certain level ('punishment') tends to decrease over time.
- The average observation in most 'normal families' corresponds to this case. Along such an equilibrium path, a parent increases and maintains her 'fair' control over the child and the probability of actual punishment decreases.

- In the above discussion we have looked at only the partial effect of the change in the level of uncertainty on the parent's interaction.
- Eq. (13) also shows the effect of the estimated human capital level.
- In the case where the parent's belief is convergent, the child is likely to develop his or her human capital rapidly with increasing effort and time inputs (see (8a)–(8c)).
- When the child's human capital is developing quickly and the parent is estimating the child's human capital in an unbiased manner, punitive interactions will be even less likely because the child will achieve a large discount factor more quickly.
- Therefore, the above result need not be altered fundamentally.

- Additionally, it is important to assume that the estimate of the child's human capital is unbiased to show this result.
- If the estimate is biased, it affects the parent's estimate of the child's effort, and the distribution of the parent's interactions.
- In particular, if it is positively biased, the parent would underestimate the child's effort and a negatively biased interaction is more likely than in the case of an unbiased belief.
- In fact, such a negative bias may prevail when the belief is diverging—an even worse combination—as shown in the next case.

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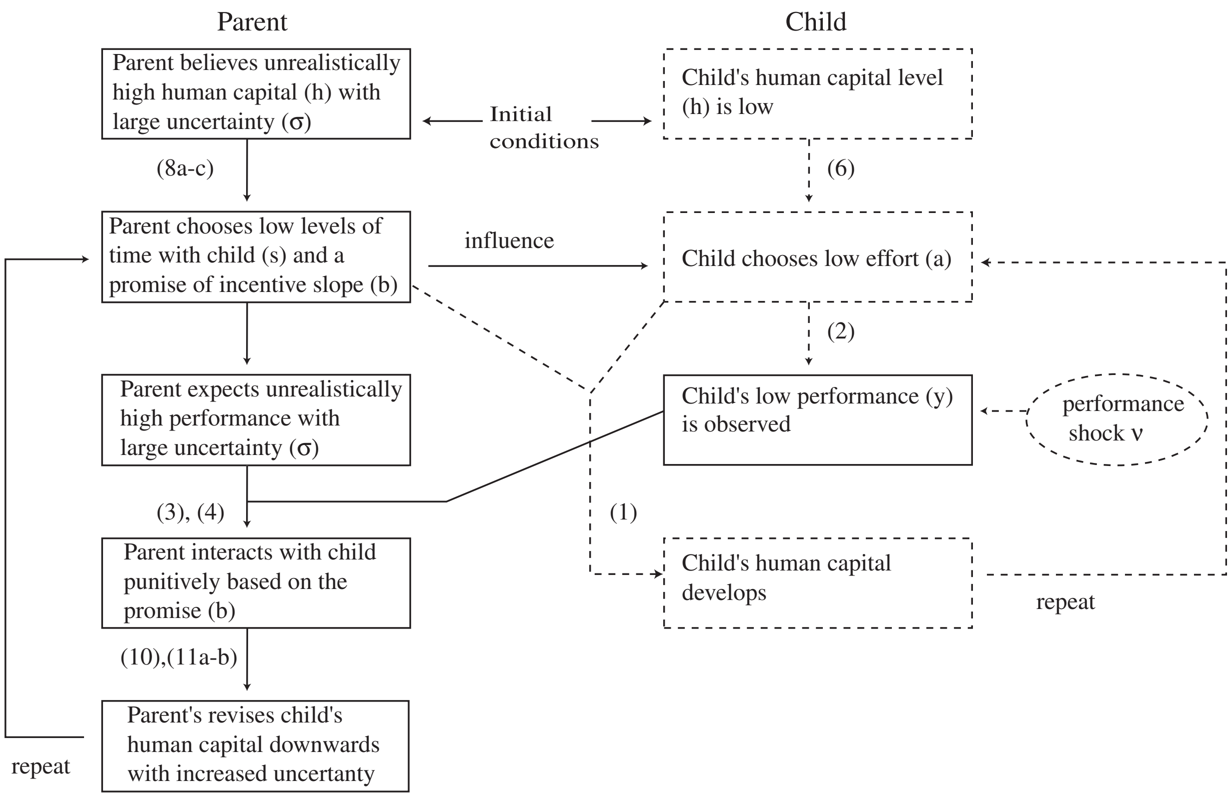
Figure 2: A sequence of actions when the risk of child maltreatment increases



Note: The dotted boxes and arrows indicate that the states and the actions are not observed to the parent. The numbers in the parentheses indicate the relevant equations.

(b) *Case of diverging belief — pathological family.*

- Using Fig. 1, we now illustrate how a unrealistically high expectation may lead to negatively biased interactions—child maltreatment.
- To clarify the process of child maltreatment in our model, Fig. 2 shows a sequence of actions that leads to maltreatment with the relevant equation numbers.
- As shown by $\hat{B}$ in Fig. 1, suppose that, initially, or after observing a couple of the child's 'lucky' performances, the parent's expectation of the child's human capital becomes unreasonably high relative to the true level of human capital while there is still a high level of uncertainty.

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- The parent chooses small b and s according to (8a)—(8b).
- Since the child's true human capital level is low, he chooses the effort level according to (6) that is lower than the parent's expectation. The parent then observes the child's unexpectedly poor performances, administers punishment increasingly, and revises her belief of human capital downwards (the arrow $\boldsymbol{S_b}$ in Fig. 1).

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- The child's human capital develops according to (1), but its speed is slow since parental time and child's effort are small.
- As the parent lowers her expectation, the expectation process becomes less stable because the child's effort is more sensitive to changes in the level of the child's human capital than before.
- If Φ becomes greater than 1, the beliefs regarding the child's human capital tend to diverge endogenously and the parent loses confidence in the child's characteristics and unobserved effort. (*"What is that kid thinking!"*)

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- As long as the belief lies in the unstable domain, the parent's expectation ($\hat{\boldsymbol{B}}$) cannot converge to the true value ($\boldsymbol{B}$) and easily reverts to and remains in the stable domain (the arrow $\boldsymbol{S}_{\boldsymbol{b}}'$).
- The belief will be stable even though it is an upward biased false belief.
- In this way, the parent tends to overestimate the child's human capital.
- She tends to be disappointed by the child's performance and tends to punish him more often until the true value of human capital actually progresses into the stable domain.

- As the true level of human capital develops and grows beyond h_2^*, the parent's expectation tends to stabilize because the child's characteristics become less sensitive to changes in the level of human capital, and it becomes easier for the parent to have a correct expectation.
- There is still some risk of getting into a bad cycle if the child experiences many unfortunate shocks that push the parental expectation downward beyond the $\Phi = 1$ curve, but it becomes increasingly less likely as the child grows.
- When the parent remains unsure about the child's characteristics, Proposition 2 predicts that the probability that the parent's interactions become negatively biased remains large if there is no bias in the parent's belief.

- If there is a positive bias in the belief, the result is worse, as is predicted above; the evaluation of the child's effort becomes negatively biased, and the probability of punitive interactions is even larger than in the case without bias.
- Thus, during the process of a diverging belief, it is highly likely that we will observe a higher frequency of punitive interactions over time unless shocks to the child's performance quickly lead the parent's expectation and the actual development of the child toward a stable state.

- The expectation may not stabilize very quickly if the development of the child's human capital is slow or the parent's initial uncertainty is large for some reason (bad luck, bad environment, etc.).
- For example, if the expectation process starts at $\hat{C}$ in Fig. 1, the process inevitably passes through the unstable domain and takes a long time to escape from a situation of false beliefs about the child.
- After a series of positive observation shocks, the parent is easily trapped in the belief that the immature child is already mature and is likely to have a negatively biased opinion afterward. (*"This kid must be mature enough not to do such a stupid thing!"*)

- Then, a divergent belief implies a delay in the child's development, because the child's effort and time spent with the parent decrease if uncertainty regarding the child's human capital is increasing (see (8a)).
- These are frequently observed actual phenomena in families with abuse (Wolfe, 1987, p. 34).

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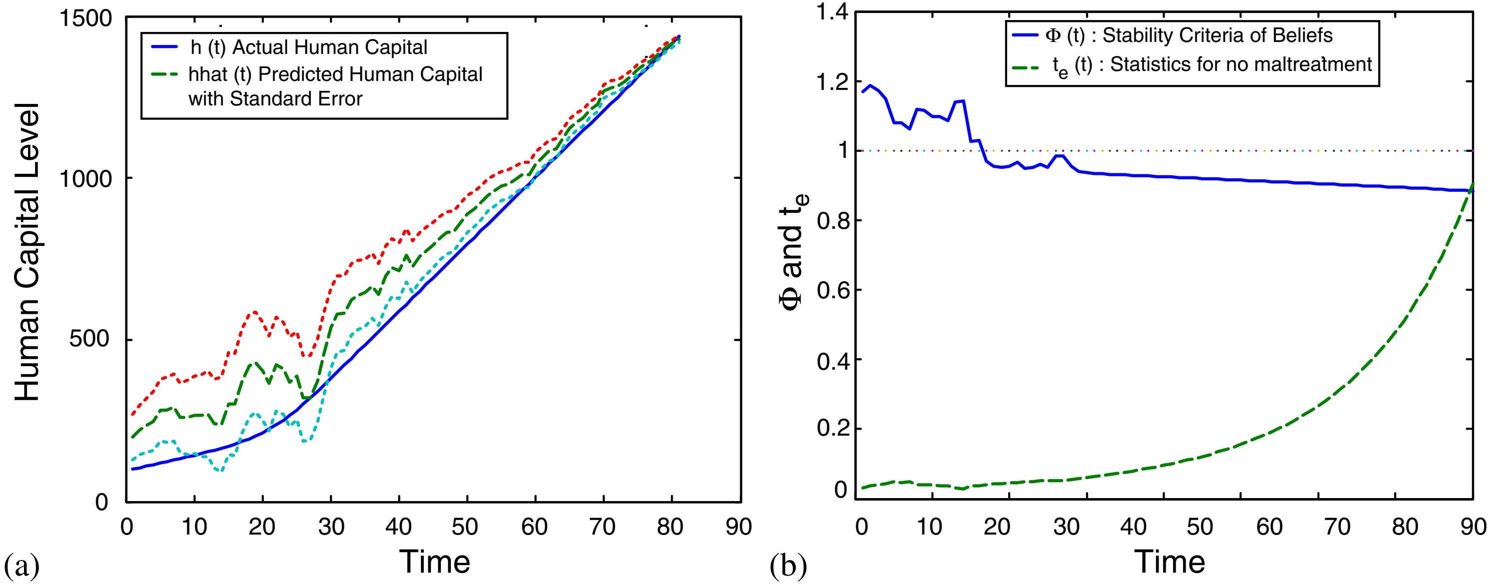
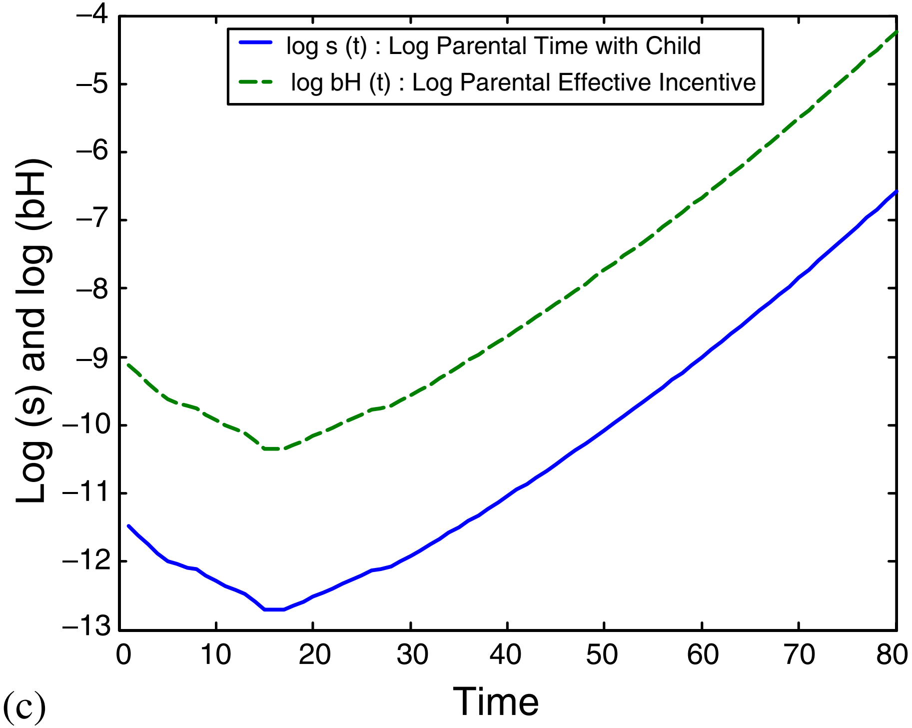
Figure 3: A simulated dynamics of parental interactions, child development, and parental beliefs



Notes: (a) Dynamics of actual and predicted child human capital development; (b) Dynamics of Φ_t (stability criteria of parental beliefs) and t_e, (non-punishment statistic); and (c) Dynamics of parental interactions.

Figure 3: A simulated dynamics of parental interactions, child development, and parental beliefs, Cont.



(c)

Notes: (a) Dynamics of actual and predicted child human capital development; (b) Dynamics of Φ_t (stability criteria of parental beliefs) and t_e, (non-punishment statistic); and (c) Dynamics of parental interactions.

- The endogeneity of and the parental uncertainty over the child's time-preference play key roles in the emergence of child maltreatment.
- Since a child's effort depends on his or her own time-preference, the transmission of uncertainty from the child's preferences to effort and human capital development implies that parental uncertainty regarding the child's hidden characteristics may be magnified over time.
- In the absence of endogenous discounting, there is no uncertainty over the child's effort, new information on the child's performance always improves the knowledge of the child's human capital, and parental expectations and interactions are stable and normal under the standard production process.

(c) *Numerical simulations.*

- In order to give a visual illustration of the converging and diverging beliefs, Fig. 3 shows results from a numerical simulation of the solution of child human capital development and parental behavior and expectations.
- Fig. 3(a) shows the equilibrium development of h_t and $\hat{h}_t$ over time with the parental estimation error, σ_h, when the initial expectation is higher than the true value, that is $h_1 < \hat{h}_1$ at period 1.
- Until about period 20, the expectation process exhibits an unstable path, the parental beliefs tend to be persistently positively biased, and the child's human capital develops slowly.
- As the child's human capital develops beyond period 20, both the estimation error and the parental expectation become converging, and the speed of human capital development increases.

- The driving forces of this dynamics are depicted in Fig. 3(b).
- Each solid curve and dotted curve shows the evolution of Φ_t and t_e, the stability criteria and the non-punishment statistic, respectively.
- It is confirmed that, as predicted by the lemma, if and only if Φ_t is greater than one, the expectation process diverges, and that under such a circumstance, t_e may decrease, i.e., the parental utility transfer may become lower and punitive over time.

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- Fig. 3(c) illustrates the dynamics of parental behavior, b_t and s_t.
- As predicted in Section 4.3, when parental uncertainty increases over time, the levels of b_t and s_t decrease.
- However, once the parental expectation enters into a stable cycle, the levels of parental incentive and time to be spent with the child increase over time, and the development of child human capital takes off as shown in Fig. 3(a).
- Once the development enters this phase, as suggested by the evolution of t_e in Fig. 3(b), the risk of being maltreated becomes extremely low.

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5.2. The characteristics of a parent and a child at risk of abuse

- When

$$\Phi = \frac{F'(\hat{h}^u)^2}{A'(\hat{h}^u)^2(\sigma_h^2/\sigma_v^2)+1} = \frac{F'(\hat{h}^u)^2}{A'(\hat{h}^u)^2(s^* \cdot \sigma_h^2/K)+1} < 1,$$

- Then the parent's expectation process converges and the probability of punitive interactions decreases over time.
- Therefore, a family with parameters that reduce Φ_t is unlikely to fall into an unstable equilibrium path, or 'a cycle of abuse.'
- Since the parameters that have a positive effect on s^* will have such a property, most of the results in the following proposition are straightforward from the discussion in Section 4.3.

Proposition 3

All else equal, a family is less likely to follow a path with persistently punitive interventions if (i) the productivity of time in the child's human capital (ρ) is large, (ii) the wage rate (π) is low, (iii) the parent is good at observing the child (small K), (iv) the child is old ($T - t$ is small), or (v) the child's natural level of effort is high (large $\underline{a}$).

Proof.

(i), (ii), (iv), and (v) are clear from the expression of s_t^* in (14) in Appendix C. To prove (iii), notice that

$$\Phi_t = \frac{F'(\hat{h}_{t|t})^2}{A(\hat{h}_{t|t})^2(s_t^* \cdot \sigma_{ht}^2/K) + 1} = \frac{[(1-\delta) + \phi^2\psi D_t'(\hat{h}_{t|t})]^2}{1 + [1 + \phi\psi D_t'(\hat{h}_{t|t})]^2 \frac{\underline{a} + \phi\psi D_t(H) - KR\alpha Q_t}{KQ_t(R\alpha - \psi/\sigma_{ht}^2)}}$$

Φ_t tends to be small, and the parent's equilibrium process is likely to be stable for a small K. □

- Among the above results, (i), (ii), (iv), and (v) would encourage the parent to stay with the child longer, and thereby work to reduce the variance of the observation error (σ_v^2).
- (iii) would make the parent choose to spend less time with the child, but knowledge about the child would improve due to the superior observation ability.
- A couple of remarks follow.
- First, some of the above results may seem counterfactual.
- For example, we usually observe that parents tend to spend less time with older children in normal families.
- However, if we interpret schooling as an extension of education at home, the implication of (iv) is not inconsistent with the fact that older children spend a longer time at school under teachers' supervision.

- As is discussed in Section 4.2, our prediction is driven by the fact that, in the absence of income effect, a higher wage rate generates a substitution away from being with children and toward paid work, resulting in increased parental uncertainty.

- The uncertainty about the child's human capital will tend to remain large for a long time, and the parent will tend to maintain a negatively biased view of the child.
- Third, as we have seen in the previous paragraph, the stabilizing effect is not the only way through which these characteristics affect the risk of child abuse.
- For example, a larger $\underline{a}$ not only has a stabilizing effect on the parent's expectation process but also a positive effect on the child's development (both s and a become larger).
- Obviously such a child is at less risk of facing abuse because he can reach a high level of human capital faster.

- Finally, it may be surprising to note that the parent's level of human capital (H) has theoretically mixed effects.
- However, if H is larger, (i) the speed of a child's development is faster because the parent's teaching is more effective, (ii) K is likely to be small (less observation errors), and (iii) such a family tends to have fewer children (more time to spend with each child), then it would be reasonable to think that the child's development tends to be faster and that there will be less risk of a divergent parental belief and child abuse.
- Moreover, it must be noted that the effects of some other key parameters are, surprisingly, uncertain.

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- For instance, the effect of the degree of altruism, a, is unclear because, as we discussed in Section 4.3, a more altruistic parent does not necessarily choose to spend more time with the child.
- Parental love may lead to a more relaxed level of control (smaller b) so that the parent does not choose to improve the accuracy of observations by spending large amounts of time with the child.
- As a result, the parent's expectation process can become divergent.
- Our model provides several insights into the prevention of child maltreatment.
- First, although the parent always maximizes her subjective expected utility, if we could exogenously remove the abnormal bias and error in the parent's expectation of the child with low costs, a family's welfare could be improved.

- This is particularly important when an unstable relationship lasts long relative to the finite length of the parent-child relationship.
- Obvious intervention methods implied would include (i) providing the parent with the correct knowledge about child development or a shock to change her belief or (ii) providing monitoring and service (child care) to reduce the monitoring error (σ_v^2).
- These have been recently mentioned by several researchers in the literature.
- For example, Wolfe (1991) wrote, "Many [young, socially disadvantaged parents] lack knowledge about infant development and therefore have inappropriate expectations for their infants' behavior..." (p. 90) and advocated a training program for parents at risk of abuse that includes "setting reasonable expectations for children's emotional and behavioral development" (p. 120).

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6. Summary

- Shed light on the role of expectations in the interaction between a parent and a child, which may lead to child maltreatment in equilibrium.
- Construct a model that explains the dynamic interrelationship between the development of the child's human capital, endogenous rate of time preference, and the parent's interactions and beliefs.
- The equilibrium dynamics are state-dependent, because a child's chosen effort both affects and is affected by the development of the child's characteristics.

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- The model predicts that if the initial uncertainty regarding the child's human capital is large, the parent's expectation process of the child's human capital might diverge and be negatively biased due to the endogenous nature of the parent's expectation formation process.
- It is shown, both analytically and numerically, that the divergent beliefs may lead to the parent's unrealistically high expectations and persistently punitive interactions with the child, which explain the emergence of child maltreatment or abuse.
- The model also identifies the characteristics of families at risk of diverging beliefs and persistent negative interactions that can be empirically testable.

- In this section we simulate the model that Akabayashi (2006) develops in his paper *An Equilibrium Model of Child Maltreatment*.
- The model is constructed to rationalize the way in which parents treat their children in terms of both the amount of time they choose to spend with them and the amount of punishment or praise they choose to provide them.
- These decisions affect the children's human capital development.
- In particular, the author shows that persistent punishment and abuse (i.e. child maltreatment) is a possible equilibrium of the model.

- The layout of the model is the following:
- The author considers a single parent household with a single child.
- The child's human capital development in each period follows a linear low of motion:

$$h_{\tau+1} = (1 - \delta)h_\tau + \varphi s_\tau H^\gamma + \phi a_\tau \tag{14}$$

for $\tau = 1, \ldots, T$.

- δ is a human capital depreciation parameter.
- φ, ϕ, γ are technology parameters,
- H is the given and fixed parent's human capital.
- $s_\tau \in [0,1]$ and a_τ are endogenous variables.
- The former is the normalized time that parents spent with their children and the latter is the effort that the children makes to acquire human capital.

- The parents are not able to observe either the true level of human capital of the children or the effort they make.
- Instead, at each time τ, they observe an outcome variable, y_τ, which evolves according to the following linear rule:

$$y_\tau = h_\tau + a_\tau + \nu_\tau \tag{15}$$

where $\nu_\tau \sim N(0, \sigma^2_{\nu\tau})$ for $\tau = 1, \ldots, T$.

- The author lets $\sigma^2_{\nu\tau} \equiv \frac{K}{s_\tau}$ because more time spent with the child reduces the uncertainty of the observation error.
- The author uses the Extended Kalman Filter to model how the parent updates his beliefs: (14) is interpreted as the state equation and (15) as the observation equation.
- Moreover, the author postulates a linear incentive schedule, through which the parent motivates his child to increase her effort and, therefore, drive up her human capital.
- The linear schedule is induced through a service that the parent provides to the child and which the kid values in her utility function.

- The service, d_τ, is as follows:

$$d_\tau = (s_\tau + b_\tau \mathbb{E}[a_\tau | I_\tau]) \, H^\gamma \tag{16}$$

for $\tau = 1, \ldots, T$ where b_τ is defined as the slope of the incentive schedule and is an endogenous decision of the parent.

- The incentive slope measures how strict the parental discipline is: with a higher b_τ the parent's service is more responsive to his expectation of the child's efforts.
- $I_t \equiv \{y_t, \ldots y_1\}$ is the information set at t.
- To build intuition on how the scheme works, we use (15) to obtain an alternative expression for (16).
- Particularly: $\mathbb{E}[a_\tau | I_\tau] = \mathbb{E}[y_\tau - h_\tau - v_\tau | I_\tau] = y_\tau - \mathbb{E}[h_\tau - v_\tau | I_\tau]$ $= h_\tau + a_\tau + \nu_\tau - \hat{h}_\tau$.

- Thus

$$d_\tau = (s_\tau + b_\tau(h_\tau + a_\tau + \nu_\tau - \hat{h}_\tau))H^\gamma \tag{17}$$

for $\tau = 1, \ldots, T$.

- The author shows that in equilibrium b_τ is positive.
- Now, suppose that a parent picks a relatively high b_τ and that the observation of his daughter's performance, y_τ, deviates from his human capital forecast, $\hat{h}_\tau$, by a lot such that $y_\tau - \hat{h}_\tau$ is very negative.
- Then, d_τ is relatively low and the child suffers from a low service, which the author interprets as abuse (instead, a high value of service is interpreted as a praise).

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- The child's utility function is

$$\max_{\{a_\tau\}_{\tau=1}^{T}} \mathbb{E}\left[u\left(\sum_{\tau=t}^{T}\frac{1}{1+\rho_{ct}}^{\tau-t}(d_\tau - v(a_\tau)) + \frac{1}{1+\rho_{ct}}^{T-t+1} Bh_{T+1}\right)|I_{t-1}\right] \tag{18}$$

subject to (14) and (16) and given $\{s_\tau, b_\tau\}$.

- B is a positive constant.
- v is a strictly convex function ($v(a) \equiv \frac{1}{2\psi}(a-\underline{a})^2$) that measures the cost of effort.
- The discount rate, p_{ct}, is a decreasing function of the human capital of the child: a kid with a higher level of human capital is less myopic and more patient.

- The parent's utility function is

$$\max_{\{b_\tau, s_\tau\}_{\tau=1}^{T}} \mathbb{E}\left[U\left(\sum_{\tau=t}^{T} \frac{1}{1+\rho_{pt}}^{\tau-t} [c_\tau + \alpha u(\cdot, a_\tau))] \right) | I_{t-1} \right] \quad (19)$$

subject to $c_\tau = \pi(1 - s_\tau)H^\gamma$, (14), (15), (16) and to the child's optimal decision rule, and where c_τ is consumption at τ, and π efficiency unit wage.

- The Nash Equilibrium of the model can be solved easily.
- In particular, the child decides the optimal level of effort to solve her utility maximization problem by taking parent's choices, $\{b_\tau, s_\tau\}_{\tau=1}^{T}$, as given.
- Then the parent solves his utility maximization problem to choose the optimal level of time and incentive slope by taking the optimal decision rule of the kid as given.

- The equilibrium of the model can be summarized through the following equations:

$$b^* H^\gamma = \frac{1}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \tag{20}$$

$$s^* = \frac{1}{Q_t(R\alpha\sigma_h^2 - \psi)}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \tag{21}$$

$$a^* = \underline{a} + \phi\psi D_t(h) + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \tag{22}$$

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- Where

$$D_t(h) = \left(\frac{1}{1+\rho_{pt}(h)}\right)^{T-t-1}(1-\delta)^{T-t}B \tag{23}$$

$$Q_t = \left(\frac{2H^\gamma}{R\alpha^2 K}[\pi - \alpha - \alpha\varphi D_t(H)]\right)^{\frac{1}{2}} \tag{24}$$

- K, R are constants and we neglect the time indices.
- The equilibrium values of b^*, s^*, a^* are plugged into (14),(15) to obtain the equilibrium human capital development dynamics and observation paths.
- During the whole process, the parent forms and updates expectations about the child's human capital as follows.

- First, when the parent receives a new observation, y, he updates his contemporaneous belief about the child's current level of human capital by taking an average of the previous belief and the new observation weighted by the degree of uncertainty:

$$\hat{h}^u = \hat{h} + \frac{A'(\hat{h})\sigma_h^2}{A'(\hat{h})^2\sigma_h^2 + \sigma_v^2}(y - A(\hat{h}) - C(\sigma_h^2)) \tag{25}$$

(26)

- Then, the parent uses the updated belief, $\hat{h}^u$, and the human capital formation rule to forecast the child's human capital level in next period:

$$\hat{h}' = F(\hat{h}^u) + G(\sigma_h^2) \tag{27}$$

(28)

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- Finally, the parent updates the uncertainty regarding the child's human capital:

$$(\sigma_h^2)' = \Phi\sigma_h^2 \tag{29}$$

Where

$$A(\hat{h}) = \hat{h} + \phi\psi D_t(\hat{h}) \tag{30}$$

$$A'(\hat{h}) = 1 + \phi\psi D_t'(\hat{h}) \tag{31}$$

$$C(\sigma_h^2) = \underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \tag{32}$$

$$F(\hat{h}^u) = (1-\delta)\hat{h}^u + \phi^2\psi D_t(\hat{h}^u) \tag{33}$$

$$G(\sigma_h^2) = \phi[\underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t)] + \psi s^* H^\gamma \tag{34}$$

$$\Phi = \frac{F'(\hat{h}^u)^2\sigma_v^2}{A'(\hat{h}^u)^2\sigma_h^2 + \sigma_v^2} \tag{35}$$

$$F'(\hat{h}^u) = (1-\delta) + \phi^2\psi D_t'(\hat{h}^u) \tag{36}$$

Simulation Parameterization

- We use the following parameters and functional forms to simulate the model:
  - Time Horizon: $T = 80$.
  - Technology parameters: $\phi = .7, \varphi = .01, \delta = .001, \gamma = 0.5$
  - Preference parameters: $\psi =, 7, \underline{a} = 7, K = 1.5, R = 2,$ $\alpha = .9, B = 50$
  - Human Capital Parameters: $h_1 = 100, H = 40000$.
  - Filter Initial Values: $\hat{h_1} = 200, \sigma^2_{h1} = 5000$.
  - Other parameters: $K = 1.5, \pi = 2$.
  - Discount Rates: $\rho_i(h) = \exp(-.02 * h)$ for $i = p, c$.

Simulation Results

- We plot the results of the simulation for the three choice variables of the model as well as for the equilibrium paths of the realized and the forecasted human capital.
- We also plot Φ because, as explained below, it is fundamental on how the equilibrium beliefs evolve.
- Our simulation exercise successfully replicates Figure 3 in Akabayashi (2006).
- The purpose of this simulation exercise is to show the channel through which an unusually high expectation on a child's human capital may result in child maltreatment.

- The channel could be explained clearly by considering the three different stages of the process:
    1. The initial period.
    2. The unstable equilibrium stage.
    3. The stable equilibrium stage.

- In the initial period, the parent starts with an unreasonably high expectation on the child's human capital, which is 200 in the simulation exercise while the true level of human capital is 100.
- Also, the parent is, initially, very uncertain about the child's initial endowment: σ_{h1}^2 is very high (See Figure 6).
- Thus, the parent, who is risk averse, chooses not to be very strict towards the child's observed performance in the initial period (i.e., a low b_1) because the likelihood of an inappropriate punishment/praise due to a wrong expectation on the child's human capital is high (See Figure 4).

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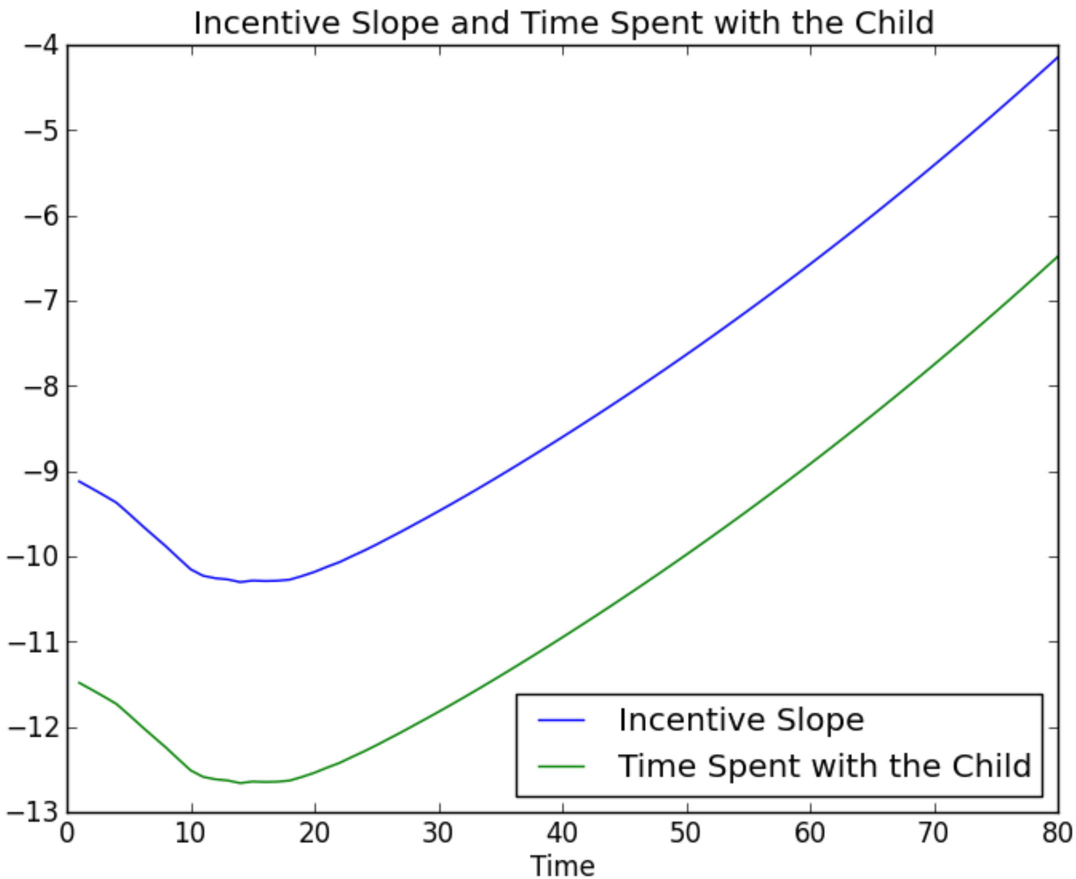
- Accordingly, the parent chooses to spend a small amount of time with the child (i.e., a low s_1), since an important incentive for spending time with the child is to gain a better measurement of the child's human capital and thus to conduct punishment/praise, whereas a lower b reduces the marginal benefit of gaining a better measurement.
- The child's effort in the first period is also low for two reasons.
- First, given that the parent does not spend too much time with the child, it is more likely for her to hide her "laziness".
- Second, with relatively low initial human capital, the child is relatively myopic toward her future utility, which depends on her level of human capital in the last period.
- Thus, she is less motivated to accumulate human capital today.

- Φ is a fundamental driver of the model because its magnitude determines if the parent's belief on the child's level of human capital is stable and converges to the true value.
- According to (29), the uncertainty of the parent's expectation increases over time when $\Phi > 1$ and decreases over time when $\Phi < 1$.
- As (36) illustrates, Φ is increasing in the uncertainty of the measurement, σ_ν^2, and decreasing in the uncertainty of the parent's expectation, σ_h^2.
- It is useful to keep in mind that both σ_ν^2 and σ_h^2 affect the level of Φ, which in turn determines the stability and convergence of parent's belief.

- The combination of the simulation parameters in this exercise is such that $\Phi > 1$ in period one.
- Therefore, the parent becomes less certain with respect to the child's human capital over time, relative to how certain he is about the outcome measure.
- This is the stage with diverging parent's belief.
- Due to his risk aversion, the parent chooses to decrease the incentive slope (he chooses to discipline her less).

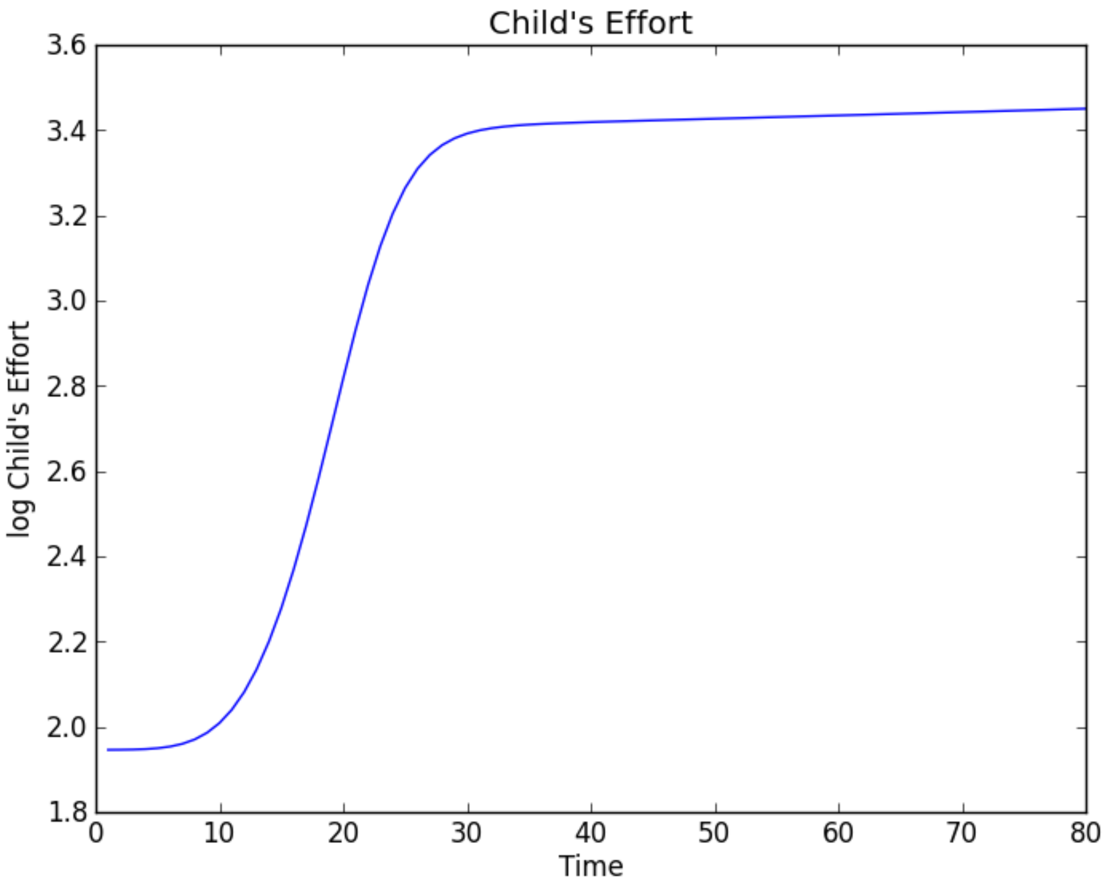
- As explained above, a lower level of b reduces the marginal benefit of spending time with the child.
- Thus s decreases together with b in Figure 4.

Figure 4: log Incentive Slope and log Time Spent with the Child



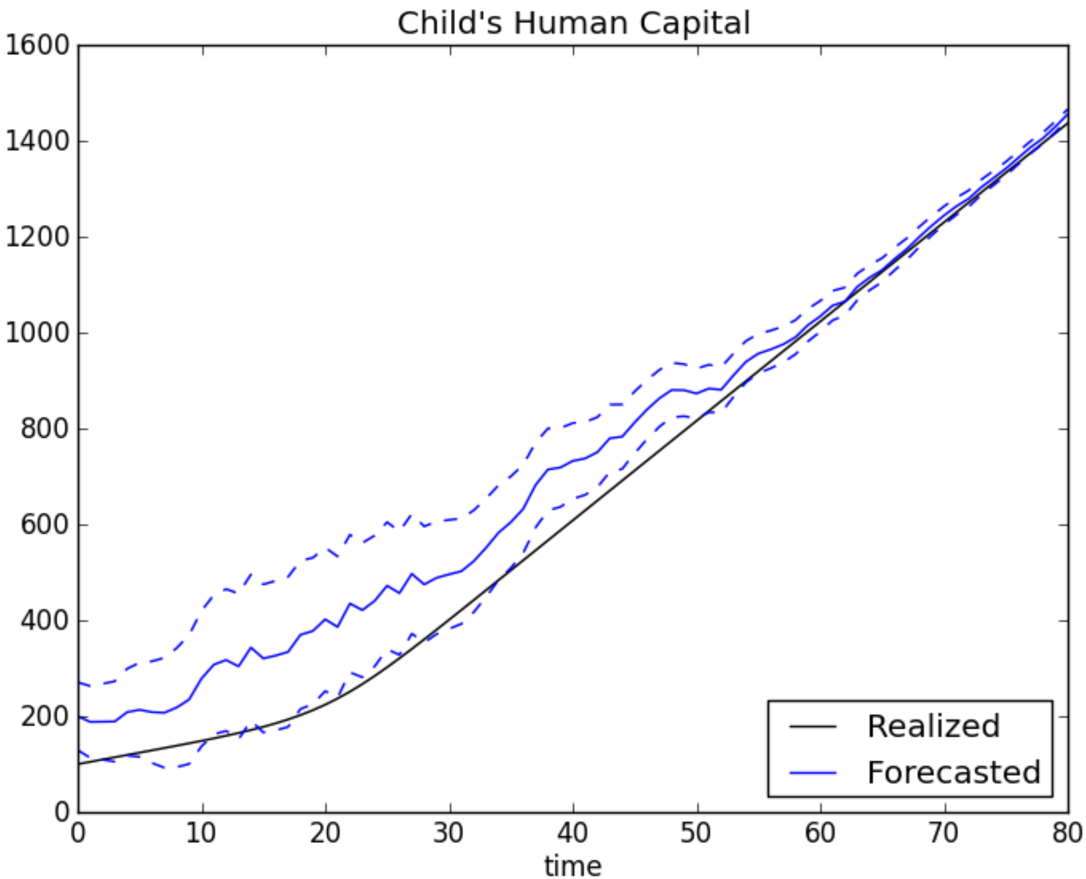
- The effort of the child is very stable in that period because, on the one hand, the child's human capital slightly increases and thus she is slightly less myopic towards her future utility, and, on the other hand, the higher measurement uncertainty, which is caused by the smaller amount of time the parent spends with her, gives the child more incentives to be "lazy" (See Figure 5).

Figure 5: Child's Effort



- Finally, as Figure 6 shows, the child's human capital is slowly accumulated in this stage of unstable and diverging parent's belief.

Figure 6: Human Capital



- Although both σ_h^2 and σ_v^2 increase in the previous stage, the former increases faster than the latter, which makes Φ less than one at after some time.
- This is the critical point at which the parent's belief enters into the stage of stability and convergence.
- When $\Phi < 1$, the uncertainty in the parent's belief decreases over time, and, thus, the parent increases the discipline, through b, and the time spent with the child, s, by following exactly the same argument as mentioned before (See Figure 4).
- Since the outcome uncertainty is reduced due to a higher s, the child increases her effort.
- Figure 5 shows that a sharply increases when the parent increases s, while the level of chosen effort is maintained at a high level after several periods because the child matures.

- Put differently, she is forward looking enough and achieves the amount of effort that the parent expects her to make.
- The child's human capital is accumulated very fast in the stage of converging parent's belief, as Figure 6 shows.
- The parent's belief is persistently higher than the true level of the child's human capital in the stage of divergence and slowly converges from above to the realized dynamics afterwards.
- This is important because a positive bias of the estimation on the child's human capital leads to a negative bias of the estimation on the child's effort.
- This results in a low level of service to the child, which is interpreted as child maltreatment in the paper.
- The simulation exercise shows that a combination of parameters makes child maltreatment a possible equilibrium.

Appendix E:
Proof of Proposition 1

- The stability condition in the lemma is

$$\Phi_t = \frac{F'(\hat{h}_{t|t})^2}{A'(\hat{h}_{t|t})^2(\sigma_{ht}^2/\sigma_{\nu t}^2) + 1} < 1.$$

- Since $F' < 1$ automatically implies the stability, for a divergence to occur, it is necessary to have $F' > 1$.

- Therefore, the divergence requires the following inequality to hold given a belief $(\hat{h}_{t|t}, \sigma^2_{ht})$:

$$\sigma^2_{\nu t}/\sigma^2_{ht} > \frac{[1 + \phi\psi D'_t(\hat{h}_{t|t})]^2}{[1 - \delta + \phi^2\psi D'_t(\hat{h}_{t|t})]^2 - 1} \equiv Z_t(\hat{h}_{t|t}).$$

- The question is whether there exists a combination of parameters with which there is a non-empty set of beliefs that can cause the expectation process to diverge.

- Let the two real roots of the equation, $F'(h) = (1-\delta) + \phi^2 \psi D_t'(h) = 1$, be h_1^* and h_2^*, which exist if $\rho_{ct} = \exp(-\eta h_t)$ with $\eta > 0$.
- It can be verified that for the range (h_1^*, h_2^*), $Z_t(h)$ is continuous, its minimum is attained at $h^{**} \in (h_1^*, h_2^*)$, and the value of $Z_t(h^{**})$ is $(1-(1-\delta-\phi)^2)/\phi^2$.
- Clearly $\lim_{h \to h_1^*+0} Z_t(h) \to +\infty$ and $\lim_{h \to h_2^*-0} Z_t(h) \to +\infty$.

- Therefore, for the above inequality to hold, it is necessary that the following inequality holds for a given value of σ_h^2 and the given parameter values:

$$(K/s_t^*)/\sigma_{ht}^2 = \sigma_{\nu t}^2/\sigma_{ht}^2 > Z_t(h^{**}) = (1-(1-\delta-\phi)^2)/\phi^2. \quad (37)$$

- If (37) holds, since both sides of (37) are independent of h and $Z_t(h)$ is continuous in h for $h \in (h_1^*, h_2^*)$, there exists a range of $\hat{h}_{t|t}$, $R(\sigma_h^2) = (h_1, h_2)$, such that any value of $\hat{h}_{t|t}$ in $R(\sigma_h^2)$ will cause the beliefs to diverge.

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- It can be verified that

$$\sigma_{\nu t}^2/\sigma_{ht}^2 = \frac{KQ_t(R\alpha\sigma_{ht}^2 - \psi)}{\underline{a} + \phi\psi D_t(H) - KR\alpha Q_t}$$

- is increasing in σ_h^2 since an increase in uncertainty of human capital discourages the monitoring of the child.
- Therefore, if the given set of parameter values satisfies

$$\lim_{\sigma_{ht}^2\to\infty}(\sigma_{\nu t}^2/\sigma_{ht}^2) = \frac{KQ_tR\alpha}{\underline{a} + \phi\psi D_t(H) - KR\alpha Q_t} > (1-(1-\delta-\phi)^2)/\phi^2,$$

- we can find a value of σ_h^2 and the associated range of $\hat{h}_{t|t}$ for the divergence.

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- It is possible because the LHS of the inequality infinitely increases in K as long as the denominator is positive (the requirement for s^* to be positive) for any combination of the other parameters.
- This proves (i) and (iii).
- Using the result of the lemma, (ii) follows naturally.

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