The Normal Generalized Roy Model

James J. Heckman
University of Chicago

Extract from: Building Bridges Between Structural and Program Evaluation Approaches to Evaluating Policy
James Heckman (JEL 2010)

Econ 312, Spring 2021

Example: The Generalized Roy Model for the Normal Case

$$
\begin{aligned}
Y_1 &= \mu_1(X) + U_1 \\
Y_0 &= \mu_0(X) + U_0 \\
C &= \mu_C(Z) + U_C \\
\text{Net Benefit: } I &= Y_1 - Y_0 - C \\
I &= \underbrace{\mu_1(X) - \mu_0(X) - \mu_C(Z)}_{\mu_D(Z)} + \underbrace{U_1 - U_0 - U_C}_{-V}
\end{aligned}
$$

$$
\begin{aligned}
(U_0, U_1, U_C) &\perp\!\!\!\perp (X, Z) \\
E(U_0, U_1, U_C) &= (0, 0, 0) \\
V &\perp\!\!\!\perp (X, Z)
\end{aligned}
$$

- Assume normally distributed errors.
- Assume Z contains X but may contain other variables (exclusions)

$$\text{Observed } Y: \qquad Y = DY_1 + (1-D)Y_0$$
$$D = 1(I \geq 0) = 1(\mu_D(Z) \geq V)$$

- Assume $V \sim N(0, \sigma_V^2)$

- Propensity Score:

$$\Pr(D = 1 \mid Z = z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$E(Y \mid D = 1, X = x, Z = z) = \mu_1(X) + \underbrace{E(U_1 \mid \mu_D(z) \geq V)}_{K_1(P(z))}$$

because $(X, Z) \perp\!\!\!\perp (U_1, V)$.

- Under normality we obtain

$$E\left(U_1 \middle| \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = \frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

- Why?

$$U_1 = \text{Cov}\left(U_1, \frac{V}{\sigma_V}\right)\frac{V}{\sigma_V} + \varepsilon_1$$

$$\varepsilon_1 \perp\!\!\!\perp V$$

$$E\left(\frac{V}{\sigma_V} \mid \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = \frac{\int\limits_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} t \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt}{\int\limits_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt} = \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$= \frac{\frac{-1}{\sqrt{2\pi}} e^{\left(-\frac{1}{2}\right)\left(\frac{\mu_D(z)}{\sigma_V}\right)^2}}{\Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)} = \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = \frac{-\phi\left(\frac{\mu_D(z)}{\sigma_V}\right)}{\Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)}$$

- Notice

$$\lim_{\mu_D(z)\to\infty} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = 0$$

$$\lim_{\mu_D(z)\to-\infty} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right) = -\infty$$

- Propensity score:

$$P(z) = \Pr(D = 1 \mid Z = z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\therefore \left(\frac{\mu_D(z)}{\sigma_V}\right) = \Phi^{-1}\left(\Pr(D = 1 \mid Z = z)\right)$$

- Thus we can replace $\frac{\mu_D(z)}{\sigma_V}$ with a known function of $P(z)$

- Notice that because $(X, Z) \perp\!\!\!\perp (U, V)$, Z enters the model (conditional on X) only through $P(Z)$.
- This is called *index sufficiency.*

- Put all of these results together to obtain

$$E\left(Y \mid D=1, X=x, Z=z\right) = \mu_1(x) + \left(\frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right)\tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$= E\left(Y_1 \mid D=1, X=x, Z=z\right) = \mu_1(x) + \left(\frac{\mathrm{Cov}(U_1, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right)\tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\tilde{\lambda}(z) = E\left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} < \frac{\mu_D(z)}{\sigma_V}\right) < 0$$

$$\lambda(z) = E\left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} \geq \frac{\mu_D(z)}{\sigma_V}\right) > 0$$

$$E\left(Y \mid D=0, X=x, Z=z\right) = \mu_0(x) + \left(\frac{\mathrm{Cov}(U_0, \frac{V}{\sigma_V})}{\mathrm{Var}(\frac{V}{\sigma_V})}\right)\lambda\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$\mathrm{Var}\left(\frac{V}{\sigma_V}\right) = 1$$

$$\frac{V}{\sigma_V} = -\frac{(U_1 - U_0 - U_C)}{\sigma_V}$$

$$\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) = -\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) + \text{Cov}\left(U_0, \frac{V}{\sigma_V}\right) + \text{Cov}\left(U_C, \frac{V}{\sigma_V}\right)$$

In Roy model case ($U_C = 0$),

$$\begin{aligned}\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) &= -\text{Cov}\left(U_1, \frac{U_1 - U_0}{\sigma_V}\right) \\ &= -\frac{\text{Cov}\left(U_1 - U_0, U_1\right)}{\sqrt{\text{Var}(U_1 - U_0)}}\end{aligned}$$

- We can identify $\mu_1(x), \mu_0(x)$
- From Discrete Choice model we can identify

$$\frac{\mu_D(z)}{\sigma_V} = \frac{\mu_1(x) - \mu_0(x) - \mu_C(z)}{\sigma_V}$$

- If we have a regressor in X that does not affect $\mu_C(z)$ (say regressor x_j, so $\frac{\partial \mu_C(z)}{\partial x_j} = 0$), we can identify σ_V and $\mu_C(z)$.
- $\therefore$ We can identify the net benefit function and the cost function up to scale.
- $\therefore$ We can compute *ex-ante* subjective net gains up to scale.

- Method generalizes: Don't need normality
- Need "Large Support" assumption to identify ATE and TT
-

$$E\left(Y \mid D=1, X=x, Z=z\right) = \mu_1(x) + \overbrace{K_1(P(z))}^{\text{control function}}$$
$$E\left(Y \mid D=0, X=x, Z=z\right) = \mu_0(x) + \underbrace{K_0(P(z))}_{\text{control function}}$$

$$\lim_{P(z)\to 1} E\left(Y \mid D=1, X=x, Z=z\right) = \mu_1(x)$$
$$\lim_{P(z)\to 0} E\left(Y \mid D=0, X=x, Z=z\right) = \mu_0(x)$$

- If we have this condition satisfied, we can identify ATE

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- ATE is defined in a limit set. This is true for any model with selection on unobservables (IV; selection models)

- What about treatment on the treated?

$$E(Y_1 - Y_0 \mid D = 1, X = x, Z = z)$$

1. From the data, we observe

$$E(Y_1 \mid D = 1, X = x, Z = z)$$

2. Can also create it from the model
3. $E(Y_0 \mid D = 1, X = x, Z = z)$ is a counterfactual

We know

$$E(Y_0 \mid D = 0, X = x, Z = z) = \mu_0(x) + \text{Cov}\left(U_0, \frac{V}{\sigma_V}\right)\lambda\left(\frac{\mu_D(Z)}{\sigma_V}\right)$$

(this is data)

(d) We seek

$$E(Y_0 \mid D = 1, X = x, Z = z) = \mu_0(x) + \text{Cov}\left(U_0, \frac{V}{\sigma_V}\right) \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

- But under normality, we know $\text{Cov}\left(U_0, \frac{V}{\sigma_V}\right)$
- We know $\frac{\mu_D(Z)}{\sigma_V}$
- $\tilde{\lambda}(\cdot)$ is a known function.
- Can form $\tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$ and can construct counterfactual.

- More generally, without normality (but with $(X, Z) \perp\!\!\!\perp (U, V)$)

$$E(Y_1 \mid D = 1, X, Z) = E(Y \mid D = 1, X = x, Z = z) = \mu_1(x) + K_1(P(z))$$

$$E(Y_0 \mid D = 0, X, Z) = E(Y \mid D = 0, X = x, Z = z) = \mu_0(x) + \tilde{K}_0(P(z))$$

where $K_1(P(z)) = E(U_1 \mid D = 1, X = x, Z = z) = E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V}\right)$

$$\tilde{K}_1(P(z)) = E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} \leq \frac{V}{\sigma_V}\right)$$

$$\tilde{K}_0(P(z)) = E\left(U_0 \mid \frac{\mu_D(z)}{\sigma_V} \leq \frac{V}{\sigma_V}\right)$$

- Use the transformation

$$\frac{F_V}{\sigma_V}\left(\frac{\mu_D(z)}{\sigma_V}\right) = P(z)$$

$$\frac{F_V}{\sigma_V}\left(\frac{V}{\sigma_V}\right) = U_D \qquad \text{(a uniform random variable)}$$

$$D = 1\left(\frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = 1\left(P(z) \geq U_D\right)$$

$$K_1(P(z)) = E(U_1 \mid P(z) > U_D)$$

$$K_1(P(z))P(z) + \tilde{K}_1(P(z))(1 - P(z)) = 0$$

$$\therefore \text{ we can construct } \tilde{K}_1(P(z))$$

- Symmetrically

$$\tilde{K}_0(P(z)) = E(U_0 \mid P(z) \leq U_D)$$
$$K_0(P(z)) = E(U_0 \mid P(z) > U_D)$$
$$(1 - P(z))\tilde{K}_0(P(z)) + P(z)K_0(P(z)) = 0$$

- $\therefore$ If we have "identification at infinity," we can construct

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- We can construct TT

$$\begin{aligned} &E(Y_1 - Y_0 \mid D = 1, X = x, Z = z) = \\ &\quad = \underbrace{[\mu_1(x) + K_1(P(z))]}_{\text{factual}} - \underbrace{[\mu_0(x) + K_0(P(z))]}_{\text{counterfactual}} \end{aligned}$$

- We can form $\mu_1(x) + K_1(P(z))$ from data
- We get $\mu_0(x)$ from limit set $P(z) \to 0$ identifies $\mu_0(x)$
- We can form $K_0(P(z)) = -\tilde{K}_0(P(z))\frac{P(z)}{1-P(z)}$
- $\therefore$ Can construct the desired counterfactual mean.

- Notice how we can get Effect of Treatment for People at the Margin:

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z)$$

- Under normality we have (as a result of independence)

$$\begin{aligned} & E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) \\ & \quad = \mu_1(x) - \mu_0(x) + E\left(U_1 - U_0 \mid \frac{\mu_D(z)}{\sigma_V} = \frac{V}{\sigma_V}, X = x, Z = z\right) \\ & \quad = \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \frac{\mu_D(z)}{\sigma_V} \end{aligned}$$

In the Roy model case where $U_C = 0$ but $\mu_C(z) \neq 0$

$$\begin{aligned} & = \mu_1(x) - \mu_0(x) - \sigma_V \left(\frac{\mu_D(z)}{\sigma_V}\right) \\ & = \mu_1(x) - \mu_0(x) - \mu_D(z) \\ & = \mu_C(z) \end{aligned}$$

(marginal gain = marginal cost)

- MTE for Normal Model:

$$E(Y_1 - Y_0 \mid V = v, X = x, Z = z) =$$
$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) v$$

- Effect of Treatment for People at the Margin picks $v = \frac{\mu_D(z)}{\sigma_V}$

- Remember

$$\frac{V}{\sigma_V} = -\frac{\{U_1 - U_0 - U_C\}}{\sigma_V}$$

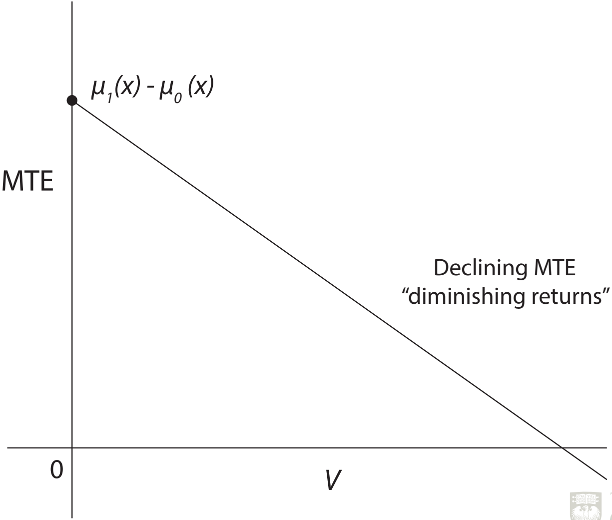
$$\text{Cov}(U_1 - U_0, V)$$

$$= -\frac{(\text{Var}(U_1 - U_0)) + \text{Cov}(U_1 - U_0, U_C)}{\sigma_V}$$

- Roy Model: $U_C \equiv 0$

Meritocratic Society (Becker, 1967)

$$\text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \leq 0$$



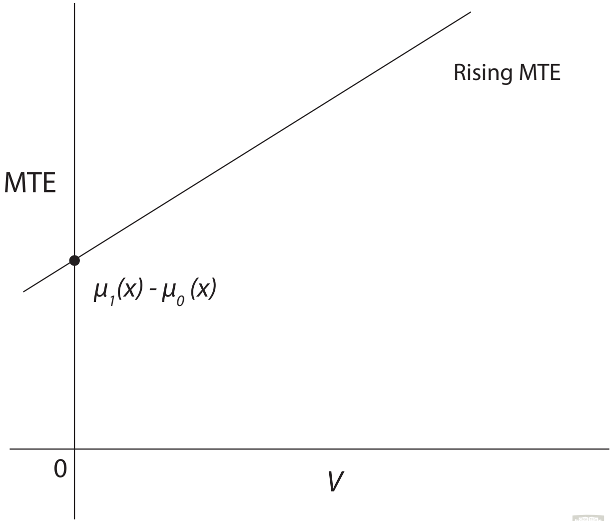
- Suppose

$$\text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \geq 0$$

$$\therefore \frac{\text{Cov}(U_1 - U_0, U_C)}{\sigma_V} > 0$$

- Unobserved components of costs rise with gross gains

Anti-Meritocratic Society



- Notice we can use the result that

$$\frac{\mu_D(z)}{\sigma_V} = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(P(z))$$

$$V = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(U_D)$$

- Effect of Treatment for People at Margin of Indifference Between Taking Treatment and Not:

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) =$$

$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \underbrace{F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(\overset{\substack{\text{Propensity}\\ \text{score}\\ \downarrow}}{P(z)})}_{\text{can estimate this}}$$

- MTE:

$$E(Y_1 - Y_0 \mid U_D = u_D, X = x, Z = z) =$$

$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(u_D)$$

- Notice from definition of TT

$$
\begin{aligned}
E(Y_1 - Y_0 | D = 1, X = x, Z = z)P(z) \\
&= [\mu_1(x) - \mu_0(x)]P(z) \\
&+ E(U_1 - U_0 | D = 1, X = x, Z = z)P(z)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial [E(Y_1 - Y_0 | D = 1, X = x, Z = z)P(z)]}{\partial P(z)} \\
&= \mu_1(x) - \mu_0(x) + E(U_1 - U_0 | X = x, P(z) = U_D) \\
&= \text{MTE}
\end{aligned}
$$

- Marginal change in TT
- Also MTE $= \frac{\partial E(Y|Z=z)}{\partial P(z)}$

Problem: Prove this claim

- Hint: Read “Building Bridges”

- Recent Advances in Econometrics:
  a. Relax normality
  b. Do not assume linearity of $\mu_1(X)$ and $\mu_0(X)$ in terms of X
  c. Do not require identification at infinity but only because they abandon pursuit of ATE, TT, TUT or else assume that $(Y_1, Y_0) \perp\!\!\!\perp D \mid X$ (matching assumption)
  d. Identification at infinity in some version or the other is required for ATE, TT, TUT as long as there is selection on unobservables (i.e., $(Y_1, Y_0) \not\perp\!\!\!\perp D \mid X$)