Using Matching, Instrumental Variables and Control Functions to Estimate Economic Choice Models

Review of Economics and Statistics 86(1) (2004)

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Econon 312, Spring 2021

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Two potential outcomes (Y_0, Y_1).
$D = 1$ if Y_1 is selected. $D = 0$ if Y_0 is selected.
Let V be utility.

$$V = \mu_V(Z, U_V) \quad D = 1\,(V > 0) \tag{1}$$

Z : observed factors determining choices, U_V unobserved

$$Y_1 = \mu_1(X, U_1) \tag{2a}$$

$$Y_0 = \mu_0(X, U_0). \tag{2b}$$

U_0, U_1, U_V are (absolutely) continuous

$$\Delta = Y_1 - Y_0.$$

Additively Separable Case: For Familiarity. Not essential.

$$V = \mu_V(Z) + U_V \quad E(U_V) = 0 \tag{1'}$$

$$Y_1 = \mu_1(X) + U_1 \quad E(U_1) = 0 \tag{2a'}$$

$$Y_0 = \mu_0(X) + U_0 \quad E(U_0) = 0. \tag{2b'}$$

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$$ATE \quad : \quad E(Y_1 - Y_0|X) \text{ (Average Treatment Effect)}$$

$$TT \quad : \quad E(Y_1 - Y_0|X, D = 1) \text{ (Treatment on the Treated)}$$

$$MTE \quad : \quad E(Y_1 - Y_0|X, Z, V) \text{ (Marginal Treatment Effect)}$$

These are familiar but by no means the only parameters we could consider

From *MTE*, can identify many other parameters

(Recall IV Lectures)

Samples generated by choices:

$$E(Y|X,Z,D=1)=E(Y_1|X,Z,D=1)$$

$$E(Y|X,Z,D=0)=E(Y_0|X,Z,D=0)$$

Data:

$$\Pr(D=1|X,Z)$$

$$E(Y_1|X,D=1) \text{ and } E(Y_0|X,D=0).$$

From raw means, we get biases.
Can form $E(Y_1|X,D=1)-E(Y_0|X,D=0)$.

In General This Produces BIASES

TT :

$$\begin{aligned}\text{Bias } TT &= [E(Y|X,D=1)-E(Y|X,D=0)] \\ &\quad -E(Y_1-Y_0|X,D=1) \\ &= [E(Y_0|X,D=1)-E(Y_0|X,D=0)]\,.\end{aligned}$$

Under Additive Separability

$$\text{Bias } TT = E(U_0|X,D=1)-E(U_0|X,D=0)$$

ATE :

$$\begin{aligned}\text{Bias } ATE &= E(Y|X,D=1)-E(Y|X,D=0) \\ &\quad -E(Y_1-Y_0|X)\,.\end{aligned}$$

Under Additive Separability

$$\begin{aligned} \text{Bias } ATE &= [E(U_1|X, D=1) - E(U_1|X)] \\ &\quad - [E(U_0|X, D=0) - E(U_0|X)] \end{aligned}$$

MTE :

$$\begin{aligned} \text{Bias } MTE &= E(Y|X, Z, D=1) - E(Y|X, Z, D=0) \\ &\quad - E(Y_1 - Y_0|X, Z, V) \end{aligned}$$

Under Additive Separability

$$\begin{aligned} \text{Bias } MTE &= E(U_1|X, Z, D=1) - E(U_1|X, Z, V) \\ &\quad - [E(U_0|X, Z, D=0) - E(U_0|X, Z, V)] \end{aligned}$$

Matching

$$W = (X, Z)$$

$$(Y_1, Y_0) \perp\!\!\!\perp D | W \qquad \text{(M-1)}$$

"$\perp\!\!\!\perp$" denotes independence given W

$$0 < \Pr(D = 1|W) = P(W) < 1, \qquad \text{(M-2)}$$

Rosenbaum and Rubin (1983) show (M-1) and (M-2) imply

$$(Y_1, Y_0) \perp\!\!\!\perp D | P(W). \qquad \text{(M-3)}$$

$$E(Y_1|D=0,P(W)) = E(Y_1|D=1,P(W)) = E(Y_1|P(W))$$

$$E(Y_0|D=1,P(W)) = E(Y_0|D=0,P(W)) = E(Y_0|P(W)).$$

Dependence between U_V and (U_1, U_0) is eliminated by conditioning on W:

$$U_V \perp\!\!\!\perp (U_1, U_0)|W.$$

"Selection on Observables"

If $P(W) = 1$ or $P(W) = 0$, method breaks down for those values.

Extensions (Heckman, Ichimura, Smith and Todd)

Distinction between X and Z

Introducing Z allows one to solve the breakdown problem arising from

$$P(X, Z) = 1 \text{ or } P(X, Z) = 0$$

Thus if outcomes are defined in terms of X and

$$Support\,(X|Z) = Support\,(X)$$

If we can find another value Z' such that

$$\Pr(X, Z') \neq 1,$$

can match using this (IV assumption)

Require only weaker mean independence assumptions

$$
\begin{aligned}
E(Y_1|W, D=1) &= E(Y_1|W) \\
E(Y_0|W, D=0) &= E(Y_0|W).
\end{aligned}
$$

Can be used for Means.

Matching is "for free" (Gill and Robins (2001)):

$E(Y_0|D=1,W)$ is not observed.

Can just as well replace it by

$$E(Y_0|D=1,W) = E(Y_0|D=0,W)$$

However, the implied economic restrictions are not "for free".

Imposes that, conditional on X and Z, the marginal person is the same as the average person.

This is the same as a flat $MTE(X,U)$ in U.
(MTE does not depend on U)

Additively Separable Case

We observe left-hand sides of

$$E(Y_1|X,Z,D=1)=\mu_1(X)+E(U_1|X,Z,D=1)$$

$$E(Y_0|X,Z,D=0)=\mu_0(X)+E(U_0|X,Z,D=0).$$

If $(U_1,U_V) \perp\!\!\!\perp X,Z$

$$E(U_1|X,Z,D=1)=E(U_1|\mu_V(Z)>U_V)=K_1(P(X,Z)).$$

If $(U_1,U_V) \perp\!\!\!\perp X,Z$

$$E(U_0|X,Z,D=0)=E(U_0|\mu_V(Z)\leq U_V)=K_0(P(X,Z))$$

So, key assumption

$$(U_1, U_0, U_V) \perp\!\!\!\perp (X, Z).$$

Under this condition

$$E(Y_1|X, Z, D = 1) = \mu_1(X) + K_1(P(X, Z))$$

$$E(Y_0|X, Z, D = 0) = \mu_0(X) + K_0(P(X, Z))$$

Need Limit Set Results

$$\lim_{P \to 1} K_1(P) = 0 \text{ and } \lim_{P \to 0} K_0(P) = 0$$

- If there are limit sets $\mathbb{Z}_0$ and $\mathbb{Z}_1$ such that $\lim_{Z \to \mathbb{Z}_0} P(X, Z) = 0$ and $\lim_{Z \to \mathbb{Z}_1} P(X, Z) = 1$, then we can identify the constants.
- There are semiparametric versions of these estimators.
- Use polynomials in P; Local Linear Regression in P.

From this model can obviously identify

$$ATE = \mu_1(X) - \mu_0(X)$$

(As we have seen)

Plus,

$$\begin{aligned} TT &= \mu_1(X) - \mu_0(X) + E(U_1 - U_0 | X, Z, D = 1) \\ &= \mu_1(X) - \mu_0(X) + K_1(P(X, Z)) \\ &\quad + \left(\frac{1 - P}{P}\right) K_0(P(X, Z)) \end{aligned}$$

$$
\begin{aligned}
MTE &= \mu_1(X) - \mu_0(X) \\
&\quad + \frac{\partial \left[E\left(U_1 - U_0 | X, Z, D = 1\right) P\left(X, Z\right) \right]}{\partial P\left(X, Z\right)} \\
&= \mu_1(X) - \mu_0(X) \\
&\quad + \frac{\partial \left[P\left(X, Z\right) \left\{ K_1\left(P(X, Z\right) + \frac{1-P}{P} K_0\left(P(X, Z\right) \right\} \right]}{\partial P\left(X, Z\right)}.
\end{aligned}
$$

Marginal and Average are allowed to be different.

Problem: Show this for $P(X, Z) = U(p)$

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Both Matching and Control functions are defined only over

$$\text{Support}\,(X|D=1)\cap\text{Support}\,(X|D=0)$$

Method of control functions does not require

$$(U_0, U_1) \perp\!\!\!\perp U_V|\,(X, Z)$$

But Matching does.

Matching is a special case of control functions in the additively separable case.

Additive separability and control functions assumptions are central to this claim.

$$
\begin{aligned}
E(U_1|X,Z,D=1) &= E(U_1|X,Z) = E(U_1|P(W)) \\
E(U_0|X,Z,D=0) &= E(U_0|X,Z) = E(U_0|P(W)).
\end{aligned}
$$

If

$$\mu_1(W) = E(Y_1|W) \text{ and } \mu_0(W) = E(Y_0|W)$$

then

$$E(U_1|P(W)) = 0 \text{ and } E(U_0|P(W)) = 0$$

However, this is not strictly required

In the method of control functions,

$$\text{If } (X, Z) \perp\!\!\!\perp (U_0, U_1, U_V)$$

$$
\begin{aligned}
& E(Y|X, Z, D) \\
= \;& E(Y_1|X, Z, D = 1) D + E(Y_0|X, Z, D = 0)(1 - D) \\
= \;& \mu_0(X) + (\mu_1(X) - \mu_0(X)) D + E(U_1|X, Z, D = 1) D \\
& + E(U_0|P(X, Z), D = 0)(1 - D) \\
= \;& \mu_0(X) + (\mu_1(X) - \mu_0(X)) D + E(U_1|P(X, Z), D = 1) D \\
& + E(U_0|P(X, Z), D = 0)(1 - D) \\
= \;& \mu_0(X) \\
& + [\mu_1(X) - \mu_0(X) + K_1(P(X, Z)) - K_0(P(X, Z))] D \\
& + K_0(P(X, Z)).
\end{aligned}
$$

To identify

$$\mu_1(X) - \mu_0(X)$$

must isolate it from

$$K_1(P(X,Z))$$

and

$$K_0(P(X,Z)).$$

Tables 1 and 2 present sensitivity analysis for the case of

$$
\begin{aligned}
(U_1, U_0, U_V)' &\sim N(0, \Sigma) \\
corr(U_j, U_V) &= \rho_{jV} \\
var(U_j) &= \sigma_j^2; \quad j = \{0, 1\}.
\end{aligned}
$$

so

$$
\begin{aligned}
\text{Bias } TT(P(Z) = p) &= \sigma_0 \rho_{0V} M(p) \\
\text{Bias } ATE(P(Z) = p) &= M(p)\left[\sigma_1 \rho_{1V} \ (1 - p) + \sigma_0 \rho_{0V} \ p\right]
\end{aligned}
$$

where $M(p) = \frac{\phi\left(\Phi^{-1}(1-p)\right)}{p(1-p)}$

Problem: Using the Generalized Roy model derive these results for bias

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Table 1

Mean Bias for Treatment on the Treated

ρ_{0V}	Average Bias ($\sigma_0 = 1$)	Average Bias ($\sigma_0 = 2$)
-1.00	-1.7920	-3.5839
-0.75	-1.3440	-2.6879
-0.50	-0.8960	-1.7920
-0.25	-0.4480	-0.8960
0.00	0.0000	0.0000
0.25	0.4480	0.8960
0.50	0.8960	1.7920
0.75	1.3440	2.6879
1.00	1.7920	3.5839

$\text{BIASTT} = \rho_{0V} * \sigma_0 * M(p)$

$M(p) = \frac{\varphi(\Phi^{-1}(p))}{[p*(1-p)]}$

Table 2

Mean Bias for Average Treatment Effect

$(\sigma_0 = 1)$

ρ_{0V}	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
					$\rho_{1V}(\sigma_1 = 1)$				
-1.00	-1.7920	-1.5680	-1.3440	-1.1200	-0.8960	-0.6720	-0.4480	-0.2240	0
-0.75	-1.5680	-1.3440	-1.1200	-0.8960	-0.6720	-0.4480	-0.2240	0	0.2240
-0.50	-1.3440	-1.1200	-0.8960	-0.6720	-0.4480	-0.2240	0	0.2240	0.4480
-0.25	-1.1200	-0.8960	-0.6720	-0.4480	-0.2240	0	0.2240	0.4480	0.6720
0	-0.8960	-0.6720	-0.4480	-0.2240	0	0.2240	0.4480	0.6720	0.8960
0.25	-0.6720	-0.4480	-0.2240	0	0.2240	0.4480	0.6720	0.8960	1.1200
0.50	-0.4480	-0.2240	0	0.2240	0.4480	0.6720	0.8960	1.1200	1.3440
0.75	-0.2240	0	0.2240	0.4480	0.6720	0.8960	1.1200	1.3440	1.5680
1.00	0	0.2240	0.4480	0.6720	0.8960	1.1200	1.3440	1.5680	1.7920
					$\rho_{1V}(\sigma_1 = 2)$				
-1.00	-2.6879	-2.2399	-1.7920	-1.3440	-0.8960	-0.4480	0	0.4480	0.8960
-0.75	-2.4639	-2.0159	-1.5680	-1.1200	-0.6720	-0.2240	0.2240	0.6720	1.1200
-0.50	-2.2399	-1.7920	-1.3440	-0.8960	-0.4480	0	0.4480	0.8960	1.3440
-0.25	-2.0159	-1.5680	-1.1200	-0.6720	-0.2240	0.2240	0.6720	1.1200	1.5680
0	-1.7920	-1.3440	-0.8960	-0.4480	0	0.4480	0.8960	1.3440	1.7920
0.25	-1.5680	-1.1200	-0.6720	-0.2240	0.2240	0.6720	1.1200	1.5680	2.0159
0.50	-1.3440	-0.8960	-0.4480	0	0.4480	0.8960	1.3440	1.7920	2.2399
0.75	-1.1200	-0.6720	-0.2240	0.2240	0.6720	1.1200	1.5680	2.0159	2.4639
1.00	-0.8960	-0.4480	0	0.4480	0.8960	1.3440	1.7920	2.2399	2.6879

BIASATE $= \rho_{1V} * \sigma_1 * M_1(p) - \rho_{0V} * \sigma_0 * M_0(p)$

BIASMTE $=$ BIASATE $-\Phi^{-1}(1-p) * (\rho_{1V} * \sigma_1 - \rho_{0V} * \sigma_0)$

$M_1(p) = \frac{\varphi(\Phi^{-1}(p))}{p}$

$M_0(p) = \frac{-\varphi(\Phi^{-1}(p))}{[1-p]}$

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Matching and Method of control functions work with $E(Y|X,Z,D)$ and $\Pr(D=1|X,Z)$.

$$\begin{aligned} Y &= DY_1 + (1-D)Y_0 \\ &= \mu_0(X) + (\mu_1(X) - \mu_0(X) + U_1 - U_0)D + U_0 \\ &= \mu_0(X) + \Delta(X)D + U_0 \end{aligned}$$

If $U_1 = U_0$

$$E(U_0|P(X,Z),X) = E(U_0|X) \quad \text{(IV-1)}$$

$$\Pr(D=1|X,Z) \text{ is a nontrivial function of } Z \text{ for each } X. \quad \text{(IV-2)}$$

When

$$U_1 \neq U_0, \text{ but } D \perp\!\!\!\perp (U_1 - U_0) | X$$

or alternatively

$$U_V \perp\!\!\!\perp (U_1 - U_0 | X),$$

we have all three mean treatment effects are the same

$$\begin{aligned} ATE &= E(Y_1 - Y_0 | X) = E(\Delta(X) | X) \\ TT &= E(Y_1 - Y_0 | X, D = 1) = E(Y_1 - Y_0 | X) \\ &= MTE \end{aligned}$$

Analytically More Interesting Case

$U_1 \neq U_0$ and $D \not\perp\!\!\!\perp (U_1 - U_0)$

For ATE :

$$E\left(U_0 + D\left(U_1 - U_0\right)|P\left(X, Z\right), X\right) = E\left(U_0 + D\left(U_1 - U_0\right)|X\right) \qquad \text{(IV-3)}$$

For TT :

$$\begin{aligned} &E\left(U_0 + D\left(U_1 - U_0\right) - E\left(U_0 + D\left(U_1 - U_0\right)|X\right)|P\left(X, Z\right), X\right) \\ &\quad = E\left(U_0 + D\left(U_1 - U_0\right) - E\left(U_0 + D\left(U_1 - U_0\right)|X\right)|X\right) \end{aligned}$$

For ATE we can rewrite:

$$\begin{aligned} &E\left(U_0|P\left(X, Z\right), X\right) + E\left(U_1 - U_0|D = 1, P\left(X, Z\right), X\right)P\left(X, Z\right) \\ &\quad = E\left(U_0|X\right) + E\left(U_1 - U_0|D = 1, X\right)P\left(X, Z\right) \end{aligned}$$

All mean parameters are the same if $U_1 = U_0$, or $(U_1 - U_0) \perp\!\!\!\perp D|P\left(X, Z\right), X$

1. Method of Control Functions Models Dependence between (U_1, U_0) and V.
2. Matching assumes $(U_1, U_0) \perp\!\!\!\perp V \mid X, Z$.
3. Z independent of U_0, U_1 conditional on X.

Local Instrumental Variables (LIV) require that

$$\mu_D(Z) \quad \text{be a non-degenerate random variable given } X \text{ (existence of an exclusion restriction)} \tag{2}$$

$$(U_0, U_1, U_V) \perp\!\!\!\perp Z|X \tag{LIV-2}$$

$$0 < \Pr(D|X) < 1 \tag{LIV-3}$$

$$\text{Support } P(D|(X,Z)) = [0,1] \tag{LIV-4}$$

Under these conditions,

$$\frac{\partial E(Y|X, P(X,Z))}{\partial(P(Z))} = MTE(X, P(Z), V)$$

Table 3
Identifying Assumptions and Implicit Economic Assumptions Underlying the Four Methods Discussed in this Paper
Conditional on X and Z

Method	Exclusion Required?	Separability of Observables and Unobservables in Outcome Equations?	Functional Forms Required?	Marginal = Average? (Given X, Z)	Key Identification Condition for Means (assuming separability)
Matching*	No	No	No	Yes	$E(U_1\|X, D=1, Z) = E(U_1\|X, Z)$ $E(U_0\|X, D=0, Z) = E(U_0\|X, Z)$
Control Function**	Yes (for nonparametric identification)	Conventional, but not required	Conventional, but not required	No	$E(U_0\|X, D=0, Z)$ and $E(U_1\|X, D=1, Z)$ can be varied independently of $\mu_0(X)$ and $\mu_1(X)$, respectively and intercepts can be identified through limit arguments or symmetry assumptions
IV (conventional)	Yes	Yes	No	No (Yes in standard case)	$E(U_0 + D(U_1 - U_0)\|X, Z)$ $= E(U_0 + D(U_1 - U_0)\|X)$ (ATE) $E(U_0 + D(U_1 - U_0) - E(U_0 + D(U_1 - U_0)\|X)\|P(Z), X)$ $= E(U_0 + D(U_1 - U_0) - E(U_0 + D(U_1 - U_0)\|X)\|X)$ (TT)
LIV	Yes	No	No	No	$(U_0, U_1, U_v) \perp\!\!\!\perp Z\|X$ $\Pr(D=1\|Z, X)$ is a nontrivial function of Z for each X.

*For propensity score matching, (X, Z) are replaced with $P(X, Z)$ in defining parameters and conditioning sets.

**Conditions for writing the control function in terms of $P(X, Z)$ are given in the text.

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Fundamental Problem: Information of the Analyst often less than that of the Agent.

Definition 1

We say that $\sigma\left(I_{R^*}\right)$ is a **relevant information set** if its associated random variable, I_{R^*}, satisfies (M-1) so

$$\left(Y_1, Y_0\right) \perp\!\!\!\perp D | I_{R^*}$$

Definition 2

We say that $\sigma(I_R)$ is a **minimal relevant information set** if it is the intersection of all sets $\sigma(I_{R^*})$ and $(Y_1, Y_0) \perp\!\!\!\perp D | I_R$. The associated random variable I_R is the minimum amount of information that guarantees that (M-1) is satisfied. Intersection may be empty. May not be a unique minimal information set.

Definition 3

The agent's information set, $\sigma(I_A)$, is defined by the information I_A used by the agent when choosing among treatments. Accordingly, we call I_A the **agent's information**.

Definition 4

The econometrician's **full information set**, $\sigma\left(I_{E^*}\right)$, is defined by **all** the information **available** to the econometrician, I_{E^*}.

Definition 5

The econometrician's **information set**, $\sigma\left(I_E\right)$, is defined by the information **used** by the econometrician when analyzing the agent's choice of treatment, I_E.

Obvious Inclusions: $\sigma(I_R) \subseteq \sigma(I_{R^*}), \sigma(I_R) \subseteq \sigma(I_A)$ and $\sigma(I_E) \subseteq \sigma(I_{E^*})$

This assumes I_R exists.

Matching implies

$$\sigma(I_R) \subseteq \sigma(I_E)$$

Generalized Roy Examples (Assume Factor Structure for error terms)

- Consider bias from matching with different information sets (i.e., different p specifications).
- Remember the Rosenbaum and Rubin result.

$$\begin{aligned} V &= Z\gamma + U_V \\ &= Z\gamma + \alpha_{V1} f_1 + \alpha_{V2} f_2 + \varepsilon_V, \\ D &= 1 \text{ if } V \geq 0, \quad = 0 \text{ otherwise} \end{aligned}$$

$$\begin{aligned} Y_1 &= \mu_1 + U_1 = \mu_1 + \alpha_{11} f_1 + \alpha_{12} f_2 + \varepsilon_1 \\ Y_0 &= \mu_0 + U_0 = \mu_0 + \alpha_{01} f_1 + \alpha_{02} f_2 + \varepsilon_0, \end{aligned}$$

$(f_1, f_2, \varepsilon_V, \varepsilon_1, \varepsilon_0)$ mean zero random variables, mutually independent of each other and Z

The minimal relevant information set when factor loadings are not zero:

$$I_R = \{f_1, f_2\}.$$

Agent information sets may include different variables. If shocks to the outcomes not known, but other terms are:

$$I_A = \{f_1, f_2, Z, \varepsilon_V\}$$

Under perfect certainty, $I_A = \{f_1, f_2, Z, \varepsilon_V, \varepsilon_1, \varepsilon_0\}$.

Construct examples using:

$$(f_1, f_2, \varepsilon_V, \varepsilon_1, \varepsilon_0) \sim N(0, \Sigma),$$

$$diag(\Sigma) = \left(\sigma^2_{f_1}, \sigma^2_{f_2}, \sigma^2_{\varepsilon_V}, \sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_0}\right)$$

Suppose $I_E = \{Z, f_1, f_2\}$

$$\begin{aligned} & E(Y_1 | D = 1, I_E) - E(Y_0 | D = 0, I_E) \\ = \ & \mu_1 - \mu_0 + (\alpha_{11} - \alpha_{01}) f_1 + (\alpha_{12} - \alpha_{02}) f_2 \end{aligned}$$

Knowledge of (Z, f_1, f_2) and of $P(Z, f_1, f_2)$ equivalent

$$\begin{aligned} P(I_E) &= \Pr\left(\frac{\varepsilon_V}{\sigma_{\varepsilon_V}} > \frac{-Z\gamma - \alpha_{V1} f_1 - \alpha_{V2} f_2}{\sigma_{\varepsilon_V}}\right) \\ &= 1 - \Phi\left(\frac{-Z\gamma - \alpha_{V1} f_1 - \alpha_{V2} f_2}{\sigma_{\varepsilon_V}}\right) = p \end{aligned}$$

$$E\left(Y_1|D=1,P\left(I_E\right)=p\right)-E\left(Y_0|D=0,P\left(I_E\right)=p\right)$$

$$\begin{aligned}
&= \mu_1-\mu_0+E\left(U_1|D=1,P\left(I_E\right)=p\right) \\
&\quad -E\left(U_0|D=0,P\left(I_E\right)=p\right) \\
&= \mu_1-\mu_0+E\left(U_1|\frac{\varepsilon_V}{\sigma_{\varepsilon_V}}>\Phi^{-1}\left(1-p\right)\right) \\
&\quad -E\left(U_0|\frac{\varepsilon_V}{\sigma_{\varepsilon_V}}\leq\Phi^{-1}\left(1-p\right)\right) \\
&= \mu_1-\mu_0
\end{aligned}$$

All the treatment parameters equal

$(MTE = ATE = LATE = TT)$

$$E\left(U_1 | \frac{\varepsilon_V}{\sigma_{\varepsilon_V}} > \Phi^{-1}(1-p)\right) = \frac{COV(U_1, \varepsilon_V)}{\sigma_{\varepsilon_V}} M_1(P)$$
$$E\left(U_0 | \frac{\varepsilon_V}{\sigma_{\varepsilon_V}} \leq \Phi^{-1}(1-p)\right) = \frac{COV(U_0, \varepsilon_V)}{\sigma_{\varepsilon_V}} M_0(P)$$

where

$$M_1(P) = \frac{\phi(\Phi^{-1}(1-p))}{p} \quad \text{and} \quad M_0(P) = \frac{\phi(\Phi^{-1}(1-p))}{1-p}$$

because

$$COV(U_i, \varepsilon_V) = COV(\alpha_{i1} f_1 + \alpha_{i2} f_2 + \varepsilon_i, \varepsilon_V) = 0, \quad i = 0, 1.$$

$$I_E = \{Z\}$$

$$\frac{z\gamma + \alpha_{V1} f_1 + \alpha_{V2} f_2 + \varepsilon_V}{\sqrt{\alpha_{V1}^2 \sigma_{f_1}^2 + \alpha_{V2}^2 \sigma_{f_2}^2 + \sigma_{\varepsilon_V}^2}} = \Phi^{-1}(1-p).$$

$$\text{Bias } TT(P(Z) = p) = \beta_0 M(p),$$

$$M(P) = M_1(P) - M_0(P)$$

$$\text{Bias } ATE(P(Z) = p) = M(p)\left[\beta_1 (1-p) + \beta_0 p\right]$$

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$$\begin{aligned} \text{Bias } MTE\left(P\left(Z\right)=p\right) &= M(p)\left[\beta_1\left(1-p\right)+\beta_0 p\right] \\ &\quad -\Phi^{-1}\left(1-p\right)\left[\beta_1-\beta_0\right] \end{aligned}$$

where

$$\begin{aligned} M(p) &= \frac{\phi\left(\Phi^{-1}\left(1-p\right)\right)}{p\left(1-p\right)} \\ \beta_1 &= \frac{\alpha_{V1}\alpha_{11}\sigma_{f_1}^2+\alpha_{V2}\alpha_{12}\sigma_{f_2}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2+\alpha_{V2}^2\sigma_{f_2}^2+\sigma_{\varepsilon_V}^2}} \\ \beta_0 &= \frac{\alpha_{V1}\alpha_{01}\sigma_{f_1}^2+\alpha_{V2}\alpha_{02}\sigma_{f_2}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2+\alpha_{V2}^2\sigma_{f_2}^2+\sigma_{\varepsilon_V}^2}} \end{aligned}$$

Problem: Verify the equations on all slides.

$$I'_E = \{Z, f_2\}$$

May raise or lower the bias.

Link

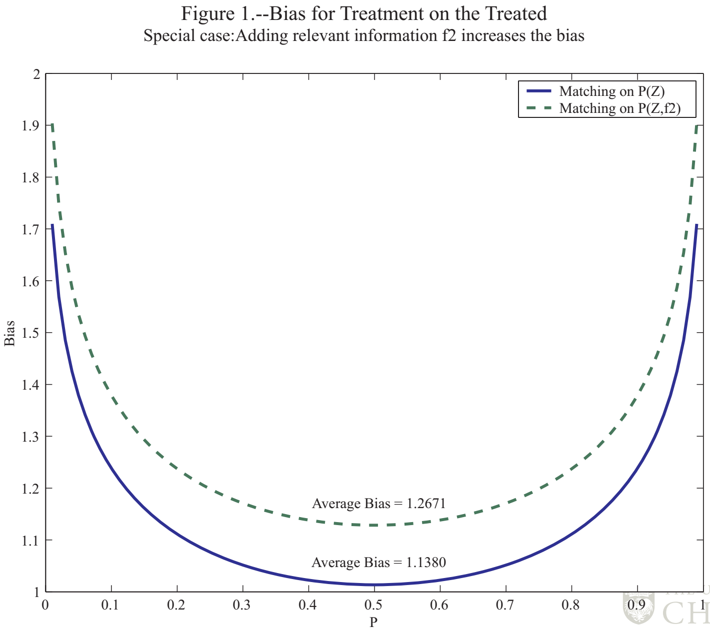

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Figure 2.--Bias for Average Treatment Effect
Special case: Adding relevant information f2 increases the bias

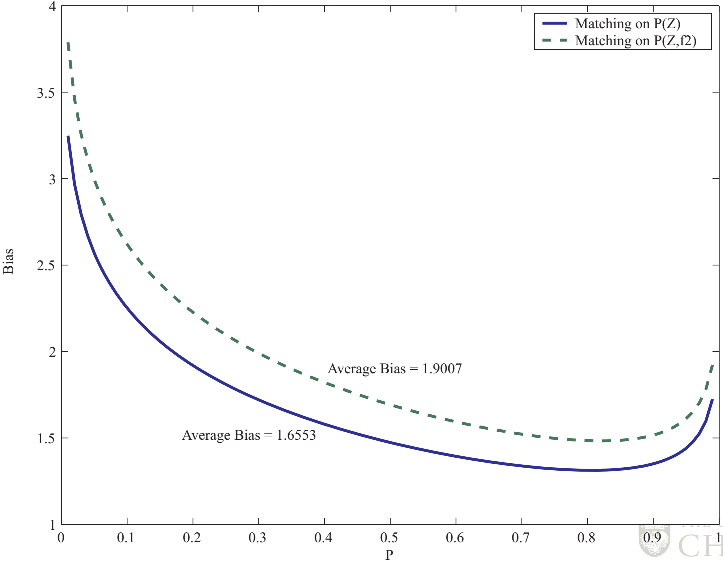
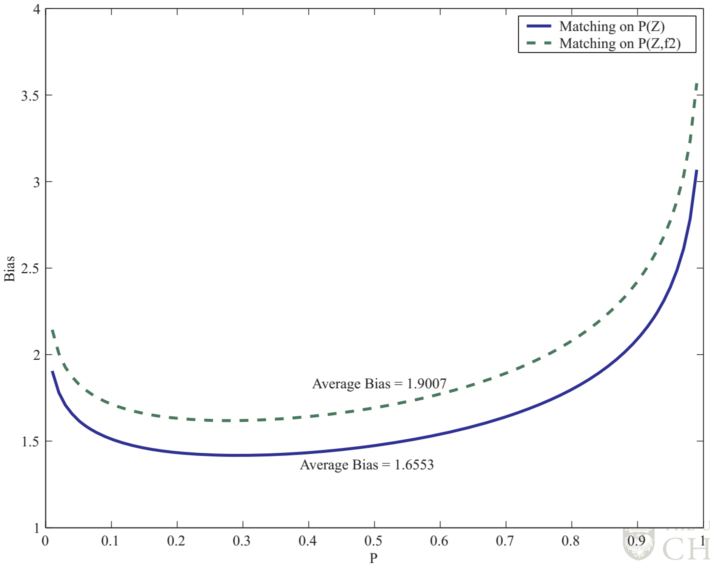


Figure 3.--Bias for Marginal Treatment Effect
Special case: Adding relevant information f2 increases the bias



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Control Function Method Models the Bias

In control function method, adding f_2 we simply change the control function. We go from

$$\begin{aligned} K_1\left(P\left(Z\right)=p\right) &= \beta_1 M_1\left(p\right) \\ K_0\left(P\left(Z\right)=p\right) &= -\beta_0 M_0\left(p\right) \end{aligned}$$

to

$$\begin{aligned} K_1'\left(P\left(Z,f_2\right)=p\right) &= \beta_1' M_1\left(p\right) \\ K_0'\left(P\left(Z,f_2\right)=p\right) &= -\beta_0' M_0\left(p\right) \end{aligned}$$

where $M_1\left(p\right)=\frac{\phi\left(\Phi^{-1}(1-p)\right)}{p}$ and $M_0\left(p\right)=\frac{\phi\left(\Phi^{-1}(1-p)\right)}{1-p}$

This protects us against misspecification.

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Suppose we do not know f_2, just proxy it by $\widetilde{Z}$

$$\widetilde{I}_{E^*} = \left\{ Z, \widetilde{Z} \right\}.$$

$$\text{Suppose } I_E = \widetilde{I}_{E^*}$$

Suppose further that

$$\begin{aligned} \widetilde{Z} &\sim N\left(0, \sigma^2_{\widetilde{Z}}\right) \\ corr\left(\widetilde{Z}, f_2\right) &= \rho, \text{ and } \widetilde{Z} \perp\!\!\!\perp \left(\varepsilon_0, \varepsilon_1, \varepsilon_V, f_1\right). \end{aligned}$$

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Expressions corresponding to β_0 and β_1 :

$$\widetilde{\beta}_1 = \frac{\alpha_{11}\alpha_{V1}\sigma_{f_1}^2 + \alpha_{12}\alpha_{V2}\left(1-\rho^2\right)\sigma_{f_2}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \alpha_{V2}^2\sigma_{f_2}^2\left(1-\rho^2\right) + \sigma_{\varepsilon_V}^2}}$$

$$\widetilde{\beta}_0 = \frac{\alpha_{01}\alpha_{V1}\sigma_{f_1}^2 + \alpha_{02}\alpha_{V2}\left(1-\rho^2\right)\sigma_{f_2}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \alpha_{V2}^2\sigma_{f_2}^2\left(1-\rho^2\right) + \sigma_{\varepsilon_V}^2}}$$

- By substituting I_E' for $\widetilde{I}_E$ and β_j' for $\widetilde{\beta}_j$ $(j = 0, 1)$ into Conditions (1), (2) and (3) we obtain equivalent results for this case. Whether $\widetilde{I}_E$ will be bias reducing depends on how well it spans I_R and the signs of the terms in the absolute values.
- In this case, there is another parameter ρ $(\rho = 0)$.

- The bias generated when the econometrician's information is $\widetilde{I}_E$ can also be smaller than when it is I_E'. It can be the case that knowing *the proxy* variable $\widetilde{Z}$ is *better* than knowing the actual variable f_2. Take treatment on the treated as the parameter. The bias is reduced when $\widetilde{Z}$ is used instead of f_2 if

$$\left| \frac{\alpha_{01}\alpha_{V1}\sigma_{f_1}^2 + \alpha_{02}\alpha_{V2}\left(1-\rho^2\right)\sigma_{f_2}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \alpha_{V2}^2\sigma_{f_2}^2\left(1-\rho^2\right) + \sigma_{\varepsilon_V}^2}} \right| < \left| \frac{\alpha_{01}\alpha_{V1}\sigma_{f_1}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \sigma_{\varepsilon_V}^2}} \right|.$$

Problem: Prove this

Figure 4.--Bias for Treatment on the Treated
Special case: Adding irrelevant information $\widetilde{Z}$ increases the bias
correlation($\widetilde{Z}$,f2)=0.5

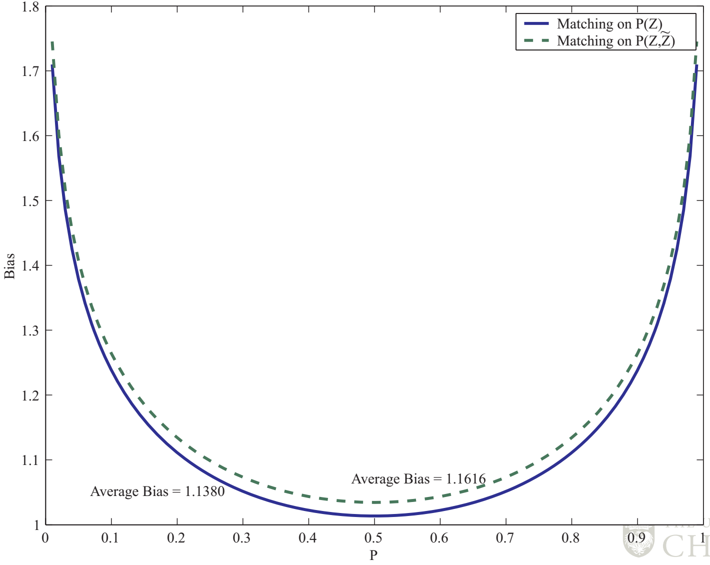

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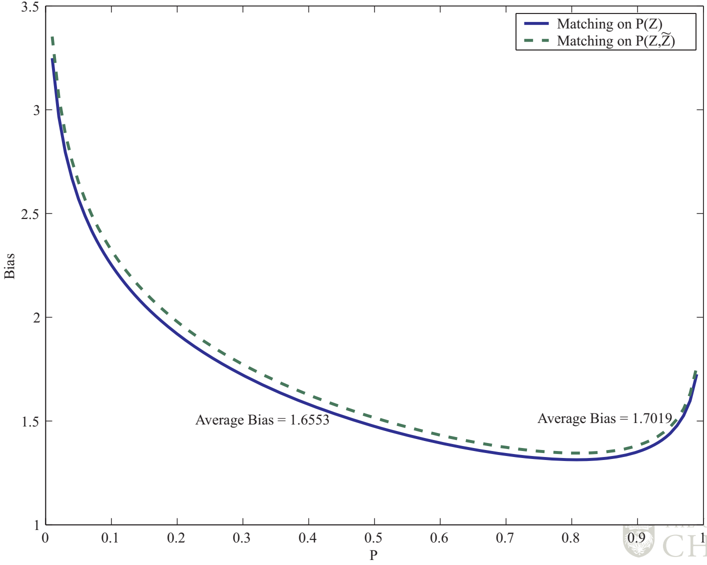
Figure 5.--Bias for Average Treatment Effect
Special case: Adding irrelevant information $\widetilde{Z}$ increases the bias
correlation($\widetilde{Z}$,f2)=0.5



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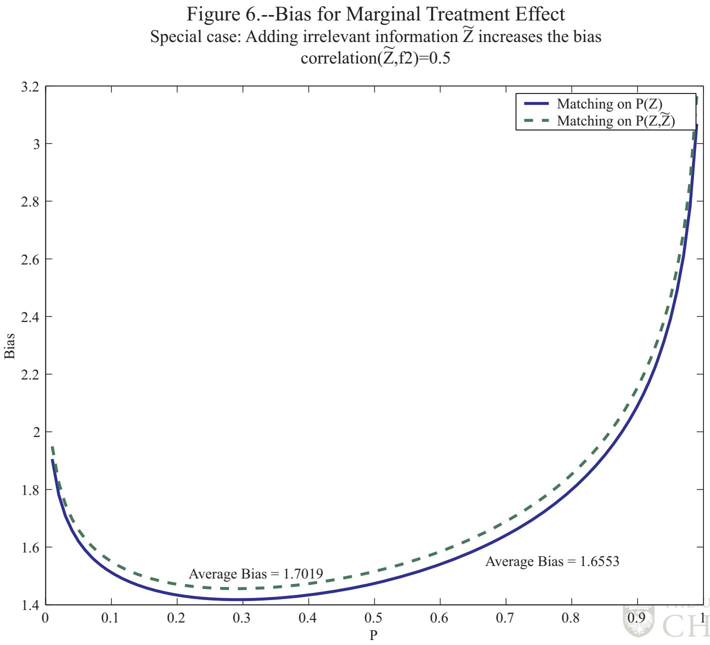

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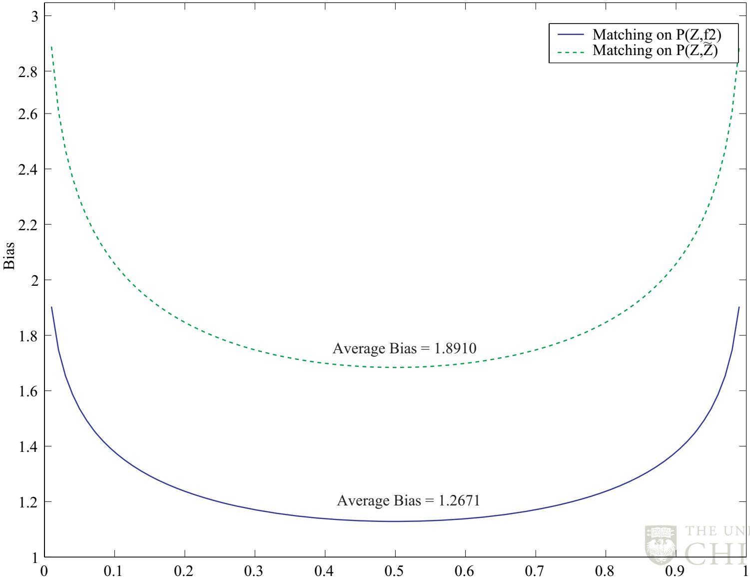
Figure 7.--Bias for Treatment on the Treated
Using proxy $\widetilde{Z}$ for f2 increases the bias
correlation $(\widetilde{Z}, f2)=0.5$



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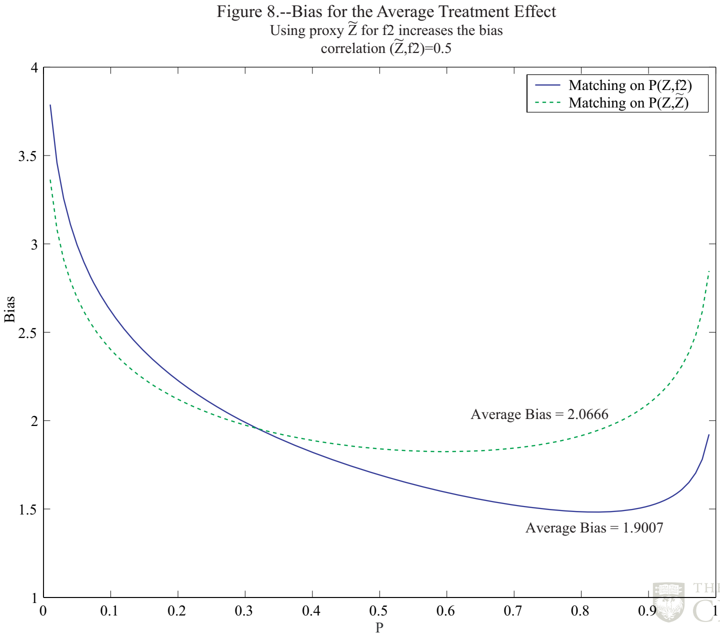

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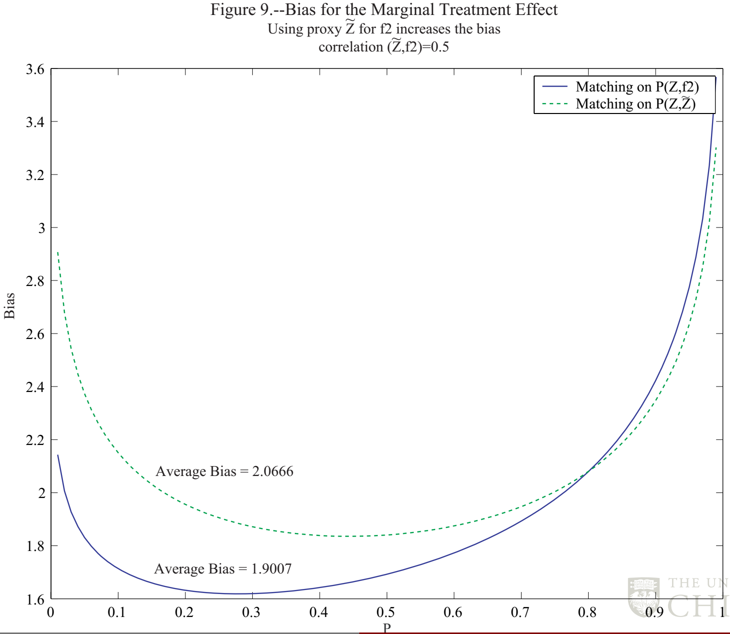

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- Adding more variables to the Information Set may increase Bias.
- How to choose the relevant W variables?
- Standard methods on model selection criteria fail.
- An implicit assumption underlying such procedures is that the added conditioning variables C are exogenous in the following sense

$$(Y_0, Y_1) \perp\!\!\!\perp D | I_E, C \tag{M-4}$$

(I_E is the list of initial variables used as conditioning variables.)

- Sometimes procedures suggested "Add variables when t ratios big in propensity score"
- Improve Fit
- Such procedures can raise the bias.

Consider the following example:

$$\widetilde{\widetilde{I}}_E = \{Z, S\}$$

where

$$\begin{aligned} & S = V - Z\gamma + \eta \\ & \eta \sim N\left(0, \sigma_\eta^2\right) \\ & \eta \perp\!\!\!\perp \left(f_1, f_2, \varepsilon_0, \varepsilon_1, \varepsilon_V\right). \end{aligned}$$

S might be an elicitation from a questionaire.

Same expressions for the biases using $\tilde{\tilde{\beta}}_j$ $(j = 0, 1)$ instead of β_j where:

$$\tilde{\tilde{\beta}}_1 = \frac{\left(\alpha_{11}\alpha_{V1}\sigma^2_{f_1} + \alpha_{12}\alpha_{V2}\sigma^2_{f_2}\right)}{\sqrt{\alpha^2_{V1}\sigma^2_{f_1} + \alpha^2_{V2}\sigma^2_{f_2} + \sigma^2_{\varepsilon_V}} + \sigma^2_{\eta}}$$

$$\tilde{\tilde{\beta}}_0 = \frac{\left(\alpha_{01}\alpha_{V1}\sigma^2_{f_1} + \alpha_{02}\alpha_{V2}\sigma^2_{f_2}\right)}{\sqrt{\alpha^2_{V1}\sigma^2_{f_1} + \alpha^2_{V2}\sigma^2_{f_2} + \sigma^2_{\varepsilon_V}} + \sigma^2_{\eta}}$$

In general, these expressions are not zero. Bias can be increased *or* decreased.

Problem: Derive these conditions

When $\sigma_\eta^2 = 0$ we can perfectly predict D. (Will pass a goodness-of-fit criterion)
Thus for

$$2\varepsilon > \sigma_\eta^2 > \varepsilon > 0$$

It follows that

$$\begin{aligned}
\lim_{\varepsilon \to 0} \Pr\left(D = 1 | V - Z\gamma + \eta\right) &= 1 \text{ for } V > Z\gamma \\
\lim_{\varepsilon \to 0} \Pr\left(D = 1 | V - Z\gamma + \eta\right) &= 0 \text{ for } V < Z\gamma.
\end{aligned}$$

Assumption (M-2) is violated and matching breaks down
Making σ_η^2 arbitrarily small, we can predict D arbitrarily well.
Can improve over the fit with $(f_{1,} f_2)$ in the set which produces no bias.

Table 4

	Goodness of fit statistics		Average Bias		
Variables in Probit	Correct in-sample prediction rate	Pseudo R^2	TT	ATE	MTE
Z	66.88%	0.1284	1.1380	1.6553	1.6553
Z, f_2	75.02%	0.2791	1.2671	1.9007	1.9007
Z, f_1, f_2	83.45%	0.4844	0.0000	0.0000	0.0000
Z, S_1	77.38%	0.3282	0.9612	1.3981	1.4070
Z, S_2	92.25%	0.7498	0.9997	1.4541	1.4590

Model:

$$\begin{array}{llll} V = Z + f_1 + f_2 + \varepsilon_v & \varepsilon_v \sim N(0,1) & S_1 = V + u_1 & u_1 \sim N(0,4) \\ Y_1 = 2f_1 + 0.1f_2 + \varepsilon_1 & \varepsilon_1 \sim N(0,1) & S_2 = V + u_2 & u_2 \sim N(0,0.25) \\ Y_0 = f_1 + 0.1f_2 + \varepsilon_0 & \varepsilon_0 \sim N(0,1) & & \\ & f_1 \sim N(0,1) & & \\ & f_2 \sim N(0,1) & & \end{array}$$

A More General Example

Considers use of a proxy regressor

$$Q = \alpha_{Q_Z} Z + \alpha_{Q_1} f_1 + \alpha_{Q_2} f_2 + \tau + \eta$$

- $Z \perp\!\!\!\perp (f_1, f_2, \tau, \eta)$;
- (f_1, f_2, τ, η) has mean zero
- $f_1 \perp\!\!\!\perp f_2$, $\tau \perp\!\!\!\perp \eta$ and $(f_1, f_2) \perp\!\!\!\perp (\tau, \eta)$;
- τ possibly dependent on ε_V in the latent variable generating the treatment choice
- η is measurement error.
- For different levels of dependence between τ and ε_V, and different weights on Z, f_1, f_2 and on the scale of measurement error, Q can be a better predictor of D than f_1, f_2 or even f_1, f_2, Z.

- However, in general, $(Y_1, Y_0) \not\perp\!\!\!\perp D \mid Q$ because Q is an imperfect proxy for the combinations of f_1 and f_2 entering Y_1 and Y_0.
- Thus conditioning on Q can produce a better fit for D but greater bias for the treatment parameters.

Consider the following example where Y is an outcome and I is an index

$$\begin{aligned} Y &= \theta + \varepsilon_Y \\ I &= \theta + \varepsilon_I \end{aligned}$$

where

$$\theta \perp\!\!\!\perp (\varepsilon_Y, \varepsilon_I),$$
$$\varepsilon_Y \perp\!\!\!\perp \varepsilon_I.$$

Obviously $Y \perp\!\!\!\perp I \mid \theta$.

- Suppose instead that we have a candidate conditioning variable $Q = \alpha_\theta \theta + \eta + \tau$.
- Suppose that all variables are normal with zero mean and are mutually independent.
- Then we may write

$$I = \pi_I Q + \varepsilon_Q$$

where

$$\pi_I = \frac{\alpha_\theta \sigma_\theta^2 + \sigma_{\tau,\varepsilon_I}}{\alpha_\theta^2 \sigma_\theta^2 + \sigma_\eta^2 + \sigma_\tau^2}.$$

- It is assumed that η is independent of all other error components on the right hand sides of the equations for Q, I and y.
- From normal regression theory we know that conditioning is equivalent to residualizing.
- Constructing the residuals we obtain

$$I - \pi_I Q = \theta\left(1 - \alpha_\theta \pi_I\right) + \varepsilon_I - \pi_I\left(\eta + \tau\right).$$

- By a parallel argument

$$Y - \pi_Y Q = \theta\left(1 - \alpha_\theta \pi_Y\right) + \varepsilon_Y - \pi_Y\left(\eta + \tau\right)$$

$Y \perp\!\!\!\perp I \mid Q$ requires that $I - \pi_I Q$ and $Y - \pi_Y Q$ be uncorrelated, which in general does not happen.

Conclusion

- Letting the dependence between τ and ε_I get large, and setting α_θ to suitable values, we can predict I better (in the sense of R^2) with Q than with θ.
- Letting $D = 1(I > 0)$ produces a simple version of the example because better prediction of I produces better prediction of D.

Comparable to $\beta_1$and β_0 above, we can define

$$\beta_1' = \frac{\alpha_{V1}\alpha_{11}\sigma_{f_1}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \sigma_{\varepsilon_V}^2}}$$

$$\beta_0' = \frac{\alpha_{V1}\alpha_{01}\sigma_{f_1}^2}{\sqrt{\alpha_{V1}^2\sigma_{f_1}^2 + \sigma_{\varepsilon_V}^2}}.$$

Condition 1 *The bias produced by using matching to estimate TT is smaller in absolute value for any given p when the new information set* $\sigma\left(I_E'\right)$ *is used if*

$$|\beta_0| > |\beta_0'| .$$

Condition 2 *The bias produced by using matching to estimate ATE is smaller in absolute value for any given p when the new information set* $\sigma\left(I_E'\right)$ *is used if*

$$|\beta_1(1-p) + \beta_0 p| > |\beta_1'(1-p) + \beta_0' p| .$$

Condition 3 *The bias produced by using matching to estimate MTE is smaller in absolute value for any given p when the new information set $\sigma\left(I'_E\right)$ is used if*

$$\begin{aligned} & \left| M(p)\left[\beta_1\left(1-p\right)+\beta_0 p\right]-\Phi^{-1}\left(1-p\right)\left[\beta_1-\beta_0\right]\right| \\ > & \left| M(p)\left[\beta'_1\left(1-p\right)+\beta'_0 p\right]-\Phi^{-1}\left(1-p\right)\left[\beta'_1-\beta'_0\right]\right| \end{aligned}$$

Proof of Condition 1

Suppose

$$\beta_0 = \underbrace{\frac{\alpha_{V1}\alpha_{01}\sigma^2_{f_1} + (\alpha^2_{V2})\left(\frac{\alpha_{02}}{\alpha_{V2}}\right)\sigma^2_{f_2}}{\sqrt{\alpha^2_{V1}\sigma^2_{f_1} + \alpha^2_{V2}\sigma^2_{f_2} + \sigma^2_{\varepsilon_V}}}}_{\text{when } f_2 \text{ is in information set}} > \underbrace{\frac{\alpha_{V1}\alpha_{01}\sigma^2_{f_1}}{\sqrt{\alpha^2_{V1}\sigma^2_{f_1} + \sigma^2_{\varepsilon_V}}}}_{\substack{\text{when } f_2 \text{ is not} \\ \text{in information set}}} = \beta_0'.$$

When $\left(\frac{\alpha_{02}}{\alpha_{V2}}\right) = 0$, $\beta_0 < \beta_0'$.

$$\frac{\partial \beta_0}{\partial\left(\frac{\alpha_{02}}{\alpha_{V2}}\right)} = \frac{\alpha^2_{V2}\sigma^2_{f_2}}{\sqrt{\alpha^2_{V1}\sigma^2_{f_1} + \alpha^2_{V2}\sigma^2_{f_2} + \sigma^2_{\varepsilon_V}}} > 0.$$

There is some critical value α_{02}^{*} beyond which $\beta_0 > \beta_0'$

Assume

$$\begin{aligned} \alpha_{01} &= \alpha_{V_1} = \alpha_{V_2} = 1 \\ \alpha_{02} &= \alpha_{12} = 1 \\ \alpha_{11} &= 2 \end{aligned}$$

Return to Text

$$
\begin{aligned}
Y_j^* &= \mu_j + U_j \\
U_j &= \alpha_{j1} f_1 + \alpha_{j2} f_2 + \varepsilon_j \ , \ j = 0, 1 \\
Y_j &= 1 \text{ if } Y_j^* \geq 0, \quad = 0 \text{ otherwise,}
\end{aligned}
$$

People receive treatment according to the rule

$$
\begin{aligned}
V &= \mu_V + U_V \\
U_V &= \alpha_{V_1} f_1 + \alpha_{V_2} f_2 + \varepsilon_V \\
D &= 1 \text{ if } V \geq 0, \quad = 0 \text{ otherwise;}
\end{aligned}
$$

$$
f_1 \perp\!\!\!\perp f_2, \ \varepsilon_0 \perp\!\!\!\perp \varepsilon_1 \perp\!\!\!\perp \varepsilon_V, \qquad (f_1, f_2) \perp\!\!\!\perp (\varepsilon_0, \varepsilon_1, \varepsilon_V)
$$

The effect of treatment is given by:

$$\Delta_1 (I_E) = \frac{\Pr(Y_1 = 1, D = 1 | I_E)}{\Pr(Y_0 = 1, D = 1 | I_E)}.$$

A second definition works with odds ratios:

$$\Delta_2 (I_E) = \frac{\frac{\Pr(Y_1 = 1, D = 1 | I_E)}{\Pr(Y_1 = 0, D = 1 | I_E)}}{\frac{\Pr(Y_0 = 1, D = 1 | I_E)}{\Pr(Y_0 = 0, D = 1 | I_E)}}$$

One could also work with $\log \Delta$. Under the null hypothesis of no effect of treatment $\Delta_1 = \Delta_2 = 1$.

$$\widehat{\Delta}_1 (I_E) = \frac{\Pr(Y_1 = 1, D = 1 | I_E)}{\Pr(Y_0 = 1, D = 0 | I_E)}.$$

The denominator replaces the desired probability

$$\Pr(Y_0 = 1, D = 1 | I_E)$$

by

$$\Pr(Y_0 = 1, D = 0 | I_E).$$

Under the Null Hypothesis of no "real" effect of treatment

$$\mu_1 = \mu_0 = \mu$$
$$F_{U_1} = F_{U_0} = F_U$$

can be generated by

$$\alpha_{11} = \alpha_{01} = \alpha_1$$
$$\alpha_{12} = \alpha_{02} = \alpha_2$$
$$F_{\varepsilon_1} = F_{\varepsilon_0} = F_{\varepsilon}$$

Assume initially that

$$I_E = \{f_1, f_2\}.$$

$$\widehat{\Delta}_1(I_E) = \frac{\Pr(Y_1 = 1, D = 1 | f_1, f_2)}{\Pr(Y_0 = 1, D = 0 | f_1, f_2)} = \frac{\Pr(Y_1 = 1 | f_1, f_2)}{\Pr(Y_0 = 1 | f_1, f_2)} = \Delta_1(I_E).$$

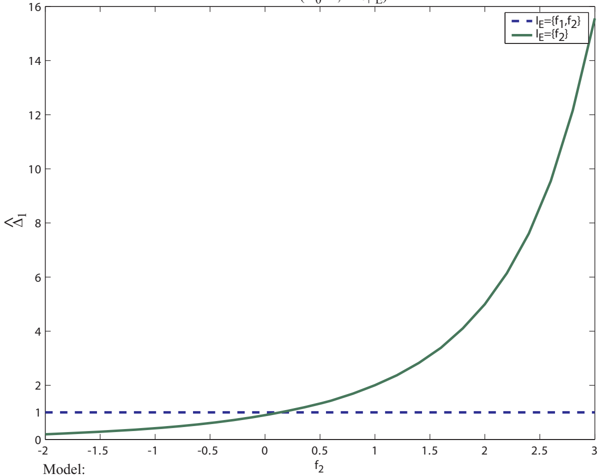
In general:

$$\Delta_1(I_E) = \frac{\Pr(Y_1 = 1, D = 1 | I_E)}{\Pr(Y_0 = 1, D = 1 | I_E)} \neq \frac{\Pr(Y_1 = 1, D = 1 | I_E)}{\Pr(Y_0 = 1, D = 0 | I_E)} = \widehat{\Delta}_1(I_E)$$

Figure 10.--Estimated Effect of Treatment under Different Information Sets

No Effect of Treatment and $\alpha_{v2}=1$

$$\hat{\Delta}_1 = \frac{\Pr(Y_1=1, D=1 | I_E)}{\Pr(Y_0=1, D=0 | I_E)}$$



Model:

$V = -1+f_1+f_2+\varepsilon_v$ $\quad \varepsilon_v \sim N(0,1)$

$Y^*_1 = -1+f_1+f_2+\varepsilon_1$ $\quad \varepsilon_1 \sim N(0,1)$

$Y^*_0 = -1+f_1+f_2+\varepsilon_0$ $\quad \varepsilon_0 \sim N(0,1)$

$Y_1 = 1(Y^*_1>0)$ $\quad f_1 \sim N(0,1)$

$Y_0 = 1(Y^*_0>0)$ $\quad f_2 \sim N(0,1)$

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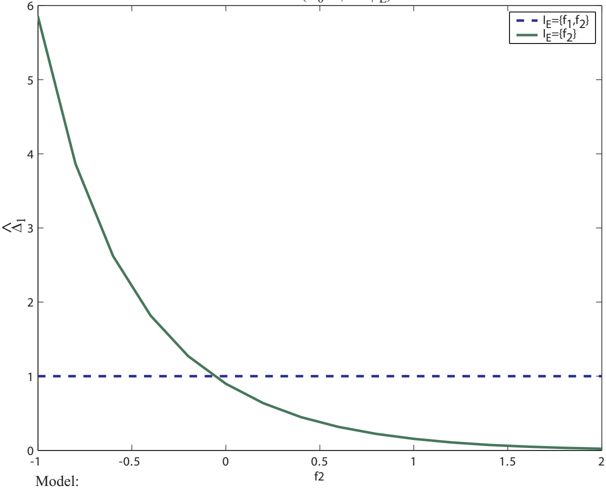
Figure 11.--Estimated Effect of Treatment under Different Information Sets

No Effect of Treatment and $\alpha_{V2}=-1$

$$\hat{\Delta}_1 = \frac{Pr(Y_1=1,D=1|I_E)}{Pr(Y_0=1,D=0|I_E)}$$



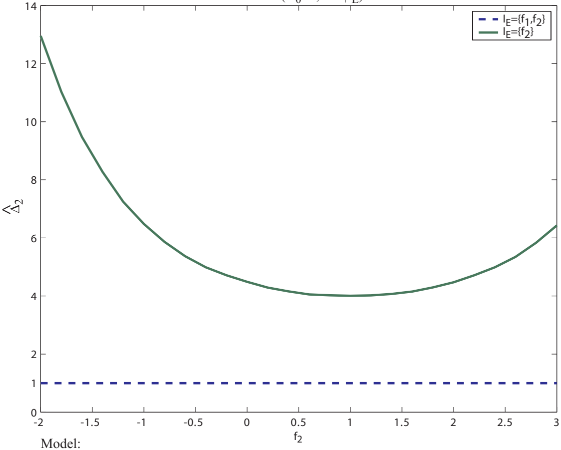
Model:

$V=-1+f_1-f_2+\varepsilon_V$	$\varepsilon_V \sim N(0,1)$
$Y^*_1=-1+f_1+f_2+\varepsilon_1$	$\varepsilon_1 \sim N(0,1)$
$Y^*_0=-1+f_1+f_2+\varepsilon_0$	$\varepsilon_0 \sim N(0,1)$
$Y_1=1(Y^*_1>0)$	$f_1 \sim N(0,1)$

Figure 12.--Estimated Effect of Treatment under Different Information Sets

No Effect of Treatment and $\alpha_{V2}=1$

$$\hat{\Delta}_2=\frac{\dfrac{\Pr(Y_1=1,D=1|I_E)}{\Pr(Y_1=0,D=1|I_E)}}{\dfrac{\Pr(Y_0=1,D=0|I_E)}{\Pr(Y_0=0,D=0|I_E)}}$$



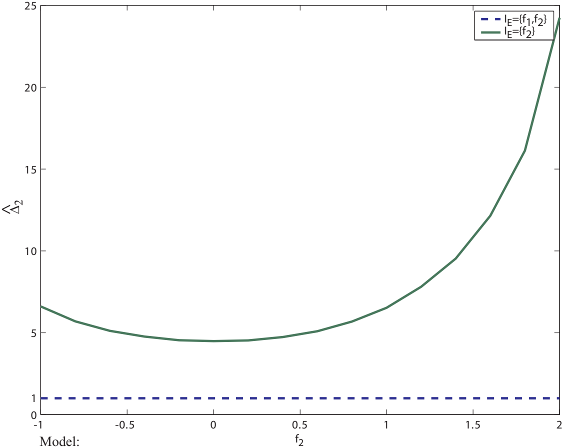
Model:

$V= -1+f_1+f_2+\varepsilon_V$	$\varepsilon_V\sim N(0,1)$
$Y^*_1=-1+f_1+f_2+\varepsilon_1$	$\varepsilon_1\sim N(0,1)$
$Y^*_0=-1+f_1+f_2+\varepsilon_0$	$\varepsilon_0\sim N(0,1)$
$Y_1=1(Y^*_1>0)$	$f_1\sim N(0,1)$
$Y_0=1(Y^*_0>0)$	$f_2\sim N(0,1)$

Figure 13.--Estimated Effect of Treatment under Different Information Sets

No Effect of Treatment and α_{V2}=-1

$$\hat{\Delta}_2 = \frac{\dfrac{\Pr(Y_1=1,D=1|I_E)}{\Pr(Y_1=0,D=1|I_E)}}{\dfrac{\Pr(Y_0=1,D=0|I_E)}{\Pr(Y_0=0,D=0|I_E)}}$$



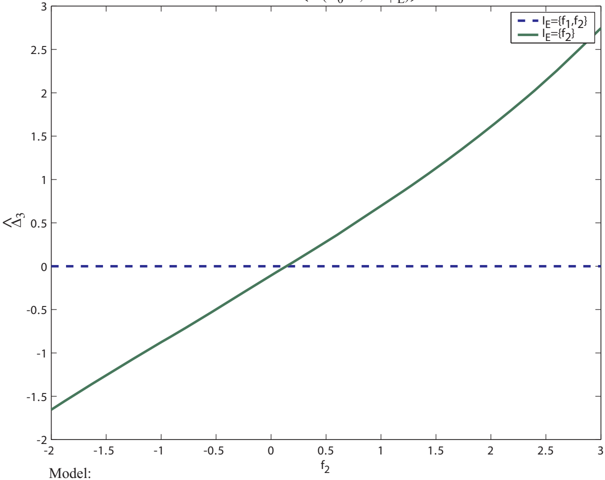
Model:

$V= -1+f_1-f_2+\varepsilon_V$	$\varepsilon_V \sim N(0,1)$
$Y^*_1=-1+f_1+f_2+\varepsilon_1$	$\varepsilon_1 \sim N(0,1)$
$Y^*_0=-1+f_1+f_2+\varepsilon_0$	$\varepsilon_0 \sim N(0,1)$
$Y_1=1(Y^*_1>0)$	$f_1 \sim N(0,1)$
$Y_0=1(Y^*_0>0)$	$f_2 \sim N(0,1)$

Figure 14.--Estimated Effects of Treatment under Different Information Sets

No Effect of Treatment and $\alpha_{v2}=1$

$$\hat{\Delta}_3 = \text{Log}\left(\frac{\Pr(Y_1=1, D=1 | I_E)}{\Pr(Y_0=1, D=0 | I_E)}\right)$$



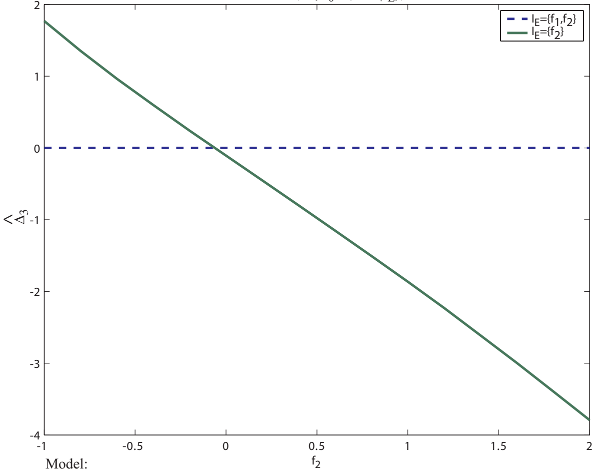
Model:

$V= -1+f_1+f_2+\varepsilon_v$	$\varepsilon_v \sim N(0,1)$
$Y^*_1=-1+f_1+f_2+\varepsilon_1$	$\varepsilon_1 \sim N(0,1)$
$Y^*_0=-1+f_1+f_2+\varepsilon_0$	$\varepsilon_0 \sim N(0,1)$
$Y_1=1(Y^*_1>0)$	$f_1 \sim N(0,1)$
$Y_0=1(Y^*_0>0)$	$f_2 \sim N(0,1)$

Figure 15.--Estimated Effects of Treatment under Different Information Sets

No Effect of Treatment and α_{V2}=-1

$$\hat{\Delta}_3 = \text{Log}\left(\frac{\Pr(Y_1=1,D=1|I_E)}{\Pr(Y_0=1,D=0|I_E)}\right)$$



Model:

$V = -1 + f_1 - f_2 + \varepsilon_V$ $\quad \varepsilon_V \sim N(0,1)$

$Y^*_1 = -1 + f_1 + f_2 + \varepsilon_1$ $\quad \varepsilon_1 \sim N(0,1)$

$Y^*_0 = -1 + f_1 + f_2 + \varepsilon_0$ $\quad \varepsilon_0 \sim N(0,1)$

$Y_1 = 1(Y^*_1 > 0)$ $\quad f_1 \sim N(0,1)$

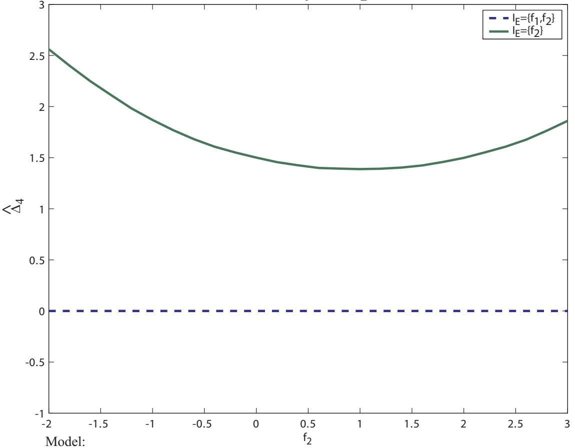
$Y_0 = 1(Y^*_0 > 0)$ $\quad f_2 \sim N(0,1)$

$D = 1(V > 0)$

Figure 16.--Estimated Effect of Treatment under Different Infomation Sets

No Effect of Treatment and $\alpha_{v2}=1$

$$\hat{\Delta}_4 = \text{Log}\left(\frac{\frac{\Pr(Y_1=1,D=1|I_E)}{\Pr(Y_1=0,D=1|I_E)}}{\frac{\Pr(Y_0=1,D=0|I_E)}{\Pr(Y_0=0,D=0|I_E)}} \right)$$



Model:

$V = -1+f_1+f_2+\varepsilon_V$ $\quad \varepsilon_V \sim N(0,1)$

$Y^*_1 = -1+f_1+f_2+\varepsilon_1$ $\quad \varepsilon_1 \sim N(0,1)$

$Y^*_0 = -1+f_1+f_2+\varepsilon_0$ $\quad \varepsilon_0 \sim N(0,1)$

$Y_1 = 1(Y^*_1>0)$ $\quad f_1 \sim N(0,1)$

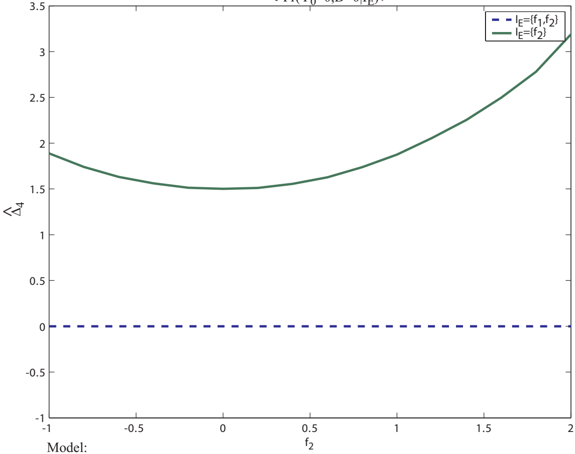
$Y_0 = 1(Y^*_0>0)$ $\quad f_2 \sim N(0,1)$

$D = 1(V>0)$

Figure 17.--Estimated Effects of Treatment under Different Information Sets

No Effect of Treatment and $\alpha_{v2}=-1$

$$\hat{\Delta}_4=\text{Log}\left(\frac{\frac{\Pr(Y_1=1,D=1|I_E)}{\Pr(Y_1=0,D=1|I_E)}}{\frac{\Pr(Y_0=1,D=0|I_E)}{\Pr(Y_0=0,D=0|I_E)}}\right)$$



Model:

$V= -1+f_1-f_2+\varepsilon_v$ $\quad\varepsilon_v\sim N(0,1)$

$Y^*_1=-1+f_1+f_2+\varepsilon_1$ $\quad\varepsilon_1\sim N(0,1)$

$Y^*_0=-1+f_1+f_2+\varepsilon_0$ $\quad\varepsilon_0\sim N(0,1)$

$Y_1=1(Y^*_1>0)$ $\quad f_1\sim N(0,1)$

$Y_0=1(Y^*_0>0)$ $\quad f_2\sim N(0,1)$

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Consider a general model of the form:

$$\begin{aligned} Y_1 &= \mu_1 + U_1 \\ Y_0 &= \mu_0 + U_0 \\ V &= \mu_V(Z) + U_V \\ D &= 1 \text{ if } V \geq 0, \quad = 0 \text{ otherwise} \\ Y &= DY_1 + (1-D)Y_0. \end{aligned}$$

where

$$\begin{aligned} (U_1, U_0, U_V)' &\sim N(0, \Sigma) \\ var(U_i) &= \sigma_i^2 \\ cov(U_i, U_j) &= \sigma_{ij} \\ i = 0; j = 1 & \\ cov(U_1, V) &= \sigma_{1V} \\ cov(U_0, V) &= \sigma_{0V} \end{aligned}$$

Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the pdf and the cdf of a standard normal random variable. Then, the propensity score for this model is given by:

$$\begin{aligned}\Pr(V > 0 | \mu_V(Z)) &= P(\mu_V(Z)) = \Pr(U_V > -\mu_V(Z)) = p \\ &= 1 - \Phi\left(\frac{-\mu_V(Z)}{\sigma_V}\right) = p\end{aligned}$$

so

$$\frac{-\mu_V(Z)}{\sigma_V} = \Phi^{-1}(1-p).$$

Since the event $\left(V \gtreqqless 0, P\left(\mu_V\left(Z\right)\right) = p \right)$ can be written as

$$\frac{U_V}{\sigma_V} \lesseqqgtr -\frac{\mu_V\left(Z\right)}{\sigma_V}$$
$$\frac{U_V}{\sigma_V} \lesseqqgtr \Phi^{-1}\left(1 - p\right)$$

we can write the conditional expectations required to get the biases as a function of p.

For U_1 :

$$E\left(U_1|V>0,P\left(\mu_V\left(Z\right)\right)=p\right)$$
$$=\frac{\sigma_{1V}}{\sigma_V}E\left(\frac{U_V}{\sigma_V}|\frac{U_V}{\sigma_V}>\frac{-\mu_V\left(Z\right)}{\sigma_V},P\left(\mu_V\left(Z\right)\right)=p\right)$$
$$=\frac{\sigma_{1V}}{\sigma_V}E\left(\frac{U_V}{\sigma_V}|\frac{U_V}{\sigma_V}>\Phi^{-1}\left(1-p\right)\right)$$
$$=\beta_1 M_1(p)$$

$$E\left(U_1|V=0,P\left(\mu_V\left(Z\right)\right)=p\right)$$
$$=\frac{\sigma_{1V}}{\sigma_V}E\left(\frac{U_V}{\sigma_V}|\frac{U_V}{\sigma_V}=\frac{-\mu_V\left(Z\right)}{\sigma_V},P\left(\mu_V\left(Z\right)\right)=p\right)$$
$$=\frac{\sigma_{1V}}{\sigma_V}E\left(\frac{U_V}{\sigma_V}|\frac{U_V}{\sigma_V}=\Phi^{-1}\left(1-p\right),P\left(\mu_V\left(Z\right)\right)=p\right)$$
$$=\beta_1\Phi^{-1}(1-p)$$

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Where

$$\beta_1 = \frac{\sigma_{1V}}{\sigma_V}$$

Similarly for U_0 :

$$\begin{aligned} E\left(U_0 | V > 0, P\left(\mu_V\right) = p\right) &= \beta_0 M_1(p) \\ E\left(U_0 | V < 0, P\left(\mu_V\right) = p\right) &= \beta_0 M_0(p) \\ E\left(U_0 | V = 0, P\left(\mu_V\right) = p\right) &= \beta_0 \Phi^{-1}\left(1 - p\right) \end{aligned}$$

Where

$$\beta_0 = \frac{\sigma_{0V}}{\sigma_V}.$$

and

$$\begin{aligned} M_1(p) &= \frac{\phi\left(\Phi^{-1}(1-p)\right)}{p} \\ M_0(p) &= -\frac{\phi\left(\Phi^{-1}(1-p)\right)}{(1-p)} \end{aligned}$$

are inverse Mills ratio terms.

Substituting these into the expressions for the biases

$$\begin{aligned} \text{Bias } TT(p) &= \beta_0 M_1(p) - \beta_0 M_0(p) \\ &= \beta_0 M(p) \end{aligned}$$

$$\begin{aligned} \text{Bias } ATE(p) &= \beta_1 M_1(p) - \beta_0 M_0(p) \\ &= M(p)\left(\beta_1(1-p) + \beta_0 p\right) \end{aligned}$$

$$\begin{aligned} \text{Bias } MTE &= \beta_1 M_1(p) - \beta_0 M_0(p) \\ &\quad -\beta_1 \Phi^{-1}(1-p) + \beta_0 \Phi^{-1}(1-p) \\ &= M(p)\left(\beta_1(1-p) + \beta_0 p\right) \\ &\quad -\Phi^{-1}(1-p)\left[\beta_1 - \beta_0\right]. \end{aligned}$$

where

$$M(p) = M_1(p) - M_0(p) = \frac{\phi(\Phi^{-1}(1-p))}{p(1-p)}$$