Identification of Social Interactions

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Abstract

While interest in social determinants of individual behavior has led to a rich theoretical literature and many efforts to measure these influences, a mature "social econometrics" has yet to emerge. This chapter provides a critical overview of the identification of social interactions. We consider linear and discrete choice models as well as social networks structures. We also consider experimental and quasi-experimental methods. In addition to describing the state of the identification literature, we indicate areas where additional research is especially needed and suggest some directions that appear to be especially promising.

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Like other tyrannies, the tyranny of the majority was at first, and is still vulgarly, held in dread, chiefly as operating through the acts of the public authorities. But reflecting persons perceived that when society is itself the tyrant-society, collectively over the separate individuals who compose it-its means of tyrannising are not restricted to the acts which it may do by the hands of its political functionary. Society can and does execute its own mandates: and if it issues wrong mandates instead or right, or mandates at all in things with which it ought not to meddle, it practices a social tyranny more formidable than many kinds of political oppression, since, though not usually upheld by such extreme penalties, it leaves fewer means of escape, penetrating more deeply into the details of life, and enslaving the soul itself. Protection, therefore, against the tyranny of the magistrate is not enough: there needs protection also against the tyranny of prevailing opinion and feeling; against the tendency of society to impose, by means other than civil penalties, its own ideas and rules of conduct on those who dissent from them; to fetter the development and, if possible prevent the formation, of any individuality not in harmony with its ways, and compel all characters to themselves upon the model of its own.

John Stuart Mill, *On Liberty* (1859)[1]

[1] *On Liberty and other Writings,* S. Collini ed., Cambridge: Cambridge University Press, p. 8.

1 Introduction

This chapter explores identification problems that arise in the study of social economics. We survey some of the existing empirical work, but do so in the context of different identification strategies. Our concern is in understanding general conditions under which the finding of evidence of social interactions is possible and when it is not; we therefore do not focus on particular contexts. A valuable complement to our chapter is Epple and Romano (forthcoming) who provide an integration of theoretical, econometric and empirical work on the specific question of peer effects in education.

2 Decision making in group contexts

Our baseline model of social interactions studies the joint behavior of individuals who are members of a common group g. The population size of a group is denoted as n_g. Our objective is to probabilistically describe the individual choices of each i, ω_i. Choices are made from the elements of some set of possible behaviors Ω_{ig}. This set is both individual- and group-specific, though the econometric literature has typically not exploited the fact that different groups may offer different choices. This is an unexplored and interesting possibility. For each i, ω_{-ig} denotes the choices of others in the group, which are one possible source of social interactions. From the perspective of econometric evaluation, it is useful to distinguish between five forms of influences on individual choices. These influences have different implications for how one models the choice problem. These forms are:

x_i	An R-vector of observable (to the modeler) individual-specific characteristics;
y_g	An S-vector of observable (to the modeler) group-specific characteristics;
$\mu_i^e(\omega_{-ig})$	A probability measure, unobservable (to the modeler), that describes the beliefs individual i possesses about behaviors of others in the group;[4]
ε_i	A vector of random individual-specific characteristics describing i, unobservable to the modeler; and
α_g	A vector of random group-specific characteristics, unobservable to the modeler.

[4]Li and Lee (2009) consider the use of survey data to render beliefs observables; we discuss their work in section 5.vi. For purposes of the elucidation of the basic theory of choice in the presence of social interactions, we focus on the case where beliefs are latent variables.

The distinction between observable and unobservable determinants of individual choices corresponds to the standard difference between observable and unobservable heterogeneity in econometrics, or even more crudely, between the data ω_{ig}, x_i, y_g and the full range of factors affecting choices. Among the different sources of unobserved heterogeneity, $\mu_i^e(\omega_{-ig})$ functions very differently from ε_i and α_g since the logic of the choice problem determines the structure of $\mu_i^e(\omega_{-ig})$ in ways that do not apply to the other terms, which are shocks from the perspective of the modeler.

Individual choices ω_{ig} are characterized as representing the maximization of some payoff function V,

$$\omega_{ig} \in \underset{\lambda \in \Omega_{ig}}{\operatorname{argmax}} V(\lambda, x_i, y_g, \mu_i^e(\omega_{-ig}), \varepsilon_i, \alpha_g). \tag{1}$$

The decision problem facing an individual, a function of preferences (embodied in the specification of V); constraints (embodied in the specification of Ω_{ig}); and beliefs (embodied in the specification of $\mu_i^e(\omega_{-ig})$).[5] Thus it is based on completely standard microeconomic reasoning. While the equilibria of these models can exhibit a range of interesting properties, such as multiple equilibria and bifurcations of the equilibrium properties of the environment around certain parameter values, these are properties of equilibria generated by this standard choice framework.[6]

[5] Throughout, probability measures are denoted by $\mu(\cdot)$.

[6] See surveys by Blume and Durlauf (2001), Brock and Durlauf (2001b) and Durlauf and Ioannides (2010) for overviews of these and other theoretical features of these models as well as the bibliographies of these papers for specific theoretical contributions.

As suggested above this choice model with social interactions is closed by the assumptions under which $\mu_i^e(\omega_{-ig})$ is determined. Without some structure on these beliefs, the model is consistent with any observed pattern of undominated choices. The standard assumption in the theoretical and econometric literatures, which we follow, closes the model by imposing an equilibrium condition: self-consistency between subjective beliefs $\mu_i^e(\omega_{-ig})$ and the objective conditional probabilities of the behaviors of others given i's information set F_i,

$$\mu_i^e(\omega_{-ig}) = \mu(\omega_{-ig}|F_i). \tag{2}$$

The requirement in (2) is usually called self-consistency in the social interactions literature and is nothing more than an equilibrium condition, from the perspective of empirical analysis. We assume that for each i, F_i consists of, for all $(x_j)_{j\in g}$, y_g, ε_i and α_g. In other words, each agent knows his own characteristics x_i, as well as those of others in the group, the observed and unobserved group-level characteristics of his group (and of other groups), and his idiosyncratic error. Agents do not observe the ε_j's of others.

From the perspective of modeling individual behaviors, it is typically assumed that agents do not account for the effect of their choices on the decisions of others via expectations formation. The equilibrium in this model can be seen as a Bayes-Nash equilibrium of a simultaneous-move incomplete-information game. The individual decisions as described by

$$\omega_{ig} = \psi(x_i, y_g, \mu(\omega_{-ig}|F_i), \varepsilon_i, \alpha_g). \tag{3}$$

Existence of an equilibrium for the group-wide vector of choices ω_g is equivalent to establishing that there exists a joint probability measure of these choices such that (3) is consistent with this joint probability measure. In applications in the literature, this is typically assured by a standard fixed point theorem, e.g. Brock and Durlauf (2001a), Cooley (2008). Notice that it is possible for y_g and $\mu(\omega_{-ig}|F_i)$ to appear in equation (1) but not in equation (3). In this case, group behaviors and characteristics act as externalities but do not influence individual behaviors. This distinction is discussed in Cooper and John (1988). From the perspective of the empirical study of social interactions, equation (3) has been the main object of interest. Typically, (3) is assumed to exhibit a form of supermodularity in the sense that the redistribution of probability mass of $\mu(\omega_{-ig}|F_i)$ towards larger (in an element-by-element pairwise comparison sense) vectors of choices of others increases ω_{ig}. Milgrom and Roberts (1990) and Vives (1990) launched the now immense literature in economics on how supermodularity affects equilibrium outcomes for a wide range of environments; ideas from this literature often indirectly appear in the empirical social interactions literature, but with the exception of Aradillas-Lopez (2009), discussed in section 5.vi.d, this literature has been underutilized in the study of identification.

The distinction between y_g and $\mu(\omega_{-ig}|F_i)$ is important in the social econometrics literature. Following Manski (1993), the former is known as a contextual effect whereas the latter (including the case of perfect foresight) is known as an endogenous effect. The importance of this distinction is that contextual interactions involve the interactions of predetermined (from the perspective of the model) attributes of one agent affecting another whereas endogenous interactions allow for the possibility of simultaneity of interactions in individual outcomes.

To see how identification problems arise in a social interactions explanation of inequality, consider the stylized fact that the probability that a student graduates from high school is negatively associated with growing up in a poor neighborhood. Among the many possible explanations for this bivariate relationship are the following:

1. Heterogeneity in educational outcomes is determined by family-specific investment. Poor parents, following Becker and Tomes (1979) or Loury (1981), invest fewer resources in their children's education. If parental income is a sufficient statistic for parental investment, then the mechanism for lower graduation rates among poorer individuals is observable, constituting an element of x_i. The low graduation rate/poor neighborhood relationship is due to the interfamily correlation of low incomes that defines a poor neighborhood.

2. Effort choices by students depend on their assessments of the payoff to education. Poor neighborhoods contain distributions of role models that adversely affect educational choices. If a poor neighborhood tends to contain individuals whose incomes are relatively low compared to educational levels (as would occur via self-selection of lower incomes into poor neighborhoods), then the payoff to education may appear less attractive to high school students and thereby affect effort in high school as well as graduation decisions.[7] Relative to our candidate explanations, observed occupations and educational levels of adults in a community are observable, then the low graduation/poor neighborhood relationship are observable and included in y_g. This is an example of how contextual effects can link poverty and low graduation rates.

3. High school graduation decisions are influenced by the choices of peers because of a direct desire to conform to the behaviors of others. Poorer neighborhoods have the feature that low values of the $\mu(\omega_{-ig}|F_i)$ are self-reinforcing, whereas high values of the $\mu(\omega_{-ig}|F_i)$ are self-reinforcing for more affluent neighborhoods. Thus endogenous social interactions can explain the relationship, although one has to be careful to explain why the peer interactions lead to lower graduation rates in poorer neighborhoods. We can offer three possible explanations: (i) The unique equilibrium could be characterized by a social multiplier that magnifies the consequences of income differences. (ii) In the spirit of Brock and Durlauf (2001a), there could be multiple equilibria in low income neighborhoods but not in more affluent neighborhoods, because the poor may face lower marginal returns to education, which would magnify the influence of peer interactions relative to education returns in the equilibrium decision rule. (iii) There could be multiple equilibria for both high and low income neighborhoods, and an (unmodeled) selection mechanism could favor different equilibria in different neighborhoods in a manner correlated with income.

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[7] Streufert (2002) formalizes this type of idea and shows that the intuitive story just given is in fact oversimplified in the sense that the mapping from neighborhood levels of parental education/outcome relationships to student assessments of the returns to education may not lead to lower estimates of the returns in poorer neighborhoods but the story we describe is possible.

4. Parents transmit a host of skills to their children. Following Cunha and Heckman (2007) and Heckman (2007), suppose that poorer parents tend to have lower cognitive and non-cognitive skills which help to explain their lower socioeconomic status and are in turn transmitted to their children. This would imply that correlations among ε_i are the reason why poor neighborhoods have lower graduation rates. This is an example of correlated unobservables and is suggestive of the standard self-selection problem in econometrics.

5. Graduation decisions are affected by the quality of schools, where quality involves a host of factors ranging from the distribution of teacher ability to safety. Poorer neighborhoods have lower unmeasured school quality, then neighborhood poverty is a proxy for a low value of α_g, i.e. the graduation finding is caused by an unobserved group effect.

The bottom line is that each of the factors we have identified as determinants of individual outcomes can produce a relationship between individual outcomes and neighborhood characteristics, even when the mechanism is individual and not socially based. Of course, no economist would ever consider arguing that the fact that poor neighborhoods are associated with lower graduation rates speaks to any of these mechanisms per se. The identification question is whether these different explanations are distinguishable given the sorts of data that are available for analysis. It is this question that motivates the methods we describe.

We close this section with the observation that the behavioral model (3) cannot be nonparametrically identified without additional assumptions on structure. One reason for this is the possible existence of the unobserved group effects α_g which cannot be disentangled from elements of y_g: Formally, there exist classes of models such that for any proposed function $\phi(\cdot)$ and associated choices of unobservables α_g, one can choose an alternative function ϕ' and alternative choice of unobservables α'_g such that all probability statements about the observables are identical. Brock and Durlauf (2007) show this for the binary choice model with social interactions which contains far more structure than (3), a model we will discuss in section 5 below.

Nonparametric identification may also fail even if one rules out unobserved group effects; Manski (1993) Proposition 3 gives various cases under which nonparametric identification fails for a version of the individual decision function eq. (3). Specifically, Manski studies an environment in which the expected value of each person's choice is determined by[8]

$$E(\omega_{ig}|y_g, x_i) = \phi\Big(E(\omega_{ig}|y_g), x_i\Big). \tag{4}$$

[8]Appendix 1 contains an example of a model where this is a Bayes-Nash equilibrium condition.

Each individual is small relative to the population, producing the rational expectations equilibrium condition

$$\mathrm{E}(\omega_{ig}|y_g) = \int \mathrm{E}(\omega_{ig}|y_g, x)\, dF_{x|y_g} \tag{5}$$

where $F_{x|y_g}$ is the conditional distribution function of x_i in group g given y_g. To say each individual is small is to say that knowledge of his own x_i does not affect the distribution function of individual characteristics within his group in a nonnegligible way. For the joint model (4) and (5) one set of conditions under which nonparametric identification fails are 1) the solution to equation (5) is unique and 2) x_i is functionally dependent on y_g. It is evident under these conditions that one cannot nonparametrically identify the separate effects of x_i and y_g in determining ω_{ig} since differences in outcomes between two individuals with differences in x_i can always be attributed to the differences in their associated values of y_g.

To make this example concrete, suppose that x_i is an individual's income and y_g is the mean income of a residential neighborhood. Functional dependence would occur if neighborhoods were perfectly segregated by income, i.e. no neighborhood contained individuals with different incomes. For this case, it would impossible to distinguish the roles of individual and neighborhood incomes on outcomes since they would coincide. Less trivially, suppose that neighborhoods are fully segregated by income, which means that the empirical supports of incomes across neighborhoods never intersect. Suppose that individual income has no direct effects on outcomes whereas average neighborhood income has a monotonic effect on equilibrium outcomes. In this case, one could not distinguish an effect of neighborhood incomes on outcomes from the case where individual incomes directly affect outcomes, but do so in a step function fashion, where the jumps coincide with income levels that define the lower endpoints of the neighborhood income supports.

Manski also shows that identification will fail when x_i and y_g are statistically independent. Non-identification follows from statistical independence because $E(\omega_{ig}|y_g)$ will not vary across groups, and so the effect cannot be distinguished from a constant term. An obvious example of this would occur if families were distributed across neighborhoods in such a way that each neighborhood had the same mean income in realization. Manski's result is in fact more general and is based on the observation that statistical independence implies that $E(\omega_{ig}|y_g) = \int \phi\Big(E(\omega_{ig}|y_g), x\Big) dF_x$, which by uniqueness means that $E(\omega_{ig}|y_g)$ must be independent of y_g.[9]

[9] The argument may be seen in Manski (1993, p. 539).

3 Linear Models of Social Interaction

i. Basic structure

a. description

Much of the empirical literature on social economics has involved variations of a general linear model, dubbed by Manski (1993) the linear-in-means model

$$\omega_{ig} = k + cx_i + dy_g + Jm^e_{ig} + \varepsilon_i \,, \tag{6}$$

where m^e_{ig} denotes the average behavior in the group, i.e.

$$m^e_{ig} = \frac{1}{n_g} \sum_{j \in g} \mathrm{E}(\omega_j | F_i) \ . \tag{7}$$

Following our definitions of the variables, note that k and J are scalars whereas c and d are R- and S- vectors, respectively.[10] Claims about social interactions are, from the econometric perspective, equivalent to statements about the values of d and J. The statement that social interactions matter is equivalent to the statement that at least some element of the union of the parameters in d and the scalar J are nonzero. The statement that contextual social interactions are present means that at least one element of d is nonzero. The statement that endogenous social interactions matter means that J is nonzero. In Manski's original formulation, $y_g = \bar{x}_g$, where $\bar{x}_g = \frac{1}{n_g} \sum_{j \in g} \mathbf{x}_j$ denotes the average across individuals i of individual characteristics x_i within a given group g, which explains the model's name. Regardless of whether they are equal, we assume that both y_g and $\bar{x}_g$ are observable to individuals, and discuss how to relax this below.

[10] Throughout, coefficient vectors such as c are row vectors whereas variable vectors such as x_i are column vectors.

We initially study the model under two assumptions on the errors. First we assume that the expected value of ε_i is 0, conditional on the information set $(x_i, \bar{x}_g, y_g, i \in g)$,[11]

$$\text{for each } g \text{ and } i \in g \quad E(\varepsilon_i | x_i, \bar{x}_g, y_g, i \in g) = 0. \tag{8}$$

Second we assume that

$$\begin{gathered} \text{for each } i, j, g, h \text{ such that } i \neq j \text{ or } g \neq h \\ \operatorname{cov}(\varepsilon_i \varepsilon_j | x_i, \bar{x}_g, y_g, i \in g, x_j, \bar{x}_h, y_h, j \in h) = 0. \end{gathered} \tag{9}$$

Equation (9) eliminates conditional covariation between the errors. The inclusion of the group memberships, e.g. $i \in g$ rules out some relationship between the identity of the group and model errors, thereby allowing us to treat groups as exchangeable.

[11] The conditioning argument $i \in g$ means that one is conditioning on the fact that i is a member of group g.

From equations (6) and (7), and assuming that each individual is small enough relative to the group that the effect of his knowledge of his own ε_i on m_{ig}^e can be ignored, equilibrium implies that each actor's expected average behavior will be equal to a common value. This common value is derived in appendix 1 and is described by

$$m_{ig}^e = m_g \equiv \frac{k + c\bar{x}_g + dy_g}{1 - J}. \tag{10}$$

This equation says that the individuals' expectations of average behavior in the group equal the average behavior of the group, and this in turn depends linearly on the average of the individual determinants of behavior, $\bar{x}_g$, and the contextual interactions that the group members experience in common, y_g. The condition $J < 1$, which is required for equation (10) to make sense, is guaranteed to hold in the game-model of appendix 1. There, J maps the marginal rate of substitution between private return and social conformity, a non-negative real number, into the interval $[0, 1)$.

b. reduced form

Substitution of (10) into (6) eliminates m_g and so provides a reduced form version of the linear in means model in that the individual outcomes are determined entirely by observables and the individual-specific error:

$$\omega_{ig} = \frac{k}{1-J} + cx_i + \frac{J}{1-J}c\bar{x}_g + \frac{d}{1-J}y_g + \varepsilon_i. \quad (11)$$

Much of the empirical literature has ignored the distinction between endogenous and contextual interactions, and has focused on this reduced form, i.e. focused on the regression

$$\omega_{ig} = \pi_0 + \pi_1 x_i + \pi_2 y_g + \varepsilon_i \, , \tag{12}$$

where the parameters π_0, π_1, π_2 are taken as the objects of interest in the empirical exercise. A comparison of (12) with (11) indicates how findings in the empirical literature that end with the reporting of π_0, π_1, π_2 inadequately address the task of fully characterizing the social interactions that are present in the data. For example, from the perspective of (12), the presence of social interactions is equivalent to $\pi_2 \neq 0$, whereas from the perspective of (6) this is neither necessary nor sufficient for social interactions to be present since $J = 0$ is neither necessary nor sufficient for $\pi_2 = 0$. To be clear, this observation does not mean that estimates of (12) are uninformative, rather that these estimates should be mapped to structural parameters in the sense of (6) when identification holds, and that if identification does not hold, then the informational limits of (12) in terms of distinguishing types of social interactions should be made explicit.

The reduced form version of the linear in means model illustrates some features of the structure that are of interest. First, the linear in means model limits the effects of reallocations of individuals across groups. To see this, suppose one thinks of each choice in the population as $\omega_{ig} = \phi(x_i, y_g) + \varepsilon_i$. Suppose that y_g is a scalar and that $y_g = \bar{x}_g$ this means that $\omega_{ig} = \phi(x_i, x_{-ig}) + \varepsilon_i$, where x_{-ig} denotes the vector of individual characteristics other than x_i among group g members with typical element x_{-ijg}. Finally, assume all groups are of equal size. Under the linear functional form (11), for all j, $\partial^2\phi/\partial x_i x_{-ijg} = 0$. This is the condition under which all allocations of individuals across groups produce the same expected population-wide average outcome for $\sum_{j\in g}\omega_{jg}$. This was first recognized in Becker's (1973) analysis of efficiency in the marriage market, in which groups are of size 2 and naturally extends to groups of any size. (See Durlauf and Seshadri (2003).) It is the case, extending an example of the type in Durlauf and Seshadri (2003), that if groups are of different sizes, the reallocation of individuals across them can affect average outcomes. This nonetheless does not diminish the qualitative point that the fact that all cross partial derivatives equal 0 in the reduced form of the linear in means model severely restricts the effects of reallocations of group memberships.

ii. instrumental variables and the reflection problem

We first consider the estimates of the regression coefficients for (6) under the expectations formation restriction (10). It is obvious that if $\bar{\omega}_g$ is projected against the union of elements of $\bar{x}_g$ and y_g, this produces the population mean m_g. Hence, we can proceed as if m_g is observable. Put differently, our identification arguments rely on the analogy principle which means that one works with population moments to construct identification arguments.[12] Since y_g appears in (10), it will not facilitate identification. As we shall see, identification via instrumental variables is determined by the informational content of $\bar{x}_g$ relative to y_g.

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[12] Goldberger (1991, p. 117) gives a concise description.

As first recognized by Manski (1993), identification can fail for the linear in means model when one focuses on the mapping from reduced form regression parameters to the structural parameters. This may be most easily seen under Manski's original assumption that $y_g = \bar{x}_g$. This means that every contextual effect is the average of a corresponding individual characteristic. In this case, equation (10) reduces to

$$m_g = \frac{k + (c + d)y_g}{1 - J}. \tag{13}$$

This means that the regressor m_g in equation (6) is linearly dependent on the other regressors, i.e. the constant and y_g. This linear dependence means that identification fails: the comovements of m_g and y_g are such that one cannot disentangle their respective influences on individuals. Manski (1993) named this failure the reflection problem. Metaphorically, if one observes that ω_{ig} is correlated with the expected average behavior in a neighborhood, (13) indicates it may be possible that this correlation is due to the fact that m_g may simply reflect the role of y_g in influencing individuals.

Under what conditions is this model identified? A necessary condition is that Manski's assumption that $y_g = \bar{x}_g$ is relaxed. This will allow for the possibility m_g is not linearly dependent on the constant and y_g. The reason for this is the presence of the term $c\bar{x}_g/(1-J)$ in equation (10). This term can break the reflection problem. This will happen if the $c\bar{x}_g/(1-J)$ term is not linearly dependent on a constant and y_g. When this is so, m_g cannot be linearly dependent on the other regressors in equation (10). This immediately leads to the argument in Brock and Durlauf (2001b) that a necessary condition for identification in the linear in means model, is that there exists at least one element of x_i whose group level average is not an element of y_g , while Durlauf and Tanaka (2008) provide a sufficient set of conditions. Necessity and sufficiency can be linked as follows. Let $\mathrm{proj}(a|b,c)$ denote the linear projection of the scalar random variable a onto the elements of the random vectors b and c.[13]

[13] Formally, this is the projection of a onto the Hilbert space generated around the elements of b and c where the inner product between any two elements is the expected value of their product so that the metric measuring the length of an element is the square root of the inner product of an element with itself.

Consider the projections $\text{proj}(\omega_g|1, y_g, \bar{x}_g)$ and $\text{proj}(\omega_g|1, y_g)$, where 1 is simply a random variable with mean 1 and variance 0, corresponding to the constant term in (6). The first projection provides an optimal linear forecast (in the variance minimizing sense) of the group average choice, $\bar{\omega}_g = (1/n_g)\sum_{i\in g}\omega_{ig}$, conditioning on the random variables defined by 1 and the elements of y_g and $\bar{x}_g$, whereas the second projection provides the optimal linear forecast when only 1 and the elements of y_g are used. The difference between the two projections thus measures the additional contribution to predicting $\bar{\omega}_g$ beyond what can be achieved using $\bar{x}_g$ in addition to 1 and y_g. When this marginal contribution is nonzero, then it is possible to estimate equation (10) using instrumental variables for $\bar{\omega}_g$ or equivalently estimate (6) when (10) is imposed by instrumenting m_g.[14] Formally,

Theorem 1. Identification in the linear in means model. *The parameters k, c, J and d are identified if and only if* $\text{proj}(\bar{\omega}_g|1, y_g, \bar{x}_g) - \text{proj}(\bar{\omega}_g|1, y_g) \neq 0$.

[14]Recall that in equilibrium, $\text{proj}(\bar{\omega}_g|1, y_g, \bar{x}_g) = \text{proj}(m_g|1, y_g, \bar{x}_g)$ and $\text{proj}(\bar{\omega}_g|1, y_g) = \text{proj}(m_g|1, y_g)$.

The intuition for the theorem is simple; identification requires that one can project $\bar{\omega}_g$ (equivalently) onto a space of variables such that the projection is not collinear with the other regressors in the model. As such, the theorem verifies that identification in the linear in means model is a species of identification of a linear simultaneous equations system, as argued above.[15]

Theorem 1 was derived under the assumption that $\bar{x}_g$ and y_g are known to the individual decisionmakers at the time that their choices are made. This assumption is a strong one and further may appear to be inconsistent with our assumption that $\bar{\omega}_g$ is unobservable to them. This latter concern is not tenable: in a context such as residential neighborhoods, it is possible for a contextual effect such as average income to be observable whereas the school effort levels of children in the neighborhood are not. However, it is important to understand the interactions of relaxing are informational assumptions on identification. This is the contribution of Graham and Hahn (2005). The models they study can be subsumed as variants of a modified version of equation (6):

$$\omega_{ig} = k + cx_i + d\,\mathrm{E}(y_g|F) + Jm_g + \varepsilon_i \tag{14}$$

[15]The conditions of the theorem do not preclude a functional dependence of x_i on y_g , which, combined with the uniqueness of m_g, means that the nonparametric analog to the model is not identified, following Manski (1993, Proposition 3). This observation builds on discussion in Manski (1993, p. 539).

where individuals are assumed to possess a common information set F. As such, it is clear that the conditions for identification in theorem 1 are easily generalized. One simply needs a set of additional instruments q_g such that the elements of q_g can jointly instrument $\mathrm{E}(y_g)$ and m_g. As they observe, the variables q_g constitute exclusion restrictions and so require prior information on the part of the analyst. For their context, y_g is a strict subset of $\bar{x}_g$, so it is difficult to justify the observability of those elements of $\bar{x}_g$ that do not appear in y_g when the others are by assumption not observable. In our view, the appropriate route to uncovering valid instruments q_g, under the Graham and Hahn information assumptions, most likely requires the development of an auxiliary model of x_i and hence $\bar{x}_g$. In other words, Graham and Hahn's concerns reflect the incompleteness of (14) in the sense that the individual characteristics are not themselves modeled. Hence, we interpret their argument as one that calls for the embedding of outcomes such as (14) in a richer simultaneous equation system, possibly one including dynamics, which describes how individual characteristics are determined. We fully agree with Graham and Hahn that in isolation, finding valid instruments for (14) is difficult, but would argue that this difficulty reflects the limitations of studying ω_{ig} in isolation rather than as one of a set of equilibrium outcomes.

a. partial linear in means models

The linear structure in (6) is typically only theoretically justified under strong function form assumptions for utility, as shown in appendix 1, which leads to the question of whether relaxation of the linearity assumption affects identification. One such relaxation is studied in Brock and Durlauf (2001b) and involves a particular nonlinear generalization of (6) under rational expectations

$$\omega_{ig} = k + cx_i + dy_g + J\mu(m_g) + \varepsilon_i. \tag{15}$$

This type of structure is known as a partial linear model. Brock and Durlauf establish that the parameters of this model are identified for those elements of the space of twice differentiable functions, for known $\mu(m_g)$, so long as $\partial^2\mu(m_g)/\partial m_g^2 \neq 0$, outside of nongeneric cases. The intuition is straightforward; the reflection problem requires linear dependence between group outcomes and certain group-level aggregates, which is ruled out by the nonlinearity in (15). Note that there does not exist any identification results, as far as we know, if the functional form for $\mu(m_g)$ is unknown, so in this sense the identification of (15) does not exploit results from the semiparametric literature on partial linear models.[16]

[16] See Tamer (2008) for a survey.

b. dynamic linear models

Similarly, dynamic analogs of the linear in means model may not exhibit the reflection problem. Brock and Durlauf (2001b) illustrate this with the dynamic social interactions model

$$\omega_{igt} = k + cx_{it} + dy_{gt} + \beta m_{g\,t-1} + \varepsilon_{it}$$

where for all $s, t \neq 0$,

$$\mathrm{cov}(\varepsilon_{i\,t}, \varepsilon_{i\,t-s}) = 0. \qquad (16)$$

This model avoids linear dependence between the contextual and endogenous variables since

$$m_{gt} = \frac{k + c\bar{x}_{gt} + dy_{gt}}{1 - \beta L} \tag{17}$$

where L is a lag operator. Equation (17) implies that m_{gt} depends on the entire history of $\bar{x}_{gt}$ and y_{gt}. This model is essentially backwards looking and is driven by the idea that current behaviors are directly affected by past beliefs. A more natural approach, of course, is to consider how beliefs about the future affect current behaviors. An example of a model in this class is

$$\omega_{igt} = k + cx_{it} + dy_{gt} + \beta m_{g\,t+1} + \varepsilon_{it} \tag{18}$$

where (16) is again assumed. This model is equivalent to the workhorse geometric discount model in rational expectations (Hansen and Sargent, 1980).

The equilibrium average choice level for a group equals, following Hansen and Sargent,[17]

$$m_{gt} = \frac{k}{1-\beta} + \sum_{s=0}^{\infty} \beta^s \mathrm{E}_t\{c\bar{x}_{g\,t+s} + dy_{g\,t+s}). \tag{19}$$

It is immediate from (19) that the regressors in (18) are linearly independent so long as $\bar{x}_{gt}$ and y_{gt} are not both random walks. Identification of this class of dynamic models was originally studied in Wallis (1980) and has recently been explored in Binder and Pesaran (2001).

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[17] In this formulation we restrict ourselves to fundamental solutions of the expected average choice level. The possibility of a nonfundamental solution, i.e. bubbles, is not germane to the discussion.

c. heirarchical models

In fields such as sociology, social interactions are typically explored using hierarchical models, i.e. models in which contextual interactions alter the coefficients that link individual characteristics to outcomes. See Bryk and Raudenbush (2001) for a full description of the method. The reason for this appears to be a different conceptualization of the meaning of social interactions in economics in comparison to other social sciences. Hierarchical models appear, in our reading, to be motivated by a view of social groups as defining ecologies in which decisions are made and matter because different social backgrounds induce different mappings from the individual determinants of these behaviors and choices, cf. Raudenbush and Sampson (1999). Economics, in contrast, regards the elements that comprise endogenous and contextual social interactions as directly affecting the preferences, constraints, and beliefs of agents and so treats them as additional determinants to individual specific characteristics, x_i. That said, there do not exist formal arguments for favoring one approach versus another at an abstract level. At the same time the additivity assumption in both approaches is ad hoc from the perspective of economic theory, even if the assumption is ubiquitous in empirical practice.

For hierarchical models, there has been no attention to the reflection problem. The only exception of which we are aware is Blume and Durlauf (2005). Here we modify the Blume and Durlauf analysis and consider a formulation that closely follows the conceptual logic of hierarchical models in that social interactions are entirely subsumed in the interactions on parameters. Formally, this means that individual outcomes obey

$$\omega_{ig} = k_g + c_g x_i + \varepsilon_i \tag{20}$$

with individual- and group-specific components obeying

$$k_g = k + d y_g + J m_g \tag{21}$$

and

$$c_g = c + y_g' \Psi + m_g \psi \tag{22}$$

respectively.

In (22), Ψ is a matrix and ψ is a vector. We omit any random terms in (21) and (22) for simplicity, although hierarchical models typically include them. This formulation assumes that the endogenous effect directly affects the individual level coefficients and so differs from the Blume and Durlauf example. Imposing rational expectations, the hierarchical model described by (20)-(22) is equivalent to the linear model

$$\omega_{ig} = k + cx_i + dy_g + Jm_g + y_g'\Psi x_i + m_g\psi x_i + \varepsilon_i. \tag{23}$$

Hence, the difference between the linear model used in economics and the hierarchical structure is the addition of the terms $y_g'\Psi x_i$ and $m_g\psi x_i$ by the hierarchical model to equation (6). Thus the hierarchical model does nothing deeper than add the cross products of variables in (6) to allow for nonlinearity. As such, the approach is far behind the econometrics literature on semiparametric methods which allows for much deeper forms of nonlinearity. On the other hand, the use of cross products of variables is still common in empirical economics.

Can this model exhibit the reflection problem? The self-consistent solution to (23) is

$$m_g = \frac{k + c\bar{x}_g + dy_g + y_g'\Psi\bar{x}_g}{1 - J - \psi\bar{x}_g}. \tag{24}$$

Recall that the reflection problem necessarily emerged in (6) when $y_g = \bar{x}_g$. If we impose this condition in the hierarchical model, (24) becomes

$$m_g = \frac{k + (c+d)y_g + y_g'\Psi y_g}{1 - J - \psi y_g}. \tag{25}$$

Equation (25) makes clear that the relationship between m_g and the other regressors is nonlinear. Further, the presence of $y_g'\Psi y_g$ in the numerator and $-\psi y_g$ in the denominator ensures that linear dependence will not hold, except for hairline cases, so long as there is sufficient variation in x_i and y_g.

Hierarchical models thus exhibit different identification properties from linear in means models because their structure renders the endogenous effect m_g a nonlinear function of the contextual interactions y_g (and also a nonlinear function of $\bar{x}_g$ if this variable is distinct from y_g). The reflection problem can thus be overcome without prior information about the relationship between $\bar{x}_g$ and y_g. However, this does not mean that users of hierarchical models of social interactions can ignore the possibility of endogenous social interactions and only focus on contextual effects. The nonlinear relationship between m_g and y_g means that the failure to account for endogenous social interactions in hierarchical models will lead to inconsistent estimates of the contextual effect parameters. Further, hierarchical models cannot be used to evaluate the interactions of changes in different variables, or the interactions on individual outcomes of altering group memberships, e.g. by changing school district boundaries.[18] These types of policy interventions will depend on the value of all the social interactions parameters and the attendant nonlinearity described by equation (25). Hierarchical models thus contrast with the linear in means example given in Manski (2010) where policy evaluation does not require knowledge of all parameters.

iii. variance-based approaches

As noted above, a second route to identification of the linear in means model may be derived from the covariance structure model errors. This approach is discussed in classic treatments of identification such as Fisher (1966) and relies on strong prior information on the covariance structure of a given model's errors. In general, this approach to identification became unpopular in economics because modern econometrics has emphasized the relaxation of assumptions on error structures, as manifested in the work on heteroskedastic and autocorrelation consistent covariance matrix estimation initiated by White (1980).[19] This emphasis on econometric analysis under weak assumptions on errors is properly regarded as a major breakthrough since in many socioeconomic contexts, assumptions such as homoskedasticity have no theoretical justification. To the extent that theory does constrain the stochastic processes for model errors, modern econometrics has focused on incorporating this dependence into the empirical analysis. Heckman (2001) gives an overview of this perspective for microeconomics, which is of course the locus of social interactions. It is therefore unsurprising that most work on empirical social economics has avoided exploiting covariance restrictions as a source of identification.

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[19] See West (2008) for an overview.

To see how this approach, which Graham refers to as the method of variance contrasts, works, we employ a simplified version of his model, which assumes that individual outcomes are affected by the realized mean outcomes in classrooms,[21]

$$\omega_{ig} = J\bar{\omega}_g + \varepsilon_i = Jm_g + \varepsilon_i + J\bar{\varepsilon}_g. \tag{26}$$

Individual and contextual interactions are thus assumed away, which renders the instrumental variable strategies we have described for identification impossible. Graham further assumes that the individual errors obey

$$\mathrm{var}(\varepsilon_i | i \in G) = \sigma_\varepsilon^2 \tag{27}$$

and

$$\text{for } i \neq j, \quad \mathrm{cov}(\varepsilon_i \varepsilon_j | i, j \in g) = 0. \tag{28}$$

[21]Graham allows for unobserved group level interactions, which we consider below.

Graham's insight, which builds on earlier work by Glaeser, Sacerdote, and Scheinkman (1996) (which will be discussed in the context of discrete choice models) is that the presence of J affects the variance of ω_{ig} and may be used for identification. For this model, one can think of the outcomes as generated by a reduced form in which the errors fully determine the outcomes, in other words, all information about the model parameters is embedded in the variance covariance matrices of the various ω_g's. Graham shows that for the model (26), under assumptions (27) and (28):

$$\operatorname{var}(\omega_g) = \left(I_{n_g} - \frac{J}{n_g}\iota_{n_g}\right)^{-2} \sigma_\varepsilon^2 \tag{29}$$

where I_{n_g} is an $n_g \times n_g$ identity matrix and ι_{n_g} is a $n_g \times n_g$ matrix of 1's. Eq. (29) implies that if there are two groups with different sizes, one can use the differences in the intergroup outcome variances to identify J. Following Graham, this result follows intuitively from the fact that for larger groups the variance in $\bar{\omega}_g$ is smaller. We should note that the assumption expressed by (28) is stronger than what can be justified by exchangeability of the individual errors per se. Durlauf and Tanaka (2008) explicitly show that Graham's results follow if one starts with exchangeability of the individual errors and further assumes that error variances are independent of classroom size.

iv. unobserved group effects

As suggested in the introduction, one of the major limits to identification of social interactions is the presence of unobserved group-level heterogeneity. To introduce this factor, we modify (6) to

$$\omega_{ig} = k + cx_i + dy_g + Jm_g + \alpha_g + \varepsilon_i \tag{30}$$

where rational expectations is imposed as in (10). The associated reduced form for (30) is

$$\omega_{ig} = \frac{k}{1-J} + cx_i + \frac{J}{1-J}c\bar{x}_g + \frac{d}{1-J}y_g + \frac{1}{1-J}\alpha_g + \varepsilon_i. \tag{31}$$

It is evident from (31) that correlation of α_g with the regressors in the equation can lead to identification problems. It is hard to see how one can rule such correlations out. For example, correlation with x_i naturally arises from self-selection and correlation with y_g naturally arises from imprecise or incomplete measurement of group contextual effects.

a. instrumental variables

One approach to dealing with unobserved group level heterogeneity in (30) or (31) is the use of instrumental variables. This approach is generally difficult to justify in addressing unobserved group characteristics for both the linear in means and other models. The reason for the difficulty is that α_g is itself undertheorized, in other words, this term captures aspects of a group that affect outcomes which the model does not explicitly describe. Beyond this, valid instrumental variables require the property that they have been excluded from (30) as either individual or contextual determinants of outcomes. It is hard to see how, in typical socioeconomic contexts, such instruments may be found, since the instruments must be known on a priori grounds to be uncorrelated with both the undertheorized α_g and ε_i. Social interactions models are typically what Brock and Durlauf (2001c) have termed openended, which means that their theoretical structure does not naturally identify variables to exclude from equations such as (30). In other words, social interactions theories are openended because the presence of a given type of social interaction does not logically preclude the empirical relevance of other theories; the econometric analog of this is that social economics models do not provide a logical basis for choosing instruments. This is quite different from rational expectations models, for example, whose logic often allows one to express linear combinations of variables as forecast errors, which must logically be orthogonal to an agent's information set; in macroeconomics a key example of this is the Euler equation in a stochastic optimization model.

b. panel data

A second standard strategy for dealing with unobserved group interactions involves the use of panel data to difference the interactions out. Supposing that the variables in (30) are indexed by t, this amounts to working with

$$\begin{aligned}\omega_{igt} - \omega_{ig\,t-1} = {} & c(x_{it} - x_{i\,t-1}) + d(y_{gt} - y_{g\,t-1}) \\ & + J(m_{gt} - m_{g\,t-1}) + \varepsilon_{it} - \varepsilon_{i\,t-1}\,.\end{aligned} \tag{32}$$

Recall that our identification theorem 1 depended on the relationship between $\bar{x}_g$, y_g and m_g. For (32), theorem 1 immediately can be applied if one considers the requirements of the theorem as they apply to $\bar{x}_{gt} - \bar{x}_{g\,t-1}$, $y_{gt} - y_{g\,t-1}$ and $m_{gt} - m_{g\,t-1}$. So long as there is temporal variation in $\bar{x}_{gt}$ and y_{gt} i.e. the first differences in (32) are not zero, then the conditions for identification will be the same as in the original linear model without α_g. Note that variation in $\bar{x}_{gt}$ and/or y_{gt} will induce variation in m_{gt} over time. An early example of this strategy is Hoxby (2000) who focuses on variation in the percentage of a student's own ethnic group in a classroom.

For those elements of x_{it} and y_{gt} that do not vary over time, differencing means that their associated coefficients will not be identified. Defining the time invariant elements of y_{gt} as y_g^1, the lack of identification of their associated parameters d^1 occurs for the obvious reason that one cannot differentiate the effect of $d^1 y_g^1$ from α_g. On the other hand, all elements of x_{it} may be identified if additional assumptions are placed on ε_{it}. As formally discussed in Graham and Hahn (2005), suppose that

$$\mathrm{E}(\varepsilon_{it}|F_{x|gt}, y_{gt}, \alpha_{gt}, i \in g \text{ at time } t)) = 0.$$

In this case, intragroup variation in x_{it} at a single point in time can identify all of the elements of c. The reason for this is that for group g at a fixed t, $k + dy_{gt} + Jm_{gt} + \alpha_{gt}$ acts as a constant term for the members of the group. Brock and Durlauf (2001b, 2006, 2007) use this same argument for cross-section identification of individual interactions coefficients in discrete choice models. As noted by Graham and Hahn, this type of argument is originally due to Hausman and Taylor (1981).

c. variance approaches and group-level unobservables

Graham (2008) provides a strategy for identifying the parameter for endogenous interactions in the presence of unobserved group interactions in parallel to the arguments that identified J in (26). Graham works with the natural generalization of (26):

$$\begin{aligned}\omega_{it} &= J\bar{\omega}_g + \alpha_g + \varepsilon_i \\ &= Jm_g + \alpha_g + \varepsilon_i + J\bar{\varepsilon}_g\end{aligned}$$

Critically, Graham assumes that α_g is a random effect, specifically requiring that the conditions

$$\text{cov}(\alpha_g \varepsilon_i | i \in g) = 0 \tag{33}$$

and

$$\text{var}(\alpha_g | i \in g) = \sigma_\alpha^2 \tag{34}$$

hold in addition to equations (27) and (28). Equation (33) states that individual and group unobservables are uncorrelated and is justified in Graham's context by the random assignment of teachers across classrooms. Equation (34) rules out any dependence of the variance of unobserved group effect on group size. In a classroom context, this means that the variance of teacher quality does not depend on the number of students. In a direct generalization of equation (29), Graham (2008) shows that

$$\text{var}(\omega_g) = \left(I_{n_g} - \frac{J}{n_g}\iota_{n_g}\right)^{-1} (\sigma_\alpha^2 \iota_{n_g} + \sigma_\varepsilon^2 I_{n_g}) \left(I_{n_g} - \frac{J}{n_g}\iota_{n_g}\right)^{-1} \tag{35}$$

which means that one can again use differences in the variance of outcomes across groups of different sizes to identify J.

v. self-selection

It is natural for many social contexts to expect individuals to self-select into groups. This is most obvious for the case of residential neighborhoods; models such as Bénabou (1993, 1996), Durlauf (1996a,b) and Hoff and Sen (2005), for example, all link social interactions to neighborhood choice. In terms of estimation, self-selection generally means that equation (8) is violated.

Following Heckman's original (1979) reasoning, one can think of individuals choosing between groups $g = 1, \ldots, G$ based on an overall individual-specific quality measure for each group:

$$I_{ig}^{*} = \gamma_1 x_i + \gamma_2 y_g + \gamma_3 z_{ig} + \nu_{ig},$$

where z_{ig} denotes those observable characteristics that influence i's evaluation of group g but are not direct determinants of ω_i and ν_{ig} denotes an unobservable individual-specific group quality term. Individual i chooses the group with the highest I_{ig}^{*}. We assume that prior to group formation, for all i and g, $\mathrm{E}(\varepsilon_i | x_i, y_g, z_{ig}) = 0$ and $\mathrm{E}(\nu_{ig} | \xi, y_g, z_{ig}) = 0$.

From this vantage point, the violation of equation (8) amounts to

$$\mathrm{E}(\varepsilon_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g) \neq 0. \tag{36}$$

Notice that equation (36) includes the characteristics of all groups. This conditioning reflects the fact that the choice of group depends on characteristics of the groups that were not chosen in addition to the characteristics of the group that was chosen. Equation (36) suggests that the linear in means model, under self-selection, should be written as

$$\omega_{ig} = cx_i + dy_g + Jm_g + \mathrm{E}(\varepsilon_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g) + \xi_i. \tag{37}$$

where by construction $\mathrm{E}(\xi_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g) = 0$. Notice that the conditioning in (36) includes the characteristics of all groups in the choice set. This is natural since the characteristics of those groups not chosen are informative about the errors.

Eqs. (36) and (37) illustrate Heckman's (1979) insight that in the presence of self-selection on unobservables, the regression residual ε_i no longer has a conditional mean of zero, yet (37) can be consistently estimated using ordinary least squares if one adds a term to the original linear in means model (6) that is proportional to the conditional expectation on the left hand side of (36), i.e., prior to estimation. Denote this estimate as

$$\widehat{\kappa\, \mathrm{E}(\varepsilon_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g)}\ . \tag{38}$$

Heckman's fundamental insight was that one can construct such a term by explicitly modeling the choice of group. From this perspective, controlling for self-selection amounts to estimating

$$\omega_{ig} = cx_i + dy_g + Jm_g + \rho\, \widehat{\kappa\, \mathrm{E}(\varepsilon_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g)} + \xi_i. \tag{39}$$

Thus, accounting for self-selection necessitates considering identification for this regression, as opposed to (6).

The property of interest for the identification of social interactions is that the addition of the term (39) can help facilitate identification. To see this, consider two possible reasons why agents choose particular groups. First, agents may choose groups on the basis of the expected average behaviors that occur. For example a family chooses a neighborhood based on its expectation of the average test score among students in the school their child will attend. In the extreme case where this is the only neighborhood factor that matters to families, the conditional expectation associated with the selection correction will be a function of the agent's characteristics and the expected outcomes in each of the neighborhoods, i.e.

$$\mathrm{E}(\varepsilon_i | x_i, \bar{x}_1, y_1, z_{i1}, \ldots, \bar{x}_G, y_G, z_{iG}, i \in g) = \varpi(x_i, m_1, \ldots, m_G) \tag{40}$$

By the same logic that rendered the partial linear model (15) identified, (40) is also identified as m_g cannot, outside of nongeneric cases, be linearly dependent on a constant term and y_g.

vi. social interactions via unobserved variables

Our discussion of the linear in means model has assumed that the variables through which social interactions operate either are directly observable or represent rational expectations forecasts of observable (to the analyst) variables.

Recent work by Arcidiacono, Foster, Goodpaster, and Kinsler (2009) considers this possibility in a panel context, which is applied to classrooms at the University of Maryland, where grades are the outcome measure. Translating their model into our notation, they analyze

$$\omega_{igt} = cx_i + d\bar{x}_{gt} + eu_i + f\bar{u}_{gt} + \varepsilon_{it} \tag{41}$$

where $y_{gt} = \bar{x}_{gt}$ is assumed. Endogenous social interactions are ruled out a priori. The key innovation in Arcidiacono et al. is that the individual variable u_i and associated group variables $\bar{u}_{gt}$ are both unobservable. Notice that x_i and u_i are time invariant whereas $\bar{u}_{gt}$ and $\bar{x}_{gt}$ are time dependent. The time dependence of the latter terms occurs because group memberships can change over time.

The identification of social interactions for this problem thus hinges upon overcoming unobservability of the contextual interactions $\bar{u}_{gt}$. In order to achieve identification, Arcidiacono *et al.* restrict the coefficients in (41) by assuming the existence of a scalar γ such that

$$e = \gamma c; \qquad f = \gamma d.$$

This assumption follows Altonji, Huang, and Taber (2005). In the spirit of simultaneous equations theory this is analogous to a coefficient restriction that facilitates identification. This assumption fixes the ratios of the coefficients of observed individual characteristics to equal those of the corresponding peer characteristics. Arcidiacono *et al.* describe this (pg. 6) as rendering the two dimensions of peer effects versus the two dimensions of individual effects "equally important". This is an ill-defined claim. The Arcidiacono *et al.* strategy is better thought of as a restriction on coefficients that, in the classical simultaneous equations sense, may help with identification and its justification should be assessed from the perspective of whether the restriction can be justified by economic theory or by some other argument.

Arcidiacano *et al.* proceed by focusing on a general notion of fixed effects for each individual, defining these fixed effects the determinants of outcomes (outside the errors ε_{it}) as

$$\kappa_i = cx_i + eu_i.$$

Letting $\bar{\kappa}_{gt} = \frac{1}{n_{gt}} \sum_{i \in g} \kappa_{igt}$, eq. (41) can be re-expressed as

$$\omega_{ig} = \kappa_i + \gamma \bar{\kappa}_{gt} + \varepsilon_{it}.$$

From this perspective, Arcidiacono *et al.* frame the identification problem for social interactions as the problem of consistently estimating γ in the presence of a large number of fixed interactions. Their theorem 1 locates a set of sufficient conditions so that a consistent and asymptotically normal estimator of γ may be found. Identification is implicit in this proof. We do not repeat their assumptions here but note that the essential substantive economic requirements are 1) the composition of an individual's peer groups change over time and 2) $\forall i, j, t$, $\mathrm{E}(\varepsilon_{it}\kappa_j) = 0$. The first condition is needed since identification requires individuals to be exposed to different peers to allow for distinguishing the influence of the fixed effects of others on a given individual. The second condition delimits the nature of self-selection into classrooms. Although Arcidiacono *et al.* argue that they allow for self-selection based on the ability of peers, this second condition appears to limit how selection can occur. This criticism does not detract from the value of their contribution, but points to an instance of the general proposition that explicit modeling of selection is essential in understanding identification conditions.

A third approach to unobservability is developed in Solon, Page, and Duncan (2000) and further studied in Page and Solon (2003a,b). This analysis assumes that one cannot observe any of the determinants of individual outcomes; only the outcome data are available. Unlike work such as Graham (2008), no assumption is made that social interactions are endogenous rather than contextual. Rather, it is assumed that individuals are influenced by family-level, group-level, and idiosyncratic influences. Individuals are distinguished by family and group (in this case residential neighborhood). A variance decomposition for individual outcomes is constructed to bound the contribution of group effects to variance of individual outcomes. The analysis may be understood in terms of a variance components model (Searle, Casella, and McCulloch, 2006, p. 14):

$$\omega_{ig} = \mu_f + \nu_g + o_{fg} + \varepsilon_i \,. \tag{42}$$

In (42), μ_f denotes a family effect, ν_g denotes a group effect, o_{fg} denotes an interactive effect between family and group, and ε_i denotes an idiosyncratic effect. A decomposition of this type always exists in which the components are orthogonal. In terms of mapping this expression back to a measure of the role of group influences, one difficulty lies with o_{fg}. Does the covariation between family and group represent a group effect or self-selection? Solon, Page and Duncan address this issue by comparing intra-family (sibling) and group (in their case residential neighborhood) variances to bound the variance of ν_g, finding the variance contribution is small. Oreopoulos (2003) finds similar results, focusing on a data set which involves adults who, as children, were randomly assigned to different public housing projects, thereby presumably eliminating o_{fg}. He finds a small role for ν_g and so concludes that neighborhood effects do not play a major role in explaining the variance of various adult outcomes.

One limitation of this approach is that it reduces the vector of social interactions to a scalar so that one cannot tell which social factors matter. In fact, it is possible for different social factors to cancel each other out. And to the extent that unobserved group effects α_g do not represent social interactions, as was the case for our example of teacher quality and classroom outcomes, it is not clear that a large variance contribution from ν_g has a social interactions interpretation. These caveats do not render such exercises uninteresting; rather they illustrate how economically substantive assumptions matter in producing economically substantive interpretations.

vii. social multipliers and information from aggregated data.

We close this section on the linear in means model by turning to how the relationship between individual and aggregated data may provide evidence of social interactions when individual actions are generated by the linear in means model. The relationship between individual and aggregated data is studied in Glaeser and Scheinkman (2002) and applied by Glaeser, Sacerdote, and Scheinkman (2003). The essential idea behind their analysis is that endogenous social interactions can generate social multipliers in the sense that a change in private incentives for every agent in a population will have an equilibrium effect that is greater than the direct effect of the incentive change on each individual because the changes in the behavior of others create additional effects on that individual. This difference is evident in the reduced form (11) since x_i and $\bar{x}_g$ have different coefficients. Focusing on a scalar case (extension to vectors is straightforward but algebraically tedious), Glaeser and Scheinkman (2002) propose comparing the coefficient b in the regression

$$\omega_{ig} = a + bx_i + \varepsilon_i$$

with the coefficient b' in its group counterpart

$$\bar{\omega}_g = a' + b'\bar{x}_g + \bar{\varepsilon}_g, \tag{43}$$

and define the social multiplier as the coefficient ratio:

$$S = \frac{b'}{b}.$$

In the context of the linear in means model, it is straightforward to compute this ratio. Our calculations differ from Glaeser and Scheinkman as we focus on the difference in the regressions as a misspecification problem. To make this calculation, notice that for the Glaeser and Scheinkman case, the reduced form (11) becomes:

$$\omega_{ig} = \frac{k}{1-J} + cx_i + \frac{cJ}{1-J}\bar{x}_g + \varepsilon_i. \tag{44}$$

Comparing (44) and (43), it is evident that the population value of b is readily calculated using the standard omitted variables formula that $b = c + \frac{cJ}{1-J}\beta$ where β is implicitly defined by $\text{proj}(\bar{x}_g|1, x_i) = \kappa + \beta x_i$, i.e. $\beta = \text{cov}(x_i, \bar{x}_g)/\text{var}(x_i)$. In contrast, it is evident from taking expected values on both sides of (44) that $b' = \frac{c}{1-J}$ which means the social multiplier is

$$S = \frac{1}{1-J+J\beta}. \tag{45}$$

Notice that if there is perfect segregation across groups, so that incomes within a group are identical, then $\operatorname{cov}(x_i, \bar{x}_g) = \operatorname{var}(x_i)$, which implies that $S = 1$ whereas under random assignment, $\operatorname{cov}(x_i, \bar{x}_g) = 0$, which implies $S = \frac{1}{1-J}$. The latter value of S takes a form that echoes the classic Keynsian income/expenditure multiplier, with the marginal propensity of consumption replacing J. Moreover, in the Bayesian game described in appendix 1, which lays out the underlying decision-theoretic framework of the linear in means model, $S = (1+\phi)/(1+\phi\beta)$ where ϕ measures the marginal rate of substitution between conformity and and the private value of the choice variable. Note the surprising fact that, as the pressure to conform increases, the social multiplier may either decrease or increase, depending upon whether β exceeds or is exceeded by 1. Both are possible since we have not specified how individuals are sampled across groups. For example, if there is systematic sampling ofx_i values below $\bar{x}_g$ in each group, then $\beta > 1$ may occur. The dependence of the social multiplier upon β makes it difficult to interpret.

As articulated in this example, the social multiplier provides a different perspective on the effects of endogenous social interactions on changes in private incentives. In terms of identification, it may also be of value. Clearly the social multiplier calculations have little to add if complete individual level data are available across the various groups of interest. On the other hand, suppose that aggregate data are incomplete; i.e. one knows about outcomes in a subset of classrooms in a school. One can imagine identification of J via analogs to (45) that compare different levels of aggregation and thus exploit variation in β to uncover J. Alternatively, one can imagine partial identification approaches that exploit the fact that different β's reflect different levels of aggregation with respect to the same underlying population.

4 Social networks and spatial models of social interactions

In defining social interactions thus far we have presumed that interactions are generated by group-specific averages. Social network models provide further focus on the microstructure of interactions among agents and allow for heterogeneity of interactions across pairs of agents. Jackson (2008) provides a thorough overview of the new social networks literature. In this section we address the identification of social interactions in social networks. In addition, we discuss the use of spatial econometrics models to study social interactions. The social networks and spatial analysis approaches are mathematically very similar, and yet, they have been until recently developed independently from one another. This similarity is not surprising as spatial econometrics approaches deal with physical space, whereas social networks address a more abstract social space, yet still a space with well posed notions of distance and the like.

i. graphical models of social networks

Before extending the model of social interactions to social networks it is useful to establish some basic terminology. For this setting, social interactions among individuals are defined by means of a social structure (or topology, the two terms are used interchangeably in the literature and here) that takes the form of a network, whose mathematical description is a graph with the vertices representing individuals and edges representing links between them. Network vertices and population members are thus identical concepts. What is of interest is the network structure that links agents. Network structure among individuals is modeled either as an undirected or a directed graph. Here we shall focus on the latter case, since it allows us to express a richer set of social relations.

Directed graphs consist of vertices (also known as nodes) and directed edges. A directed edge is an ordered pair (i, j) of vertices. A directed graph is a pair (V, E) where V is the set of nodes, with cardinality n_V, and E is the set of edges. A subgraph (V', E') of (V, E) is a graph where V' is a subset of V and E' is a subset of E. A subgraph (V', E') is induced by (V, E) if and only if E' contains all edges of E which begin and end in V'. A social network is a graph (V, E) where V is the set of individuals and the directed edges in E signify social influence: (i, j) is in E if and only if j influences i.

A social network can be represented by its adjacency matrix A, also known as its sociomatrix in the mathematical sociology literature. An adjacency matrix is an $n_V \times n_V$ matrix, with one row and one column for each individual in V. For each pair of individuals i and j, $a_{ij} = 1$ if there is an edge from i to j, and 0 otherwise. Since the network is supposed to represent social connections, it is natural to assume that no i is connected to himself. That is, for all i, $a_{ii} = 0$. A path from i to j is a sequence of individuals $i_0, \ldots, i_K$ such that $i_0 = i$, $i_K = j$, and for all $k = 1, \ldots, K-1$, there is an edge from i_{k-1} to i_k. Such a path is said to have length K. If there is a path from i to j of length exceeding 1, then i indirectly influences j. The adjacency matrix A displays all paths of length 1. The K-fold product A^K counts all paths of length K; if the ij'th element of A^K is n, then there are n paths of length K from i to j. A subgraph (W, F) of (V, E) is strongly connected if and only if for any i and j in W there is a path from i to j consisting solely of edges in F. A subgraph which is strongly connected and is a subgraph of no larger such graph is a strongly connected component of (V, E).[25] A graph (V, E) is strongly connected if and only if some power of its adjacency matrix is strictly positive. The literature also contains the less restrictive requirement of weak connectivity. Intuitively, it is the notion of connectivity that emerges when one can walk links in any direction. By suitably ordering the vertices, the adjacency matrix of a graph (V, E) can be written as a block-diagonal matrix where the rows (columns) of each block correspond to a weakly connected component.[26] A graph (V, E) is complete if for each pair i and j in V there is an edge from i to j. A graph is oriented if the existence of an edge from i to j implies that there is no edge from j back to i.

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[25] Note that a strongly connected component can have links into it from outside the component, and links to the outside. But no other node is on a path *from* the component *and* a path *to* the component.

[26] Weak connectivity is connectivity assuming that all edges can be traversed in any direction.

While social influence can be a one-way relationship, we usually think of some relationships, for example friendship, as being bidirectional and the social network is represented by an undirected graph, and the adjacency matrix is symmetric. Edges are now undirected, and so there is a path from i to j if and only if there is a path from j to i. A subgraph (V', E') of the undirected graph (V, E) is connected if and only if between any two nodes i and j in V' there is a path in (V', E') between them. A component of the graph (V, E) is, as before, a subgraph which is connected and maximal with respect to inclusion. The distance between any two nodes is the length of the shortest path between them.

Some particular network topologies are important in the social networks literature. A *star network* is an undirected graph in which one individual, the center, is connected to all other individuals while all other individuals are connected only to the center. A *group*, also known as a *complete network*, is one that contains an edge between each two of its vertices. Its adjacency matrix is all 1s. In a *bipartite graph*, the vertex set V is the union of two disjoint sets T and U, and all edges are between members of T and members of U, i.e. edges represent matches between vertices in the two sets. We will call a bipartite graph *directed* if for all $(i, j) \in E$, $i \in T$ and $j \in U$.

Sociologists allege that social relations like friendship exhibit the property of homophily — loosely but accurately described by the phrase "the friend of my friend is my friend, too." This property is described by the prevalence of transitive triads. Triads are connected subgraphs consisting of three nodes. Transitivity is the property that the existence of an edge from node i to j and an edge from j to k implies the existence of an edge from i to k. A graph is transitive if it contains no intransitive triads. The linear in means model is specified by assuming A is symmetric, that edges are bidirectional, and that the graph is transitive. If this is true, then the graph is the union of completely connected components. The nodes of the component containing i constitute i's group.

While the linear in means model is a good starting point for the study of social interactions, social networks allow for a much richer specification of social relations. The model can be enriched still further by allowing the elements of adjacency matrices to be arbitrary real numbers. In such models, the magnitude of the number a_{ij} measures the degree of influence j has on i, and the sign expresses whether that influence is positive or negative. Throughout this section we will assume that all elements a_{ij} are non-negative except as noted, and that that contextual variables are weighted averages of the individual characteristics. This generalizes the contextual effects in the linear in means model case in which $y_g = \bar{x}_g$.

Note that in this section, we defined choices as ω_i. No additional subscript is employed to denote a person's social environment. The reason for this is that the networks literature focuses on members of a common population and introduces social structure for the population as a whole via the matrix A.

ii. identification in social networks: basic results

Cohen-Cole (2006) appears to be the first analysis of the linear in means model under richer interactions structures. That is, he posits that an individual reacts to multiple reference groups, such as a teenage boy might care differently about what other teenage boys do than about what teenage girls do. He shows that the model with agents' beliefs about actions in multiple other groups as well as observables for each group can be fully identified provided that there are more observed linearly independent group level effects than there are groups in the sample and that there is some pair of groups for which agents in one group care about the actions in the other. This type of reasoning is extended in an analysis by De Giorgi, Pellizari, and Redaelli (2010) of peer effects in the choice of college education. From the econometric perspective, Bramoullé, Djebbari, and Fortin (2009), Lee, Liu, and Lin (2010) and Lin (forthcoming) constitute the most systematic explorations of social interactions in social and spatial contexts respectively, but there are several other contributions which we will discuss below.

The network model employed by Bramoullé *et al.* assumes that each individual i is influenced by the average behavior of a set of peers $P(i)$ and that, like Moffitt (2001) but unlike Manski (1993), individual i is not his own peer. The peer relationship is not assumed to be symmetric, so the social network is represented by a directed graph. The social interactions are described by a weighted adjacency matrix:

$$a_{ij} = \begin{cases} \frac{1}{|P(i)|} & \text{if } j \in P(i), \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Individual outcomes are then described by the behavioral equation system

$$\omega_i = k + cx_i + d \sum_{j \neq i} a_{ij} x_j + J \sum_j a_{ij} \omega_j + \varepsilon_i \tag{47}$$

with the error restriction

$$\mathrm{E}(\varepsilon_i | (x_i)_{i \in V}, A) = 0\,. \tag{48}$$

The reduced form for this system may be described in vector notation as

$$\omega = k(I - JA)^{-1}\iota + (I - JA)^{-1}(cI + dA)x + (I - JA)^{-1}\varepsilon \qquad (49)$$

where I refers to the $n_V \times n_V$ identity matrix and ι is a $n_V \times 1$ vector of 1's. (Recall that n_V is the number of individuals in the network.) Bramoullé *et al.* focus on identification by studying this reduced form. Recognizing, as did Moffitt (2001), that systems of this type are examples of linear simultaneous equations models in which one can think of the outcomes for the members of the overall network as the endogenous variables and the individual effects as the exogenous variables.[27] The important insight of Bramoullé *et al.* is that the ideas concerning the averages of behaviors and characteristics of groups carry over into more general social-network settings. Bramoullé *et al.* provide a fundamental algebraic result with respect to identification of models like equations (47) and (48), which does not rely on the constraint (46). The theorem assumes that J can take values in an arbitrary parameter set $\mathcal{J}$ in $\mathbf{R}$.

[27] If y_g were included in the system, this vector would also represent a set of exogenous variables.

Theorem 2. Identification of social interactions in linear network models. *For the social interactions model described by eqs.* (47), *and* (48), *assume that* $Jc + d \neq 0$ *and that for all values of* $J \in \mathcal{J}$, $(I - JA)^{-1}$ *exists.*

i. *If the matrices* I, A, *and* A^2 *are linearly independent, then the parameters* k, c, d *and* J *are identified.*

ii. *If the matrices* I, A, *and* A^2 *are linearly dependent, if for all* i *and* j, $\sum_k a_{ik} = \sum_k a_{jk}$, *and if* A *has no row in which all entries are 0, then parameters* k, c, d *and* J *are not identified.*

The condition that $Jc + d \neq 0$ requires, in the network setting, that endogenous and contextual effects do not cancel out in the reduced form. Theorem 2.i is a purely algebraic result. This is to say, it does not rely on the specific structure of A which arises from its network context. It applies to any linear system of the form (47) for which $|J| \cdot ||A|| < 1$ for all possible parameter values J. An interesting feature of this result is that it does not rely on exclusion restrictions. This should not be surprising. Although the number of equations in the system is n_V, the size of the population, there are only four parameters to estimate. There are thus many cross-equation and within-equation linear equality constraints: The independence condition describes when these constraints satisfy the appropriate rank condition. Theorem 2.ii identifies an important case for which linear independence of I , A and A^2 is necessary as well as sufficient for identification. The requirements on A mean that each individual averages in some way over those who influence him, and that no one is isolated. The result can be thought of as a converse to theorem 2.i.

The analysis of group interaction is the leading case in the econometric literature on networks. It is also appealing from the perspective of existing data sets such as the National Longitudinal Study of Adolescent Health (Add Health).[28] Suppose that the peer relation is symmetric, $j \in P(i)$ if and only if $i \in P(j)$. Suppose too that the peer relation is transitive: If $j \in P(i)$ and $k \in P(j)$, then $k \in P(i)$. As discussed in section 4.i, the graph is the union of a finite number G of completely connected components, that is, groups. Suppose that component g has n_g members. We will consider two ways to average over the group: *Exclusive averaging* excludes i from $P(i)$. In this case, for $i \in g$,

$$a_{ij} = \begin{cases} \frac{1}{n_g - 1} & \text{if } j \neq i \text{ and } j \in g, \\ 0 & \text{otherwise.} \end{cases}$$

Inclusive averaging includes i in $P(i)$. In this case, for $i \in g$,

$$a_{ij} = \begin{cases} \frac{1}{n_g} & \text{for all } j \in g, \\ 0 & \text{otherwise.} \end{cases}$$

[28] The Add Health data set is the outcome of a longitudinal data collection exercise designed to facilitate study of health-related behaviors of adolescents in grades 7 through 12. The data set includes information, for example, on the structure of adolescent friendships via responses to questions on the identities of best friends. As such, directions of friendships are revealed but not intensity. See http://www.cpc.unc.edu/projects/addhealth for details.

With inclusive averaging, equations (47) and (48) are equivalent to our linear in means model, except that realized rather than expected outcomes affect individual outcomes. This difference is inessential since the instrumental variable projections used to replace the endogenous choices of others coincide with equilibrium formulations of beliefs.[29] Means and realizations, however, represent two distinct theoretical models. The first is a network version of the incomplete-information game developed in appendix 1. The second is a complete-information version of the same game. With exclusive averaging, the subject of Bramoullé *et al.*, an additional distinction is that the calculation of group-level contextual effects does not include i's own individual characteristics. This distinction is inessential in that the identification results for the two models are nearly the same. The following result is a corollary of the forthcoming theorem 3:

Corollary 1. Identification of social interactions in group structures with different-sized groups. *Suppose that individuals act in groups, and that the* a_{ij} *are given by either inclusive or exclusive averaging. Assume that* $Jc + d \neq 0$*. Then the parameters* k*,* c*,* d *and* J *are identified if and only if there are at least two groups of different sizes.*

[29] As we will see, the distinction is important for binary choice models.

The positive result of corollary 1 is similar to Graham's (2008) variance contrast identification strategy, but its source is different. Here identification follows the reduced form regression parameters rather than the second moments of the average group outcomes.[30] Note that in Graham's case, $Jc + d = 0$ since $c = d = 0$, so his findings allow for identification when individual and contextual effects are absent.

[30]Corollary 1 is a special case of Lee's (2007) result, without fixed effects. Lee (2007) studied the effects of group size on identification while also allowing for unobserved group fixed effects. He establishes identification of both the endogenous and exogenous social interactions provided there are sufficient variations in group sizes, but under somewhat restricted conditions relating group sizes to the total number of observations, and also provides asymptotic estimation properties. Davezies, d'Haultfoeuille, and Foughre (2009) show that the identification results hold under different conditions than Lee's. For Davezies *et al.*, identification holds so long as 1) the sizes of groups do not depend on overall sample size, as in Lee (2007), and 2) there exist at least three different group sizes. This avoids Lee's requirements that group sizes are linked to population size. Intuitively, this is possible because variations in group sizes create variations in reduced form coefficients across groups. This variation is evident from an example like that in Bramoullé, Djebbari, and Fortin (2009, p. 49), where the reduced form coefficients depend on the size of the group to which an individual belongs.

At first glance, this corollary might appear to contradict theorem 1 and indeed call into question Manski's nonidentification results on the linear in means model, since neither involved groups sizes while the corollary links groups sizes to identification. In fact, there is no contradiction. Theorem 1 and Manski's earlier analysis did not treat social interactions in the linear in means model as a simultaneous equations system that explicitly relates individual choices to one another within a group. More generally, previous studies of identification of the linear in means model have taken the effects of group averages as the objects of interest, not the pairs of cross-individual effects. In contrast, the linear in means model as appears in the econometrics literature is a large sample approximation to the solution of a particular Bayes/Nash game, as shown in appendix 1.[31] If one relaxes the approximation, then the coefficients of the linear in means model as it applies to a given group depend on the group's size. When groups sizes differ, the coefficients of their associated linear in means representations differ. When coefficients from groups of different sizes are combined, this allows one to uncover the parameters k, c, d and J.

[31] Compare equations (85) and (86) in appendix 1.

Powers of A describe the network topology. When we examine this in detail, we find that for very few networks are parameter estimates not identified in the reduced form. Bramoullé *et al.* prove that identification fails in two kinds of networks: groups and directed bipartite networks. In particular, within the class of undirected networks, it is only with a single group or two or more groups of the same size that identification fails. It is ironic that the importance of groups is due to their prevalence as a common specification in econometric studies, for it is only with groups that the identification issue even arises. By a directed bipartite network we mean a bipartite network in which all edges go one way — the vertex set is the disjoint union of two sets T and U, and all edges (i, j) have $i \in T$ and $j \in U$. Influence is one-way. While directed bipartite networks are every parent's dream — children listen only to their parents and never to their peers — they are largely irrelevant for the social networks economists study. Interestingly, however, they satisfy a set of exclusion conditions since the actions and characteristics of children do not appear in the behavioral equations for the parents, but this does not help identification. We state and prove (in appendix 2) a variant and unification of the Bramoullé *et al.* results.

Theorem 3. Nonidentification of social interactions in network models under exclusive and inclusive averaging. *Assume $Jc + d \neq 0$ and that for all values of J, $(I - JA)^{-1}$ exists.*

i. Under exclusive averaging, the parameters k, c, d and J are not identified in the reduced form (49) *for the structural model* (47) *and* (48) *if and only if the social network (V, E) is the union of weakly connected components wherein each component is either a directed bipartite network or a group, and all groups are the same size.*

ii. Under inclusive averaging, the parameters k, c, d and J are not identified in the reduced form (49) *for the structural model* (47) *and* (48) *if and only if the social network (V, E) is the union of identically-sized groups.*

This theorem depends upon the particular weighting schemes used to average across peers. It follows from the more general theorem 4, which is interesting in its own right because it applies to any weighting scheme which puts positive weight on all edges of E.[32]

[32] Bramoullé *et al.* discuss results of this type, but provide no general theorem like this.

Theorem 4. Non-identification for weighted averaging implies network transitivity. *Let (V, E) be a network with weighted adjacency matrix A as described by (46). Assume that $Jc + d \neq 0$ and that for all values of J, $(I - JA)^{-1}$ exists. If the parameters k, c, d and J are not identified, then (V, E) is transitive. If the network is undirected, then (V, E) is the union of groups.*

Finally, we note that all of this is based on the fundamental independence criterion of theorem 2, which applies to any matrix A no matter what its source, so long as it satisfies an algebraic criterion. From this general point of view, it is clear that non-identification of parameters in the reduced form is rare. We suppose without loss of generality that the parameter J takes values $[0, 1)$, and denote by S the set of all matrices A such that $(I - JA)$ is invertible. If the matrices are $n_V \times n_V$, S is a semi-algebraic set of full dimension in $\mathbf{R}^{n_V^2}$.[33]

Theorem 5. Generic identifiability of the linear social networks model. *The set of all matrices $A \in S$ such that the powers I, A and A^2 are linearly dependent, is a closed and lower-dimensional (semi-algebraic) subset of S.*

This theorem is a complement to McManus' (1992) result on the generic identifiability of non-linear parametric models. For the social networks context, the key intuition for generic identifiability is that since A is assumed to be known a priori, this knowledge is the equivalent of a large number of coefficient restrictions on the coefficients in the reduced form representation of individual behaviors. These restrictions are rich enough that, outside of nongeneric cases, they permit identification of k, c, d and J.

[33] A semi-algebraic set is a set which can be described as the solutions to a finite number of polynomial inequalities. The set of $n_V \times n_V$ matrices such that for all J, $(I - JA)^{-1}$ exists is a semi-algebraic set in $\mathbf{R}^{\mathbf{n}_V^2}$. Semi-algebraic functions are functions whose graphs are semi-algebraic sets. Every semi-algebraic set is the union of a finite number of disjoint open C^∞ manifolds. The dimension of a semi-algebraic set is the largest of the dimensions of these manifolds. For more on semi-algebraic geometry see Bochnak, Coste, and Roy (1998).

iii. unobserved component-specific fixed effects

The analog to group-level unobservables in the linear in means model in networks is component-level unobservables. If individual outcomes contain unobservables that are correlated among individuals belonging to the same component they may be treated as fixed effects in the stochastic structure of (47), producing

$$\omega_i = k + cx_i + d\sum_{j\neq i} a_{ij}x_j + J\sum_{j} a_{ij}\omega_j + \alpha_g + \varepsilon_i \tag{50}$$

with error structure

$$\mathrm{E}(\varepsilon_i | \alpha_g, (x_i)_{i\in g}, A) = 0 \tag{51}$$

where α_g is a component-specific fixed effect. This can be thought of as a model of interacting in groups, in which the groups themselves have internal social structure.

Little work has been done on this problem. We know of only the identification results of Bramoullé, Djebbari, and Fortin (2009). Since their model is linear, component-specific fixed effects can be differenced away.[34] In principle, this differencing can be done in many different ways, of which Bramoullé et. al. discuss two. "Local differencing" subtracts from each individual's behavioral equation the average of those who directly influence him. "Global differencing" subtracts from each individual's behavioral equation the average of those in the connected component to which the individual belongs. A third differencing strategy not yet studied is to subtract from each individual's behavioral equation the average of those to whom he is indirectly connected. Differencing entails loss of information, and so conditions for identification are stronger. But here too identification is determined by the network topology. In particular, Bramoullé et. al. prove the following theorem:

Theorem 6. Identification of social interactions in linear network models with component-specific fixed effects. *For the social interactions model described by equations* (46), (50) *and* (51), *assume that* $Jc + d \neq 0$. *With local differencing, a necessary and sufficient condition for identification of the parameters* k, c, d *and* J *is that the matrices* I, A, A^2 *and* A^3 *are linearly independent.*

[34] Bramoullé, Djebbari, and Fortin (2009) refer to these fixed-effects as "network-specific". "Component-specific" is a more precise description.

iv. self-selection in social network models

Investigating self-selection in social network models requires modelling the co-evolution of networks and behavior. Although the growth of networks has been studied empirically, and evidently behavior on networks is a well-established subject, the joint evolution of both has rarely been touched upon.[35] In particular, the econometric issues posed by endogenous network formation are briefly discussed by Jackson (2008, p. 437).[36]

[35] These issues have come up in the study of online commmunities. See Crandall, Cosley, Huttenlocher, Kleinberg, and Suri (2008).

[36] See Bala and Goyal (2000) and Jackson and Wolinsky (1996) for notable contributions, and Jackson (2008, ch. 6) for an extensive treatment of several other works.

v. spatial econometrics specifications of social interactions

A close relationship exists between social interactions and spatial econometrics models. Equation (47) implies the classic Cliff-Ord spatial autoregressive (SAR) model with one spatial lag, as the special case of $d = 0$ where the dimension of endogenous outcomes is equal to the number of spatial units. These can be states, counties, parcels of land, etc. When instead of spatial units, the model refers to individuals, one has a social interactions model. The social interactions literature has recently sought to exploit the relationship. See Lee (2007), who explored this link formally, and Lee, Liu, and Lin (2010).[37] In addition, the spatial econometrics literature has made important advances in terms of allowing for spatial autocorrelation in error structures: see Kapoor, Kelejian, and Prucha (2007) and Kelejian and Prucha (2010) for recent examples of advances in the study of spatial environments under weak error assumptions and Anselin (2010) for a review of the area. Spatial econometrics models have a long tradition in geography where the weights attached to different observations are motivated in terms of various distance concepts. For example, if the units of observations are counties, one may wish to account not only for adjacency but also for distance between their main population centers. Adding contextual effects, as in Lee (2007), brings the model closer to standard social interactions models.

[37] Boucher, Bramoullé, Djebbari, and Fortin (2010) estimate the full Lee (2007) model with group-level unobservables using data on student achievement from Quebec secondary schools and find evidence of endogenous peer effects while also controlling for contextual effects and group unobservables in the form of fixed effects.

Lee, Liu, and Lin (2010) is a significant advance in the econometrics of social networks and spatial models. It generalizes Lee (2007) by allowing for group unobservables and correlated disturbances of connected individuals. Spatial autocorrelations in the error structure of their model are accounted for by assuming that the vector of shocks for a given component l (comprised of n_l members), ε_l, consists of the sum of group-specific fixed effects and stochastic components that satisfy

$$\varepsilon_l = \rho A_l^* \varepsilon_l + \nu_l$$

where A^* is an exogenous and non-stochastic $n_l \times n_l$ non-negative error-interactions matrix that need not coincide with A.[39] The parameter ρ is the spatial autocorrelation coefficient, and ν_l is a n_l-vector of i.i.d. individual-specific shocks. This error specification may be thought of as a generalization of a number of previous studies. Relative to Lee (2007), in which an individual in a group interacts with all other group members with equal weights (and identification is ensured by different group sizes), Lee, Liu, and Lin (2010) allows different individuals to have their own social groups, defined by the respective social interactions matrices A_l.

[39] In their analysis, nondiagonal entries are assumed to be symmetric and positive, but diagonal entries are 0.

vi. from econometrics to applications

We end this section with two illustrations of how the types of models we have discussed have been applied in empirical work. Calvó-Armengol, Patacchini, and Zenou (2009) is a good illustration of how network methods have been employed. Using the Add Health data set, these authors estimate individual school performance as a function of the topology of their friendship networks, while controlling for individual characteristics. Individuals' friends always lie in the same school as themselves. They estimate a restricted variant of equation (47) in which $J = 0$. This model is generalized, however, by allowing the unobserved individual-level heterogeneity to also be related to the structure of the component level interactions. For each component l, the vector of errors ε_l for members of component l obeys

$$\varepsilon_l = \varrho A \iota + \rho A \varepsilon_l + \nu_l \tag{52}$$

where A is the same adjacency matrix that links observable characteristics across individuals, ϱ measures the mean effect of the number of direct neighbors for each individual ("best friends" according to the Add Health questions), which is given by $A\iota$, ρ denotes a spatial autocorrelation coefficient in the ε's, and ν_l is again an n_l-vector of i.i.d. individual-specific shocks.[40] Since the error structure in (47), ε_l, represents individual outcomes that are not explained by individual characteristics x and contextual effects Ax, these authors reason that it proxies for peer interactions. Their estimation of the stochastic structure subsumes social interactions into the estimation of (ϱ, ρ) in (52), the former because it controls for the number of best friends and the latter because it reflects how each individual's unobservable shock is affected by those of his friends. These authors interpret (ϱ, ρ) as expressing peer effects, which is not a standard use of terminology. From the perspective of the distinctions we have drawn between types of social interaction effects, ϱ and ρ parameterize the strengths of different contextual effects since neither refers to direct interdependences of choices per se.

[40] ι is an $n_l \times 1$ vector of 1's. For the Add Health data set, A is an adjacency matrix where a 1 means that at least i or j has designated the other as a best friend, so $A\iota$ is the number of friends of each member of the component.

The crucial identifying assumption here is that workers can choose residential locations down to a group of blocks, but do not purposefully choose among the individual blocks in the group because of block-specific characteristics. Therefore, conditional on sorting at the group of blocks level, the assignment of individuals to specific blocks is independent of block-specific characteristics. The authors use this conditional independence to identify local interactions with respect to labor market referrals. Specifically, let i and j denote individuals who reside in the same Census block group but do not belong to the same household. The outcome of interest is the binary variable ω_{ij} which indicates whether or not i and j work in the same Census block.[41] Further, δ_{ij}^{b} is a dummy variable that equals 1 if i and j reside in the same Census block, x_{ij} denotes a vector of socio-demographic characteristics for the pair i, j, and η_g denotes a reference group fixed effect which serves as the baseline probability of an employment match for individuals living in the same block group. The proposition that block-level interactions occurs in labor market referrals is defined via the regression

$$\omega_{ij} = \beta x_{ij} + (\alpha_0 + \alpha_1 x_{ij})\delta_{ij}^{b} + \eta_g + \varepsilon_{ij}. \tag{53}$$

The Bayer *et al.* test for the presence of social interactions due to proximity reduces to testing for the statistical significance of α_0 and α_1 in (53). The observable pair covariates term βx_{ij} controls for individual-specific reasons why two individuals work on the same block and η_g controls for any unobserved heterogeneity that occurs at the block group level and affects employment location. For example, η_g may be argued to control for features of the urban transportation network that might induce clustering in both residence and work location. The empirical strategy of Bayer *et al.* addresses several additional potential pitfalls, including possible sorting below the block level and the possibility of reverse causation due to co-workers giving referrals on desirable residential areas, and find large block-level social interactions effects on employment location, especially among individuals who are socioeconomically similar.

vii. social networks with unknown network structure[42]

All the results in this section so far have taken the social network matrix A as known. This severely restricts the domain of applicability of existing identification results on social networks. We finish this section by considering how identification may proceed when this matrix is unknown. In order to do this, we believe it is necessary to consider the full implications of the interpretation of linear social interactions models as simultaneous equations systems. While this interpretation is given in studies like Bramoullé *et al.*, the full implications of this equivalence have not been explored. This is evident if one observes that the matrix form of the general social networks model may be written as

$$(I - JA)\omega = (cI + dA)x + \varepsilon \tag{54}$$

where for expositional purposes, the constant term is ignored. From this vantage point, it is evident that social networks models are special cases of the general linear simultaneous equations system of the form

$$\Gamma\omega = Bx + \varepsilon. \tag{55}$$

[42] This section was inspired by comments by Gary Becker and especially James Heckman.

Systems of this type, of course, are the focus of the classical identification in econometrics, epitomized in Fisher (1966) and comprehensively summarized in Hsiao (1983). One can go further and observe that the assumption that the same network weights apply to both contextual and endogenous social interactions is not well motivated by theory, and regard equation (55) as the general specification of a linear social networks model where the normalization $\Gamma_{ii} = 1$ for all i is imposed. From this vantage point it is evident that the distinction between J and A is of interest only when A is known a priori, as is the case both for the linear in means model and the more general social networks framework.

Following the classical literature, one can then think of the presence or absence of identification in terms of whether particular sets of restrictions on (55) produce identification. All previous results in this section are examples of this perspective but rely on the very strong assumption of a particular way of imposing these restrictions, i.e. $\Gamma = I - JA$ and $B = cI + dA$ for known A. Note that the results we have described do not employ information on the variance covariance matrix of the reduced form error structure, which is one source of identifying information and the basis for Graham's (2008) results. The simultaneous equations perspective makes clear that the existing results on identification in linear social networks models can be extended to much richer frameworks. We consider two classes of models in which we interpret all agents $i = 1, \ldots, n_V$ as arrayed on a circle. We do this so that agents 1 and n_V are immediate neighbors of one another, thereby allowing us to work with symmetric interaction structures.

First, assume that each agent only reacts to the average behaviors and characteristics of his two nearest neighbors, but is unaffected by anyone else. This is a linear variation of the model studied in Blume (1993). In terms of the matrices Γ and B, one way to model this is to assume that, preserving our earlier normalization, $\Gamma_{ii} = 1$ and $\Gamma_{i\,i-1} = \Gamma_{i\,i+1} = \gamma_1$ for all i, $\Gamma_{ij} = 0$ otherwise; $B_{ii} = b_0$, $B_{i\,i-1} = B_{i\,i+1} = b_1$ for all i, and $B_{ij} = 0$ otherwise, where here (and for the remainder of this discussion, all indices are mod n_V). The model is identified under theorem 4 since the nearest neighbor model may be interpreted via the original social networks model via restrictions on A. For our purposes, what is of interest is that identification will still hold if one relaxes the symmetry assumptions so that $\Gamma_{i\,i-1} = \gamma_{i-1}$, $\Gamma_{i\,i+1} = \gamma_{i1}$, $B_{ii} = b_{i0}$, $B_{i\,i-1} = b_{i-1}$ and $B_{i\,i+1} = b_{i1}$. If these coefficients are nonzero, then the matrices Γ and B fulfill the classical rank conditions for identification, cf. Hsiao (1983, theorem 3.3.1) and one does not need to invoke theorem 4 at all. Notice that it is not necessary for the interactions parameters to be the same across agents in different positions in the network. Relative to Bramoullé *et al.*, what this example indicates is that prior knowledge of A can take the form of the classical exclusion restrictions of simultaneous equations theory. From the vantage point of the classical theory, there is no need to impose equal coefficients across interactions as those authors do. Imposition of assumptions such as equal coefficients may be needed to account for aspects of the data, e.g. an absence of repeated observations of individuals. But if so, then the specification of the available data moments should be explicitly integrated into the identification analysis, something which has yet to be done. Further, data sets such as Add Health, which produce answers to binary questions concerning friends, are best interpreted as providing 0 values for a general A matrix, but nothing more in terms of substantive information.

This example may be extended as follows. Suppose that one is not sure whether or not the social network structure involves connections between agents that are displaced by 2 on the circle, i.e. one wishes to relax the assumption that interactions between agents who are not nearest neighbors are 0. In other words, we modify the example so that for all i, $\Gamma_{ii} = 1$, $\Gamma_{i\,i\text{-}1} = \Gamma_{i\,i+1} = \gamma_1$, $\Gamma_{i\,i\text{-}2} = \gamma_{i\,-2}$, $\Gamma_{i\,i+2} = \gamma_{i\,2}$, $\Gamma_{ij} = 0$ otherwise, $B_{ii} = b_{i0}$, $B_{i\,i\text{-}1} = b_{i\,-1}$, $B_{i\,i+1} = b_{i1}$, $B_{i\,i\text{-}2} = b_{i\,-2}$, $B_{i\,i+2} = b_{i2}$, and $B_{ij} = 0$ otherwise. If the nearest neighbor coefficients are nonzero, then by Hsiao's theorem 3.3.1 the coefficients in this model are also identified regardless of the values of the coefficients that link non-nearest neighbors. This is an example in which aspects of the network structure are testable, so that relative to Bramoullé *et al.* one does need to exactly know A in advance in order to estimate social structure. The intuition is straightforward, the presence of overlapping network structures between nearest neighbors renders the system overidentified: so that the presence of some other forms of social network structure can be evaluated relative to it. This form of argument seems important as it suggests ways of uncovering social network structure when individual data are available, and again has yet to be explored. Of course, not all social network structures are identified for the same reason that without restrictions, the general linear simultaneous equations model is unidentified. What our argument here suggests is that there is much to do in terms of uncovering classes of identified social networks models that are more general than those that have so far been studied.

For a second example, we consider a variation of the model studied by Bramoullé *et al.*, which involves geometric weighting of all individuals according to their distance; as before we drop the constant term for expositional purposes. Specifically, we consider a social networks model

$$\omega_i = cx_i + d\sum_{j\neq i} a_{ij}(\gamma)x_j + J\sum_{j\neq i} a_{ij}(\gamma)\omega_j + \varepsilon_i.$$

The idea is that the weights assigned to the behaviors of others are functions of an underlying parameter γ. In vector form, the model is

$$\omega = cx + dA(\gamma)x + JA(\gamma)\omega + \varepsilon. \tag{56}$$

where

$$A(\gamma) = \begin{pmatrix} 0 & \gamma & \gamma^2 & \cdots & \gamma^k & \gamma^k & \gamma^{k-1} & \cdots & \gamma^2 & \gamma \\ \gamma & 0 & \gamma & & \cdots & \gamma^k & \gamma^k & \gamma^{k-1} & \cdots & \gamma^2 \\ & & & & \vdots & & & & & \\ \gamma & \gamma^2 & & \cdots & & & & & \gamma & 0 \end{pmatrix}. \tag{57}$$

Following Bramoullé *et al.*, x is a scalar characteristic. The parameter space for this model is $\mathcal{P} = \{(c, d, J, \gamma) \in \mathbf{R^2} \times \mathbf{R}_+ \times [0, 1)\}$. The reduced form for this model is

$$\omega = \left(I - JA(\gamma)\right)^{-1}\left(cI + dA(\gamma)\right)x + \left(I - JA(\gamma)\right)^{-1}\varepsilon$$

Denote by $F : \mathcal{P} \to \mathbf{R}^{\mathbf{n_V^2}}$ the map

$$F(c, d, J, \gamma) = (I - JA(\gamma))^{-1}\left(cI + dA(\gamma)\right) \tag{58}$$

The function F characterizes the mapping of structural model parameters (c, d, J, γ) to reduced form parameters. We will establish what Fisher (1959) calls complete identifiability of the structural parameters from the regression coefficients for the reduced form. By this he means that each reduced form parameter vector is derived from only a finite number of structural parameter vectors, i.e. that the map from structural models to reduced form models is finite-to-one.

The behavioral model described by (56) is nonlinear in the parameters because d, γ and J interact multiplicatively. This is nonetheless a natural model, as it is the simplest way to discount individual effects by distance. The following complete identification result holds for this model:

Theorem 7. Identification of the linear social networks model with weights exponentially declining in distance. *Suppose that the number of individuals n_V is at least 4. Then for all $(c, d, J, \gamma) \in \mathcal{P}$,*

i. if $I - JA(\gamma)$ is non-singular, $c + d \neq 0$ and $\gamma \neq 0$, then the cardinality of $F^{-1}\big(F(c, d, J, \gamma)\big)$ is no more than $2(n_V - 1)$.

ii. The events $J = d = 0$ and $\gamma = 0$ are observationally equivalent. In this case, $F(c, d, J, \gamma) = cI$.

Part i. of theorem 7 says the following: Each structural parameter vector is observationally equivalent to at most $2n_V - 3$ other structural parameter vectors in the sense that they all generate the same reduced form. As such, while point identification may not be achieved, any true structural parameter vector fails to be identified relative to at most $2n_V - 3$ alternatives. Notice that complete identification is stronger than local identification. Local identification implies that for the true structural parameters, there exists an open neighborhood of these parameters that does not contain any observationally equivalent structural parameters. The set of observationally equivalent structural parameters could nonetheless be countable. Complete identification requires that the set be finite, which implies local identification. Part ii. notes that if there are no social interactions, this imposes sufficiently strong restrictions on the reduced form parameters to identify both c and also requires that the matrix of reduced form parameters is proportional to an identity matrix. We believe that refinements of theorem 7.i are possible and leave this to future work.

These examples illustrate how the results of sections 3 and 4 are far from exhaustive in understanding the identification of linear social interactions models.

5 Discrete choice models of social interactions

In this section we consider identification for discrete choice models. Identification conditions for discrete choice models will prove to be conceptually quite different than the conditions that apply to linear models. Some reasons are trivial. For example, discrete choice models, because they involve probabilities, are inherently nonlinear and as we have discussed, nonlinear models have very different identification conditions than linear ones. Other differences will prove to be more subtle.

i. binary choice: basic structure

We first focus on binary choice models of social interactions. These have been the primary focus of theoretical work. Early theoretical studies include Blume (1993), Brock (1993), Durlauf (1993) and Glaeser, Sacerdote, and Scheinkman (1996). Recent contributions which generalize these earlier analyses in terms of the timing and network structure of interactions as well as in terms of belief formation include Bisin, Horst, and Özgür (2006), Horst and Scheinkman (2006) and Ioannides (2006).

Identification for binary choice models has been studied in detail by Brock and Durlauf (2001a,b, 2007); other contributions include Soetevent and Kooreman (2007). We follow Brock and Durlauf (2001a,b) for the development of an initial structure and indicate how subsequent analyses have relaxed assumptions relating to their formulation. Choices are coded so that they belong to the set $\{-1, 1\}$. If the context is teenage pregnancy, then $+1$ can denote *had a child while a teenager* while -1 can denote *did not have a child while a teenager.* To interpret these choices as the outcomes of a decision problem, we define individual-specific payoff functions $V_i(\omega_i)$.

An econometrically implementable choice structure is implemented by assuming the difference between the payoffs for the two choices is additive in the different factors that have been defined for the linear model, i.e.[44]

$$V_i(1) - V_i(-1) = k + cx_i + dy_g + Jm^e_{ig} - \varepsilon_i. \quad (59)$$

[44] The payoff differential is written in terms of $-\varepsilon_i$ for algebraic convenience. See the derivation of choice probabilities in appendix 3.

Note that unlike the linear in means model, it is not necessary to require $J \in [0,1)$ because here it has a different interpretation, as a utility parameter. We will almost always discuss the model as if $J > 0$ as this is the standard case of interest in the literature, but theory imposes no natural upper bound on J. Analogous to our initial analysis of the linear in means model, we make two error assumptions. First, the expected value of the unobservable ε_i term is independent of observable features of the individual and any features of his group:

$$F(\varepsilon_i|(x_j)_{j\in g}, y_g, i \in g) = F_\varepsilon(\varepsilon_i). \tag{60}$$

Second, any pair i and j of the errors are conditionally independent within and across groups:

$$\begin{gathered} F(\varepsilon_i, \varepsilon_j|(x_k)_{k\in g}, y_g, i \in g, (x_l)_{l\in h}, y_h, j \in h) = F_\varepsilon(\varepsilon_i) \cdot F_\varepsilon(\varepsilon_j) \\ \text{unless } i = j \text{ and } g = h. \end{gathered} \tag{61}$$

These conditions are the analogs of the error restrictions (8) and(9) that were initially imposed on the linear in means model. These conditions are substantially stronger than those that appear in the linear in means model as they impose conditional independence rather than set certain conditional expectations equal to 0. They are also stronger than needed for identification proofs. It is well understood in the discrete choice literature that median restrictions can play a role analogous to expected value restrictions in linear models.[45] We make them here for ease of exposition and to link directly to theoretical results as developed in Brock and Durlauf (2001a,b, 2007).

The decision problem for this binary choice context is simple: individual i chooses $+1$ if and only if $V_i(1) - V_i(-1) \geq 0$. Hence

$$\mu(\omega_i = 1|x_i, y_g, i \in g) = \mu\Big(V_i(1) - V_i(-1) \geq 0\Big) =$$
$$\mu(\varepsilon_i \leq k + cx_i + dy_g + Jm_{ig}^e) = F_\varepsilon(k + cx_i + dy_g + Jm_{ig}^e).$$

[45] See Horowitz (2009) for an extended treatment.

As before, the model is closed by imposing an equilibrium condition on beliefs. Each person is assumed to know y_g, F_ε, and $F_{x|g}$, the empirical within-group distribution of x_i. When the population size is large, equilibrium requires that the expected value of the average choice level in the population, given this information, is defined by

$$m_g = 2\int F_\varepsilon(k + cx + dy_g + Jm_g)dF_{x|g} - 1. \tag{62}$$

There typically does not exist a closed form solution for m_g.

McFadden (1974) observed that the logit, probit, and similar discrete-choice models have two interpretations. The first interpretation is that of individual random utility. A decisionmaker draws a utility function at random to evaluate a choice situation. The distribution of choices then reflects the distribution of utility, which is the object of econometric investigation. The second interpretation is that of a population of decisionmakers. Each individual in the population has a deterministic utility function. The distribution of choices in the population reflects the population distribution of preferences. Brock and Durlauf (2001a) (and theoretical models such as Blume (1993)) extend this idea to games. One interpretation of this game theoretic approach is that the econometrician confronts a population of random-utility maximizers whose decisions are coupled. These models extend the notion of Nash equilibrium to random-utility choice. The other interpretation views an individual's shock as known to the individual but not to others in the population (or to the econometrician). In this interpretation, the Brock-Durlauf model is a Bayes-Nash equilibrium of a game with independent types, where the type of individual i is the pair (x_i, ε_i). Information is such that the first component of each player i's type is common knowledge, while the second is known only to player i.

ii. identification

Identification of the parameters in the binary choice model holds for very different conditions than were seen in the linear in means case. These differences derive from the nonlinear nature of the binary choice and do not require that the functional form F_ε is known a priori. The following theorem is proved in Brock and Durlauf (2007). We emphasize that the theorem's conditions are sufficient, not necessary, and were chosen to render the sources for identification transparent.

Theorem 8. Identification of the binary choice model with social interactions. *Suppose for the binary choice model social interactions described by equations* (59) *through* (62),

i. *conditional on* (x_i, y_g), *the random payoff terms* ε_i *are distributed according to* F_ε, *and* $F_\varepsilon(0) = 0.5$;

ii. F_ε *is absolutely continuous with associated density* dF_ε. dF_ε *is positive almost everywhere on its support, the interval* (L, U), *which may be* $(-\infty, \infty)$;

iii. *for at least one group* g, *conditional on* y_g, *each element of the vector* x_i *varies continuously over all* $\mathbf{R}$ *and* $\mathrm{supp}(x_i)$ *is not contained in a proper linear subspace of* $\mathbf{R}^{\mathbf{R}}$;

iv. y_g *does not include a constant; each element of* y_g *varies continuously over all* $\mathbf{R}$; *at least one element of* d *is non-zero; and* $\mathrm{supp}(y_g)$ *is not contained in a proper linear subspace of* $\mathbf{R}^{\mathbf{S}}$.

Then k, c, d, J *and* F_ε *are identified up to scale.*

The intuition for why identification holds is as follows. Within a given group, $dy_g + Jm_g$ is constant for all agents. Assumptions i)–iii) are sufficient to ensure that within that group, the parameter vector c and density function F_ε are identified up to scale. Identification of these objects up to scale was originally established by Manski (1988). The assumptions stated here allow for a proof structure that mimics Horowitz (2009). Assumption iv) ensures that k, d and J are identified up to scale. Identification of k is trivial if the other parameters are identified. The reason why d and J are identified is that the unbounded support on the y_g element with a nonzero coefficient ensures that m_g and y_g cannot be linearly dependent. This follows simply from the fact that m_g is bounded between -1 and 1. This bound is not driven by any functional form assumption but follows from the fact that the expected choice values are functions of the choice probabilities which are bounded between 0 and 1. Hence the argument for identification is analogous to one of the basic reasons why bounds can be established on probabilities in the partial identification literature. (See Manski (2003) for a synthesis.) Note as well that this is not an identification at infinity argument.

This theorem extends Brock and Durlauf (2001a,b) who proved identification when F_ε is a negative exponential distribution of the type used in appendix 3 and Brock and Durlauf (2006) who proved identification for general F_ε when F_ε is known a priori. Clearly the conditions of this theorem can be relaxed. For example, if condition iii) holds for all groups, then one can allow for multiple $F_{\varepsilon g}$'s, i.e. different group-specific distributions. Similarly, one does not need unbounded supports for all regressors, rather what one needs is a large enough support for a nonlinear relationship between m_g and y_g to ensure identification.

iii. observability of actions

The identification results in Brock and Durlauf (2007) are sensitive to the assumption that individuals react to expected rather than realized behaviors of others. This follows from the assumption that an individual's random shock is observed only by himself. Soetevent and Kooreman (2007) build a game theoretic model with a different assumption. They assume that each individual knows the other individuals' shocks, that shocks are invisible only to the econometrician. Thus in equilibrium, each individual's expectation of the average choice of others will be the realized average choice of others. Soetevent and Kooreman have replaced the incomplete information and Bayes-Nash equilibrium of Brock and Durlauf (2001a) with complete information and Nash equilibrium. They justify their informational assumption by presuming to study interactions in relatively small groups of given sizes in which choices of other individuals are assumed to be fully observed, and therefore an individual's payoff depends on the actual choice of others in his group, as opposed to expected choices as in Brock and Durlauf. This difference in information structure and the resulting equilibrium concept makes for an interesting contrast between the identification conditions in the two models. The equilibrium payoff difference equation (59) now becomes

$$V_i(1) - V_i(-1) = k + cx_i + dy_g + \frac{J}{n_g - 1} \sum_{j \neq i} \omega_{jg} - \varepsilon_i. \quad (63)$$

Like Brock and Durlauf, Soetevent and Kooreman focus on pure Nash equilibria with binary outcomes and estimate the model in effect as a system of simultaneous equations by means of simulation methods. Each individual choice is determined by the rule

$$\omega_{ig} = \begin{cases} 1 & \text{if } k + cx_i + dy_g + \frac{J}{n_g - 1} \sum_{j \neq i} \omega_{jg} > \varepsilon_i; \\ -1 & \text{if } k + cx_i + dy_g + \frac{J}{n_g - 1} \sum_{j \neq i} \omega_{jg} \leq \varepsilon_i \,. \end{cases}$$

Unlike the Brock and Durlauf model, for given values of parameters (k, c, d, J) and data (x_i, y_g), the decision rules for the individual agents may not produce unique strategy profiles.[46] This creates a very different multiple equilibrium problem than occurs in Brock and Durlauf, since in the latter, each agent has a unique strategy profile given the expected average choice m_g. Consequently, the normal approach of forming the likelihood function would not be appropriate in their case even when the ε_i's are independent.

[46] Soetevent and Kooreman (2007, pp. 602-3). This finding verifies the claim by Krauth (2006b) that with small (finite-size) social groups, the Brock and Durlauf model can exhibit multiplicity of strategy profiles whenever observed group behavior exerts any influence. The range of equilibrium group behavior depends on the size of the social group as well as its strength of influence.

Soetevent and Kooreman employ simulation-based estimation methods to compute the likelihood that any choice pattern would be observed. Their approach accounts for the potential multiplicity of non-cooperative equilibria. For parameter values that generate multiple equilibria they assume that the equilibria are equally likely, which in turn guarantees statistical coherency of the model.[47] Simulation of the model over different regions of the parameter space allows for calculation of the number of equilibria for draws of the ε_i's which are assumed to be i.i.d. normal. The procedure skirts the issue of exact identification of the model (no proof of identification is given) but provides a practical approach for implementation of their theoretical model. Also, in their actual estimation, Kooreman and Soetevent exclude contextual effects by setting $d = 0$.

The multiplicity of equilibria in Soetevent and Kooreman is very similar to types of multiplicity that have been studied in the industrial organization literature. Tamer (2003) launched a now thriving literature on multiple equilibria and partial identification by means of bounds for industrial organization contexts. This body of work, surprisingly, has had little contact with the social interactions literature. Clearly both literatures would benefit from integration.

[47] The assumption that all equilibria are equally likely is questionable. Blume and Durlauf (2003) show that in dynamic analogs of the Brock and Durlauf model, the percentage of time spent in the vicinity of the highest average utility equilibrium exceeds that of other equilibria; similarly Brock and Durlauf (2001b), for a version of the discrete choice model of social interactions in which the conditional probabilities of each choice depend on the realized choices of others, show that the equilibrium choice configuration will assign almost all probability to the social optimum as the population becomes large. While these analyses employ different microfoundations from Soetevent and Kooreman, they suggest that not all equilibria are equally likely. We thank James Heckman for discussion on this general issue.

iv. unobserved group effects

Unobserved group effects may be introduced in a fashion directly analogous to the linear in means model. Specifically, payoff differentials are described by

$$V_i(1) - V_i(-1) = k + cx_i + dy_g + Jm_g + \alpha_g - \varepsilon_i.$$

Here α_g is a fixed effect and equilibrium is required. Recall that individual agents are assumed to observe α_g while the analyst does not.

Without any restrictions on this fixed effect, it is evident that identification breaks down. Note that the presence of a fixed effect does not affect identification (up to scale) of c and F_ε. This holds because α_g is constant within a group and so is subsumed in the constant term. To see why the other parameters are not identified, observe that parameter values k, d and J are observationally equivalent to $\bar{k}$, $\bar{d}$ and $\bar{J}$, that is, for all $y_g \in \operatorname{supp} y$,

$$k + dy_g + Jm_g + \alpha_g = \bar{k} + \bar{d}y_g + \bar{J}m_g + \bar{\alpha}_g$$

if one chooses $\bar{\alpha} = \alpha + Jm_g$ and $\bar{J} = 0$. Thus J and d are not identified. (See Brock and Durlauf (2007) for an elaboration.) We can therefore state:

Theorem 9. Non-identification with unobserved group effects. *In the presence of unobserved group interactions whose properties are unrestricted, the parameters of the binary choice model with social interactions are not identified up to scale.*

In response to unobserved group effects, instrumental variables and differencing strategies are available just as occurs for linear models. Our remarks on instrumental variables for the linear in means model apply for the binary choice context as well and so are not repeated. Instead, we focus on two strategies, one which parallels the linear in means model and one which is new and only applies in the binary choice context.

a. panel data

We first consider how panel data can be used to eliminate unobserved group effects for the binary choice model. Panel data, of course, allows one to consider differencing methods. The notion of differencing in panels for binary choice data is more subtle than was the case for the linear in means model since it involves considering differences in probabilities across time. Chamberlain (1984) provides the generalization of differencing to discrete choice contexts. Identification of social interactions with differencing is studied in Brock and Durlauf (2007) who consider

$$V_{it}(1) - V_{it}(-1) = k + cx_{it} + dy_g + ey_{gt} + Jm_{gt} + \alpha_g - \varepsilon_{it}. \tag{64}$$

The vector y_{gt} is introduced in order to distinguish between those contextual effects that are time varying and those that are not. Applying Chamberlain's ideas on quasi-differencing of discrete data to models with social interactions, Brock and Durlauf verify a corollary to theorem 8.

Corollary 2. Identification of a subset of parameters with panel of the binary choice models of social interactions with fixed effects. *For the binary choice model with social interactions described equations* (59)–(62), *assume that within period-choices are described by equation* (64) *and that the model equilibrium conditions hold period by period. If the assumptions of theorem 8 hold for all t, then c, e and J are identified up to scale whereas k and d are not identified.*

b. partial identification

For binary choice models, Brock and Durlauf (2007) have proposed partial identification approaches to social interactions which involve weak assumptions on unobservables. We consider two examples. The partial identification arguments we develop are qualitatively different from those that typically appear in the econometrics literature. The reason is that we do not establish probability bounds. Rather, we show how certain empirical observations represent evidence of social interactions, even though parameter magnitudes cannot be bounded. The approach we describe is theory-dependent in the sense that it involves asking how the introduction of unobserved heterogeneity into various models affects their properties. Put differently, we are concerned with uncovering "footprints" of social interactions in heterogeneity-filled environments using various theoretical models as the basis for the analysis.

Our first example of a weak assumption is first order stochastic monotonicity of group level unobservables. We assume that y_g is measured so that $d \geq 0$. We denote the conditional distribution of the unobservable given y_g as $F_{\alpha_g|y_g}$. Letting $\succ$ denote first order stochastic dominance, and using $>$ when comparing vectors to mean that each element of one vector is greater than the corresponding element of the other, we assume

$$\text{if } y_g > y_{g'}, \text{ then } F_{\alpha_g|y_g} \succ F_{\alpha_{g'}|y_{g'}}. \tag{65}$$

This assumption is sufficient to produce partial identification of social interactions.

Theorem 10. Pattern reversals and partial identification of endogenous social interactions. *For the binary choice model with social interactions described by equations* (59)–(62) *suppose that the distribution of fixed effects exhibits first order stochastic dominance with respect to the contextual effects as characterized by equation* (65)*. If assumptions i)–iv) in theorem 8 hold and*

$$y_g > y_{g'} \text{ and } \mathrm{E}(m_g|y_g) < \mathrm{E}(m_{g'}|y_{g'}), \tag{66}$$

then it must be he case that $J > 0$ and J is large enough to produce multiple equilibria.

The term "pattern reversals" refers to the case where the observed characteristics of two groups suggest one ordering in their expected average outcomes, while the opposite ordering in fact holds.[48] This reversal of outcomes with respect to fundamentals can occur for three reasons. One possibility is that the observed outcome ordering is due to sampling error. This is irrelevant to identification because of the analogy principle. The second reason is that the unobserved group effects reverse the ordering that is implied by dy_g. This is ruled out, in an expectations sense, by the stochastic dominance assumption. The only remaining reason for the pattern reversal is that there are multiple equilibria associated with m_g such that the low y_g group has coordinated on the high expected average outcome equilibrium whereas the high y_g group has not. This is why the theorem requires multiple equilibria. To be clear, endogenous social interactions may be present when no pattern reversal occurs. All that can be said is that a pattern reversal in the presence of stochastic dominance in the sense of (66), is evidence of social interactions.

[48]Of course, expected group values are not directly observed. Our identification analysis replaces sample means with population means, following the analogy principle.

Our second example involves restricting the conditional density of the unobserved group interactions given observed group characteristics, that is, $dF_{\alpha_g|y_g}$, via unimodality,

$$\text{for all } y_g,\ dF_{\alpha_g|y_g} \text{ is unimodal.} \tag{67}$$

This assumption is sufficient to verify

Theorem 11. Partial identification of endogenous social interactions when the density of unobservables is unimodal. *For the binary choice model with social interactions described by equations* (59)–(62), *suppose that fixed effects are added as characterized by equation* (67). *If assumptions i)–iv) in theorem 8 hold, then*

i. *if $J = 0$, then $dF_{m_g|y_g}$ is unimodal;*

ii. *if $J > 0$ is large enough to produce multiple equilibria for the binary choice model with social interactions, then $dF_{m_g|y_g}$ is multimodal.*

This result also is based on multiple equilibria. In this case, the multiple equilibria produce the multimodality described in the theorem. Two observations should be made about this result. First, no analogous result exists for the unconditional density of expected outcomes, dF_{m_g}. The reason is that integrating $dF_{m_g|y_g}$ over y_g to produce dF_{m_g} would not necessarily preserve multimodality if it is present in the conditional density and, in contrast, may spuriously produce it when it is absent from the conditional density. This follows from the nonlinear relationship between m_g and y_g. Second, multimodality is sufficient but not necessary for multiple equilibria in $dF_{m_g|y_g}$ as mixture densities are not necessarily multimodal.[49]

[49] See Lindsay (1995, p. 4–5) for a nice example of a unimodel two-part mixture.

v. self-selection

Self-selection for discrete choice models has generally been handled using instrumental variables methods. The concerns we articulate about this strategy for the linear in means model apply to the discrete choice context as well. In parallel to the case of group level unobservables, Brock and Durlauf (2007) provide a number of partial identification results which hold under relatively modest assumptions.

To do this, Brock and Durlauf (2007) treat the membership question as the outcome of a matching problem and place some restrictions on the equilibria that emerge from the matching. Matching is assumed to occur with respect to an individual index A_i and a group index T_g, defined as

$$A_i = cx_i - \varepsilon_i \tag{68}$$

$$T_g = dy_g. \tag{69}$$

In the context of peer effects in classrooms, A_i may be thought of as student ability and T_g as teacher quality. For simplicity, the individual characteristics x_i are assumed to be measured so that $c \geq 0$.

Individuals and groups are matched in the sense that higher group quality is associated with higher individual quality. With respect to the equilibrium matching process, Brock and Durlauf assume

$$\text{For any pair of groups } g \text{ and } g', T_g > T_g' \Rightarrow F_{A|T_g} \succ F_{A|T_{g'}}. \tag{70}$$

This assumption is weaker than one which imposes strict assortative matching between better groups and higher ability individuals; the latter is predicted by models such as Becker (1973). The assumption is qualitatively consistent with a range of payoff functions that relate groups and individuals, see Sattinger (1993) for a survey of equilibrium matching problems. Note that (70) places an implicit restriction on $F_{\varepsilon|y_g\, i\in g}$. This assumption on matching leads to theorem 12.

Theorem 12. Partial identification of endogenous social interactions under assortative matching. *For the binary choice model of social interactions* (59)–(62), *assume assortative matching as described by* (70). *Then* $\mathrm{E}(m_g|T_g) > \mathrm{E}(m_{g'}|T_{g'})$.

This theorem is useful as it indicates how the presence of endogenous social interactions may be inferred if $T_g > T_{g'}$ yet $\mathrm{E}(m_g|T_g) < \mathrm{E}(m_{g'}|T_{g'})$. This can only occur, under the specification we have assumed, if group g has coordinated on an equilibrium expected average choice level other than the largest of the possible equilibria associated with it while group g' has coordinated on an equilibrium other than the lowest possible expected average choice level among those it could have attained. The existence of multiple equilibria immediately implies $J > 0$.

The use of assortative matching to facilitate identification may be extended to panel data. To do this, modify (68) and (69) so that $A_{it} = k + cx_{it} + \varepsilon_{it}$ and $T_{gt} = dy_g + ey_{gt}$ and that (70) holds period by period. Brock and Durlauf show:

Corollary 3. Equality of average outcomes with equal observable contextual effects. *Assume that the binary choice model of social interactions* (59)–(62) *holds for all t with equilibrium at each date and assortative matching as described by* (70). *If $J = 0$ or $J > 0$ but is sufficiently small that m_{gt} is unique, then $y_{gt} = y_{gt'}$ implies $m_{gt} = m_{gt'}$.*

vi. beyond the binary choice model

a. multinomial choice

Little econometric work has been done on multinomial choice models with social interactions; as far as we know the only contributions are Brock and Durlauf (2002,2006) and Bayer and Timmins (2007). Nevertheless these models seem important in many contexts. We develop the analog to the binary choice model and establish identification. Multinomial choice models with social interactions can exhibit multiple equilibria and bifurcations in parallel to those found in binary choice models. Appendix 3 provides a brief discussion.

To formulate the model, we consider an environment in which each member of a common group makes a choice l from a common choice set with L discrete possibilities, i.e. $\Omega_{ig} = \{0, \ldots, L-1\}$. The same choices are assumed to be available regardless of group. The common choice set assumption is without loss of generality, since if agents face different choice sets, one can always assume their union is the common set and then specify that certain choices have payoff of $-\infty$ for certain agents. Individual utility is defined as

$$V_{ig}(l) = k_l + cx_i + d_l y_g + J p^e_{igl} + \beta^{-1} \varepsilon_{il} \tag{71}$$

Here p^e_{igl} denotes agent i's expected value for the fraction of group g that chooses l. This generalizes the preference structure of the binary choice model to account for any number of choices. As before, β indexes the degree of heterogeneity in the random payoff term ε_{il}. We assume that these unobserved utility terms are independent and identically distributed with a common distribution function F_ε. In parallel to the binary choice case

$$F\left(\varepsilon_{il} | (x_m)_{m \in g}, y_g, i \in g\right) = F_\varepsilon(\varepsilon_{il})$$

and

for all i, j, g, h, k, l such that not all of $i = j, g = h, k = l$ hold

$$F\left(\varepsilon_{ik}\varepsilon_{jl} | (x_m)_{m \in g}, i \in g, y_g, (x_n)_{n \in h}, y_h, j \in h\right) = F_\varepsilon(\varepsilon_{ik}) \cdot F_\varepsilon(\varepsilon_{jl})$$

For this model, the probability that agent i makes a particular choice l is the probability that l produces the maximum payoff among all choices according to (71). This amounts to the joint probability defined by

$$\mu\left(\begin{array}{l} \varepsilon_{i0} - \varepsilon_{il} \leq \beta(k_l + c_l x_i + d_l y_g + J_l p^e_{igl} \\ \qquad\qquad - k_0 - c_0 x_i - d_0 y_g - J_0 p^e_{ig0}) \\ \qquad \vdots \\ \varepsilon_{i\,L-1} - \varepsilon_{il} \leq \beta(k_l + c_l x_i + d_l y_g + J_l p^e_{igl} \\ \qquad\qquad - k_{L-1} - c_{L-1} x_i - d_{L-1} y_g - J_{L-1} p^e_{ig\,L-1}) \end{array}\right)$$

Following an order-statistics argument[50] the probability of choosing l conditional on a particular realization of ε_{il} is

$$\prod_{j \neq l} F_\varepsilon\Big(\beta(k_l + c_l x_i + d_l y_g + J_l p^e_{igl}) - \beta(k_j + c_j x_i + d_j y_g + J_j p^e_{igj}) + \varepsilon_{il}\Big)$$

which immediately implies that the unconditional probability of the choice l is

$$p_{igl} = \int \prod_{j \neq l} F_\varepsilon\Big(\beta(k_l + c_l x_i + d_l y_g + J_l p^e_{igl}) \\ - \beta(k_j + c_j x_i + d_j y_g + J_j p^e_{igj}) + \varepsilon_{il}\Big) dF_\varepsilon \quad (72)$$

[50] Anderson, de Palma, and Thisse (1992, p. 36) provides a clean exposition.

In equilibrium, the aggregate choice probabilities of this general multinomial choice model are the solutions to

$$p_{gl} = \int \int \prod_{j \neq l} F_{\varepsilon}\Big(\beta(k_l + c_l x_i + d_l y_g + J_l p^e_{igl}) - \beta(k_j + c_j x_i + d_j y_g + J_j p^e_{igj}) + \varepsilon\Big) dF_{\varepsilon} dF_{x|g}. \quad (73)$$

Brock and Durlauf (2006) prove a general identification theorem for the multinomial choice model.

Theorem 13. Parametric identification for the multinomial choice model. *Let the true data generating process be given by* (71)–(73) *and assume that* F_ε *is known. Under the normalization* $k_0 = 0$, $c_0 = 0$, $d_0 = 0$, *and* $J_0 = 0$, *if*

i. *the mapping defined by equation* (73) *is globally one-to-one,*

ii. *the joint support of* x_i, y_g *is not contained in a proper linear subspace of* $\mathbf{R}^{\mathbf{R+S}}$,

iii. *the support of* y_g *is not contained in a proper linear subspace of* $\mathbf{R}^{\mathbf{S}}$,

iv. *no linear combination of elements of* x_i *and* y_g *is constant,*

v. *for each individual* i, *conditional on* y_g, x_i *is not contained in a proper linear subspace of* $\mathbf{R}^{\mathbf{R}}$,

vi. *none of the elements of* y_g *has bounded support,*

vii. *for all* l, p_{gl} *is not independent of* g,

then the vector of model parameters $(k_1, c_1, d_1, J_1, \ldots, k_{L-1}, c_{L-1}, d_{L-1}, J_{L-1})$ *is identified up to scale.*

Bayer and Timmins (2007) study a variation of the multinomial choice problem which focuses on choices across locations. They thus consider a population that forms a single group. We omit the group index in describing their model. In terms of the error structure, they set $\beta = 1$ in (71) and assume that the error terms are double exponentially distributed,

$$\mu(\varepsilon_{il} \leq \zeta) = \exp\Big(-\exp(-\beta\zeta + \gamma)\Big). \tag{74}$$

In terms of preferences, they follow the industrial organization literature in allowing for coefficient heterogeneity; their implementation of this heterogeneity is the opposite of the formulation one finds for hierarchical models in that the heterogeneity is determined by individual characteristics. In addition, they allow for unobserved choice-specific fixed effects. This produces choice-specific payoffs

$$V_i(l) = d_{il} z_l + J_{il} p_{il} + \xi_l + \varepsilon_{il}$$

where

$$d_{il} = d + D_l x_i, \qquad J_{il} = J + J_l x_i$$

and ξ_l is an unobserved location-specific effect. Bayer and Timmins use the functional form assumption (74) to construct instruments for estimation of this model. Their approach is a variant of models that all fall under the approach pioneered by Berry, Levinson, and Pakes (1995). An interesting aspect of Bayer and Timmins' work is that they focus on identification power that derives from changes in substitution patterns in multinomial choice models.

b. duration models

A number of studies have sought evidence in dynamic contexts based on duration and optimal stopping problems. Brock and Durlauf (2001b) first discussed this approach to modeling social interactions, albeit briefly. Sirakaya (2006) studies recidivism under the assumption the individual hazard function for an individual probationer depends on individual and neighborhood characteristics as well as social interactions among probationers. She allows for two types of social interactions: the mean hazard rate for probationer's in i's neighborhood, m_g, and mean time to recidivate in the population, r_g. These are estimated over the entire sample, and so are not time varying. The hazard rate she employs takes the functional form

$$m_{ig}(t, x_i, y_g, m_g, r_g) = \varepsilon_0(t) \exp(k + cx_i + dy_g + J_1 m_g + J_2 r_g), \qquad (75)$$

where $\varepsilon_0(t)$ denotes the baseline hazard function. (Since the model is expressed in continuous time, t is treated as an argument rather than a subscript.) Sirakaya addresses unobserved group effects by considering frailty model variations of (75) which helps address issues of unobserved group effects. Probationers are assigned to neighborhoods, which eliminates issues of self-selection. Sirakaya finds strong evidence that endogenous social interactions effects matter.

c. uncovering social interactions via their effects on laws of large numbers and central limit theorems

When social interactions generate dependence across agent behaviors in a group, their presence will have implications for the convergence rates of sample means and so will affect laws of large numbers and central limit theorems associated with data sampled from the group. A number of authors have proposed ways to exploit these effects in order to generate social interactions. A neglected theoretical predecessor to this social interactions work is Jovanovic (1987) who studies how interdependences could lead idiosyncratic shocks to produce aggregate uncertainty.

One approach of this type is due to Glaeser, Sacerdote and Scheinkman (1996). Their objective is to examine whether endogenous social interactions contribute to cross-city variation in crime rates. One can interpret cities as groups and code the crime/no crime choice as $\omega_i = 0$ and $\omega_i = 1$ respectively, in order to preserve our binary choice notation. If one thinks of persons across all cities as having a common probability p to commit a crime, then the crime rate for the population of city g will have an associated variance of $p(1-p)/n_g$. On the other hand, the presence of social interactions may increase this variance by introducing dependence across choices. To formalize this intuition, Glaeser *et al* consider a model in which individuals are placed on a line and indexed outwards from the origin, $\{0, \pm 1, \pm 2\}$, so that a city of size $n_g = 2n + 1$ will have individuals ranging from $-n$ to n. They propose a stochastic process for choices in which individuals in a city come in three types: type 0 individuals are always law abiding, type 1's are always criminals, and the remaining type 2's mimic their predecessor in the order. The assignment of types to locations on the line is i.i.d. They show that this model produces greater cross city variance in crime rates than the model without social interactions. Specifically, the variance in the crime rate is $p(1-p)(2-\pi)/\pi n_g$, where p is the probability that an individual with fixed behavior is a criminal, $(1-p)$ is the probability that a fixed individual never commits a crime, and π is the probability that an individual is a fixed type. Without social interactions, $\pi = 1$; that is, everyone is either type 1 or type 2. The presence of a group in the population that can be influenced raises the variance. They propose independently estimating p, and testing for social interactions by comparing the variance of cross-city crime rates with $p(1-p)/n_g$, that would result from no social interaction.

de Paula and Tang (2010) provide a set for tests of social interactions that may be interpreted as extensions of the Glaeser, Sacerdote and Scheinkman approach to looking at the properties of sample moments. de Paula and Tang consider binary choices in which the payoffs in (63) are modified to

$$V_i(1) - V_i(-1) = k + cx_i + J(x_i) \sum_{j \neq i} \omega_{jg} - \varepsilon_i.$$

There are several qualitative differences with (63). First, the endogenous social interactions parameter is allowed to depend on x_i. The authors are interested in the case where the parameter is negative as well as positive. Second, the endogenous effect depends on the number of agents making the choice, not the average. Third, the while the idiosyncratic shocks are still assumed to be conditionally independent, their distribution functions are modeled as $F_{\varepsilon|x}$, so that each distribution function may depend on the individual's characteristics. The information set for agents is assumed to be the same as in Brock and Durlauf (2001a) and elsewhere, and so leads to a Bayes-Nash equilibrium of the type we have studied.

de Paula and Tang argue that even with individual level data, this model is not identified. They therefore propose to study cases where groups are composed of individuals with identical x_i values. This leads them to argue that multiple equilibria in groups with a given $x_i = \bar{x}$ can identify the sign of $J(\bar{x})$. When $J(\bar{x})$ is positive, this is easy to see, since different groups will have different expected average choice levels and so in the Glaeser *et al.* sense produces excess intergroup variance in sample means. While this was originally recognized in Brock and Durlauf (2001b), de Paula and Tang develop the argument.

Further, de Paula and Tang argue that multiple equilibria can hold when $J(\bar{x})$ is negative. Their identification argument differs from our previous arguments on how multiple equilibria facilitate identification. de Paula and Tang shift their analysis from average choice to individual choices within a group. They show how negative $J(\bar{x})$ can mean that there is a negative correlation among intragroup choices. The key to their analysis is that even though aggregate quantities such as the expected average group choice may be constant, there are multiple equilibria with respect to which agents choose 1 as opposed to -1. This represents a new view of the informational content of multiple equilibria. This approach does require individual level data, unlike Glaeser *et al*. We conjecture that if one focuses on average group behavior, a negative $J(\bar{x})$ would lead to lower variance in the sample averages of group behavior than would occur when social interactions are absent, so that the aggregate approach of Glaeser *et al.* may be applied to test for social interactions for this case as well.

Another approach to the identification of social interactions via qualitative features of sample moments was suggested in the context of financial applications in an early paper by Brock (1993) and uses bifurcations around certain parameter values of a type where the Law of Large Numbers and the Central Limit Theorem break down as in, for example, the statistical mechanics models of Amaro de Matos and Perez (1991) and Ellis (1985). The basic idea of this second approach is to explore how strong dependence between choices can lead to qualitative changes in the properties of the joint stochastic process for a set of choices. These types of breakdowns occur in variations of the binary choice model, and they have some surprising consequences. In the linear in means model, it is natural to use the sample mean as an instrument for individuals expectations. Since equilibrium requires that individuals' beliefs are the correct first moment of the population distribution of choices, this amounts to using the sample mean as an instrument for the population mean. While this approach is well-justified in the linear in means model, it creates an equilibrium selection bias in binary choice models. If $\beta J > 1$, the model has multiple equilibria. Brock and Durlauf (2001b, pp. 3364–7) show that there is a function $H : \mathbf{R} \to \mathbf{R}$ which can be thought of as a potential function. It has the property that if m is a local maximum of $H(\beta J m)$, then m is an equilibrium expectation. Nonetheless, as the population size grows, $\lim_{n\to\infty} m_g \in \operatorname{argmax} H(\beta J m)$. Generically, this set is a singleton, and so the procedure of replacing the population mean with the sample mean in effect selects one equilibrium, the equilibrium which globally maximizes $m \mapsto H(\beta J m)$.

The selection of equilibrium by an estimator should appear to be quite troubling. The argument has been made that different dynamical processes of choice revision by individuals (such as best-response and learning dynamics) select the potential-maximizing equilibrium.[51] For the economist who is aware of these results, the use of the sample mean as an instrument for beliefs in binary choice models may be a virtue rather than a vice. One implication of this selection effect is that estimates using the sample mean will behave discontinuously in the parameters of the model. The correspondence from model parameters to global maxima of H is upper hemi-continuous but not continuous — small changes in parameters can produce big changes in the location of the global maximum (although not in the maximal value of H).

Brock and Durlauf (2006, section 2.3) extend this type of argument to the multinomial case to show how a tiny change in the distribution $F_{h|g}$ of the characteristics of group g can cause a large change in the limiting value of the fraction of group g choosing choice l among possible choices $0, \ldots, L-1$, provided βJ is greater than some critical value. This approach suggests that for general social interactions structures a potential route to identification would be to estimate the sum of absolute values of correlations among members of a group, denote this S_t and look for dates t^* where S_t changes abruptly. While it is possible that the stochastic structure of the generating processes of unobservables and selection effects that have not been accounted for by estimating a model of selection into groups, e.g. equation (39) could display similar abrupt changes.

d. beyond Bayes-Nash equilibrium

Very recent work on discrete choice models of social interactions has focused on relaxing equilbrium belief restrictions. One approach is due to Li and Lee (2009) who employ an interesting data set on the 1996 Clinton versus Dole Presidential election. In this data set individuals were asked about their own intended vote and whether they thought their reference group members (where the reference group was well defined in the data set) would vote for Clinton, Dole, a third party candidate, or not vote at all; these data are trimmed to produce a binary choice between the two major party candidates. Using this data on the beliefs of each respondent about the voting choices of his reference group, Li and Lee compute a subjective expectation, which they denote by p_{ig}, for each individual i, to play the role of m_{ig}^{e} in equation (62) above, and use it to test the null hypothesis of equilibrium beliefs by the goodness of fit in the binary choice model with independent types and equilibrium beliefs, as opposed to the subjective expectations they construct. Using maximum likelihood estimation methods, Li and Lee produce two interesting results in this part of their paper. First, they show that estimation effectiveness (measured by the size of the likelihood and in-sample and out-of-sample prediction results) of the binary discrete choice model with social interactions is improved when the subjective expectation data are used in place of the equilibrium beliefs version of the model. Second, they reject the null hypothesis of rational expectations. Incorporation of group level unobservables does not qualitatively affect these findings.

6 Experimental Approaches

This section considers different approaches to the identification of social interactions that involve various forms of experiments, ranging from the laboratory experiments in which the analyst is free to specify much of the socioeconomic environment to quasi-experiments in which a change in some environment produces experimental-type conditions to social experiments in which a policy change is implemented in order to generate evidence of social interactions effects.

i. laboratory experiments

Given the difficulties involved in identification of social interactions from non-experimental data, laboratory experiments would seem to offer a promising alternative for studying social interactions. It should be possible, for instance, to create experimental designs such that $\bar{x}_g$ does not lie in the span of the elements of y_g, thus achieving differentiation of contextual from endogenous effects in the linear in means model. Unobserved group characteristics are essentially a measurement problem. By controlling what group members know about each other, and by defining the environment of the interaction, unobserved group characteristics can be eliminated. Finally, group membership can be explicitly controlled, which addresses the self-selection issues.

The trustor's strategies are illustrated in figure 1. The only Nash equilibrium in

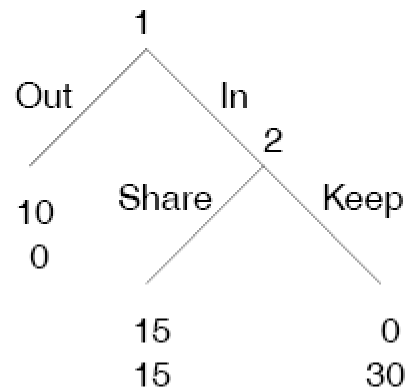

Figure 1: A Trust Game

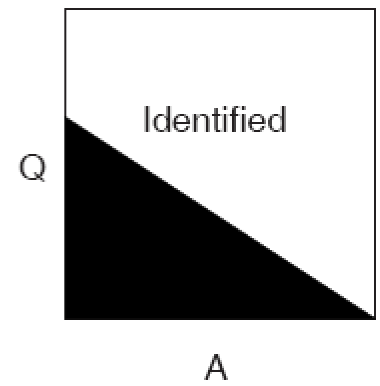

Figure 2: The Parameter Space

monetary payoffs is for the trustee to keep everything given to her, and for the trustor to stay Out. We might imagine however, that trustors and trustees both have utility functions exhibiting pro-social preferences. We suppose that the trustees are of two types: *Keepers* care only about monetary payoffs, and so keep all the money. *Sharers* are sufficiently motivated by fairness or reciprocity that they return half of the proceeds. The probability that a given trustee is a sharer is Q. This is a parameter known to the trustors but not to the econometrician, who must estimate it. Trustors also come in types. The utility of going In is the sum of the expected return, $15Q$, and a utility of being altruistic α. That is, $u_o(I) = 15Q + \alpha$. The utility of Out is just the monetary reward, $u_o(O) = 10$. The type distribution for trustors is uniform: $\alpha \sim U[0, A]$. Trustors and trustees are drawn independently from the appropriate type distributions and are matched to play. The equilibrium of this game is simple to describe.

Sharers Share, and Keepers Keep. Trustors of type $\alpha > 10 - 15Q$ pay In, while trustors of type $\alpha < 10 - 15Q$ stay Out. The econometrician wants to estimate the parameters of the type distributions, A and Q. The possible parameters are $(A, Q) \in [0, 10] \times [0, 1]$. Parameter values (A, Q) such that $A + 15Q > 10$ are point-identified. This region is labeled in figure 2. For these parameter values, the fraction of trustors who play In identifies A, and Q is identified by the fraction of trustors who Share. For parameter values on the other side of the boundary, all trustors stay Out, the trustees never get to choose, and their type is never revealed. In this game, it is the action of trustors that allows identification of the trustee type distribution parameter to be observed.

One advantage of the Bayesian framework is that it makes possible inferences across games. For instance, Dufwenberg and Gneezy (2000) consider a variant on the game of figure 1 where trustees can make any division of 20 (rather than 30) should they get the move, and trustors can choose to play In or Out, and receive x, which is varied across treatments. If it is assumed that the type distributions are independent of x, then by changing the treatment, any type distribution can be identified.

To this end, we suppose that trustors are motivated by three potential considerations: The monetary return, conformance to a sharing norm, and altruism. The utility function of a trustor who gives ω_o and receives ω_e from the trustee is

$$u_o(\omega_o, \omega_e) = 10 - \omega_o + \omega_e - \frac{\psi}{2}(\omega_o - \eta)^2 + \rho\omega_o.$$

The first term is the monetary payoff. The second term is the disutility of non-conformity to a social norm η, which is a feasible transfer. The third is altruism, the utility of giving. A type for a trustor is a triple (ψ, η, ρ), and the type space is $T_o = \mathbf{R}_+ \times [0, 10] \times \mathbf{R}_+$. A strategy for the trustor is a function $\sigma_o : T_o \to [0, 10]$.

The trustee has the utility function of the form

$$u_e(\omega_e, \omega_o) = \delta(3\omega_o - \omega_e) - \frac{\phi}{2}(\omega_e - \gamma\omega_o)^2 + \alpha\omega_o\omega_e, \tag{76}$$

where the parameter quadruple $(\delta, \phi, \gamma, \alpha)$ describes the trustee's type. For the utility function described in equation (76) the first term is the monetary payoff, the second is conformity to a social norm, and the third creates a taste for trustworthiness. Notice that the marginal utility of giving depends upon what has been received.

We will simplify our analysis by imposing a constraint on the type space for trustees, that the marginal utility of conformity is positive: $\phi > 0$. Hence utility can be renormalized so that $\phi = 1$. The marginal utility of the transfer ω_e for the trustee when the trustor transfers ω_o is then

$$u_e'(\omega_e, \omega_o) = -\bar{\delta} - \omega_e + (\bar{\gamma} + \bar{\alpha})\omega_o + \nu,$$

where $\bar{\delta}$, $\bar{\gamma}$ and $\bar{\alpha}$ are population means, and

$$\nu = \omega_o(\varepsilon_\alpha + \varepsilon_\gamma) + \varepsilon_\delta.$$

By construction, $\mathrm{E}(\nu) = 0$ since the ε's are deviations from population means. The type of a trustee is, given our normalization, the vector (α, γ, δ).

We suppose that individual trustees can differ in their perception of what the norm is, but this is not essential for our analysis.[57] The type space is defined by 1) non-negativity of the utility parameters, 2) that it is not the norm to give back more than the gross return on the transfer from the trustor, and 3) an *a priori* constraint on the transfer ω_e, that it not exceed the amount of money the trustee has been alloted to divide: $\omega_e \leq 3\omega_o$. Without loss of generality, $\gamma + \alpha - \delta \leq 3$, since behaviors of types with $\gamma + \alpha - \delta > 3$ are indistinguishable from that of types on the boundary, $\omega_e = 3\omega_o$. Thus the type space for the trustees is

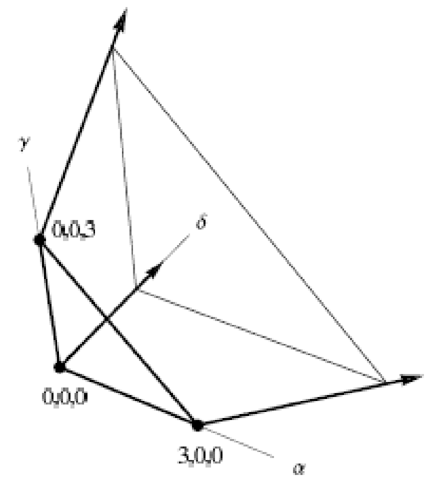

Figure 3: Type Space, $T_e = (\alpha, \delta, \gamma)$

$$T_e = \{(\alpha, \delta, \gamma) \in \mathbf{R}_+^2 \times [0,3] : \alpha + \gamma - \delta \leq 3\}$$

and trustee strategies are functions $\sigma_e : T_e \times [0,10]$ such that $\sigma_e(t, \omega_o) \leq 3\omega_o$.

[57] Perceptions of the norm come from the world external to the experiment. If we believed that individuals completely internalized the experiment, then we could impose an additional equilibrium condition on the norm. This belief, however, which we would require for observations of real social phenomena, is unnatural for the lab. Here is an example of how, by not being able to control the frame, the laboratory setting introduces additional noise not present in the world.

The specification of the Bayesian game is completed by specifying type distributions μ_o and μ_e on the type spaces T_o and T_e, respectively, for trustors and trustees. Each individual trustor and trustee knows his own type, and the distribution from which the other type is drawn. In this Bayesian game, the type of the trustor is irrelevant to the trustee since the trustee sees the trustor's action when she must choose. The trustor, however, cannot be certain about how much the trustee will return. The trustor will maximize expected utility, where the expectation will be over the type of the trustee and the trustee strategy function is known to him. The econometrician knows the structure of the game, but sees only transfers ω_o and ω_e. The econometricians task is to estimate the type distribution, thereby pinning down the relative importance of the different motivations for the transfer of money in the population of experimental subjects.

It is easy to see in this framework how identification problems arise. First, suppose that the type distributions are such that all decisions are interior.[58] The first order condition requires that $u'_e(\omega_e, \omega_o) = 0$, and so

$$\omega_e = -\bar{\delta} + (\bar{\gamma} + \bar{\alpha})\omega_o + \nu$$

and it is clear that while the marginal rate of substitution between monetary reward and conformity can be identified from the trustee's behavior, the social norm and marginal rate of substitution between altruism or trustworthiness and conformity cannot. One might argue that this is due to the excessive simplicity of the structural assumptions. A more natural assumption might be to assume that the norm is affine rather than linear. This introduces another parameter, and the consequence is that the marginal rate of substitution between private return and conformity can no longer be identified either. Another possibility is to assume, for instance, that the norm is linear while trustworthiness is quadratic. This is no less arbitrary than our linear assumption, and leads to a mismatch between norm and equilibrium strategy which is in some sense more severe than the present model, since the equilibrium strategy would be quadratic in received transfers while the social norm is presumed to be linear.

[58] The set of parameters for which this will be true has a non-empty interior. We do not derive it here, but it is worth noting that sufficient conditions involve both trustor and trustee parameters, since if $\omega_o = 0$, then of necessity $\omega_e = 0$.

The trustee's behavior does not exhaust the possibilities for identification. The trustor knows the parameter values that the econometrician does not, and they are payoff-relevant for the trustor's decision. This, unfortunately, does not help. The trustor's optimal strategy is

$$\omega_o = \frac{\rho - 1 + \bar{\gamma} + \bar{\alpha}}{\psi} + \eta$$

(recall that we have assumed that we are in a region of $T_o \times T_e$ where the right-hand side is positive), and no additional information is revealed that allows for distinguishing $\bar{\alpha}$ from $\bar{\gamma}$. It is possible however, that variation in the initial stakes provided to the trustor and the rate of return on the transfer to the trustee could lead to additional identifying restrictions on the distributions of both trustor and trustee type distributions.

ii. quasi-experiments

Other authors have focused on changes in group composition whose purpose was not to study social interactions but whose structure is potentially informative of their presence. One well-cited example is Angrist and Lang (2004), which focuses on Boston's Metropolitan Council for Educational Opportunities (METCO).[59] This is a voluntary desegregation program that involves enrolling underprivileged inner city children in suburban public schools. Angrist and Lang (2004) show that the receiving school districts, which have higher mean academic performance than the sending ones, do experience a mean decrease due to the program. However, they also show that the interactions are merely compositional in that there is little evidence of statistically significant interactions of METCO students on their non-METCO classmates. Their analysis with micro-data from one receiving district (Brookline, Massachusetts) generally confirms this finding, but also produces some evidence of negative interactions on minority students in the receiving district. Since METCO is a voluntary program for both sides and thus involves self-selection both at the individual and at the receiving end, at best it can be thought of as uncovering treatment on the treated, which does not translate naturally into claims about social interactions per se for reasons we will discuss in detail below in the next subsection.

[59]Another prominent study of this type is Sacerdote (2001). See Durlauf and Ioannides (2010) for some assessment of its information content with respect to social interactions.

iii. Moving to Opportunity

There exists one intervention in group formation that has been implemented on a large scale in order to understand social interactions. Interest in understanding the effects of poor neighborhoods on their residents led the Department of Housing and Urban Development to implement the Moving to Opportunity (MTO) demonstration in Baltimore, Boston, Chicago, Los Angeles and New York, starting in 1994.[60] The program provided housing vouchers to a randomly selected group of families from among residents of high-poverty public housing projects. Within this subsidized group, families in turn were randomly allocated between two subgroups: one which received unrestricted vouchers; and another which received vouchers that could only be used in census tracts with poverty rates below 10% (these users are termed the experimental group). Members of the experimental group also received relocation counseling. The presence of both unrestricted and restricted voucher recipients is a nice feature of the demonstration.

[60]One reason why HUD implemented the MTO demonstration was that there was a prior program in the Chicago area that had found large effects from moves from inner city public housing to more affluent suburbs of the city. The Gautreaux program, named after the lead plaintiff in a law suit against the Chicago Housing Authority dating from 1967, led to the movement of some public housing families in Chicago to other parts of the city whereas other families moved to nearby suburbs. Sociologist James Rosenbaum is responsible for the construction of data sets of the families that participated in the Gautreaux program and initiated use of these data to study neighborhood effects. Rosenbaum (1995) is a good overview of Gautreaux findings, which found that families who moved to suburbs exceeded those who stayed in Chicago for a broad range of socioeconomic outcomes. For example, the percentage of college attendees among children who families moved to suburbs was 54% whereas the percentage for children whose families moved to other locations in Chicago was 21%; when one considers only 4-year colleges the attendance rates are 27% and 4% respectively. As Rosenbaum and other students of the Gautreaux data are well aware, there are problems with the data that delimit how informative they are with respect to social interactions. Information about families who moved to suburbs and then returned to the city is missing, so comparisons of city and suburban families at the time the data were collected suffers from self-selection problems. Self-selection was also present in the initial set of families who participated in the program, as the program was restricted to families that had good track records of public housing upkeep. MTO was explicitly designed to avoid these problems.

7 Suggestions for future directions

In this section we suggest some new directions we regard as promising in developing a full econometrics of social interactions.

i. measurement

The empirical literature on social interactions suffers from serious measurement problems. This is a first important area that needs new econometric work. Here we follow the discussion in Durlauf and Ioannides (2010). Economic theory does not dictate the appropriate empirical measures of contextual variables that a researcher ought to use. As a result, one for example finds Bertrand, Luttmer, and Mullainathan (2000) using the product of welfare usage and own-ethnic group intensity to explain individual welfare usage, whereas Aizer and Currie (2004) use the utilization rate of an individual's language group to measure social interactions on public prenatal-care utilization. Similarly, the empirical literature does not typically consider how social variables should interact with individual decisions, so that linearity assumptions are too often employed without reflection. If the reason why utilization of social services depends on the usage of others is because of information transmission, as argued by Bertrand *et al.*, then it is unclear why the percentage of users is the appropriate variable, as opposed to some nonlinear transformation, as presumably one only needs one neighbor to provide the information. While considering this type of problem in studying social interactions in marriage markets, Drewianka (2003) argues that a higher marriage rate in a community may reduce the propensity of unmarried people to marry as a higher rate hampers search.

ii. social interactions and prices

Most social interactions work has ignored the informational content of prices for group membership. For example, social interactions of residential neighborhoods will be reflected in housing prices, via standard hedonic price arguments. Nesheim (2002) is a pioneering advance in this regard, social interaction effects, measured as averages of parental characteristics, can be extracted from housing prices using hedonic pricing methods. Implementation of Nesheim's approach is facilitated when assumptions are made to allow an explicit solution for the hedonic price in terms of neighborhood characteristics; see Ioannides (2008) for a very straightforward way of doing this. So far, Nesheim's methods have not received the empirical attention they warrant. Bayer, Ferreira, and McMillan (2007) report some nonstructural hedonic regressions of housing prices on neighborhood characteristics. See also Bayer and Ross (2009) who propose using neighborhood prices to construct a control function to proxy for unobserved neighborhood characteristics.

iii. group characteristics as evidence of social interactions

Another dimension along which endogenous group formation can be used to provide evidence of social interactions is the equilibrium distribution of types across groups. The informational content of this distribution was first recognized in the context of racial discrimination in Becker (1971). Becker showed that taste-based discrimination may not manifest itself in black white wage differences but rather in segregation of a subset of firms. Analogous reasoning applies in social interactions contexts. Models such as Bénabou (1993, 1996) and Durlauf (1996a,b) emphasize how social interactions can produce stratification of neighborhoods by income; work such as Epple and Sieg (1999) and Calabrese, Epple, Romer, and Sieg (2006) show how these types of effects can be incorporated into sophisticated models of locational choice; the latter paper is of particular interest since social interactions are essential to the analysis. Yet another context where group compositions are informative about interactions concerns assortative matching, where as discussed above, following Becker (1973), supermodularity in production functions can produce efficient stratification of firms by ability. In general, the tight relationship between supermodularity and stratification (see Durlauf and Seshadri (2003) and Prat (2002) for examples of a tight supermodularity/stratification link for payoff functions other than those studied by Becker as well as for some caveats) has been underutilized as a strategy for uncovering social interactions. We believe that group composition represents a potentially powerful source of evidence on social interactions.

iv. joint modeling of group memberships and behaviors

Our discussion of prices and group characteristics as sources of information on social interactions suggests yet another direction for new research: the joint modeling of group memberships and behavioral choices as facets of a general decision problem. Brock and Durlauf (2006) give an example of this perspective using a logit framework; we borrow heavily from their original presentation. The basic idea of this approach is to model individuals as making joint choices of group memberships, $g \in \{0, ...G-1\}$, and behaviors, $l \in \{0, ...L-1\}$. Group choices are denoted as δ_i while ω_i continues to denote the behavioral choice. This joint decision is sequential as groups are chosen first and then behaviors are chosen once groups form; this particular sequencing renders the model mathematically equivalent to a standard nested logit model (Ben Akiva (1973) and McFadden (1978)) with the exception of the presence of endogenous social interactions.

The sequential logit structure ensures that choice probabilities at both stages have a multinomial logit probability structure. Defining $h_{ilg} = k_l + c_l x_i + d_l y_g$, the behavioral choices conditional on a group choice g will be defined by the probabilities

$$\mu\Big(\omega_{ig} = \bar{l}\Big|(h_{ilg}, p^e_{ilg})_{l=0}^{L-1}\Big) = \frac{\exp\beta(h_{ilg} + Jp^e_{ilg})}{\sum_m \exp\beta(h_{img} + Jp^e_{img})}. \tag{77}$$

Group choices reflect the fact that choices in the stage will produce utility in the fashion of our original multinomial choice model. This is operationalized by making the group choice probabilities depend on the expected utility of the choice ω_i will produce in the second stage. Letting $\delta_i = g$ code for the choice of group by individual i, these choices are also assumed to exhibit a logit structure:

$$\mu\Big(\delta_i = \bar{g}\Big|(h_{ilg}, p^e_{ilg})_{l=0}^{L-1}{}_{g=0}^{G-1}\Big) = \frac{\exp\beta_G Z_{i\bar{g}}}{\sum_g \exp\beta_G Z_{ig}} \tag{78}$$

where β_G denotes the heterogeneity parameter for the unobservable shocks associated with group choices and

$$Z_{ig} = E\Big\{\max_l(h_{ilg} + Hp^e_{ilg} + \varepsilon_{ilg})\Big|(h_{ilg}, p^e_{ilg})_{l=0}^{L-1}\Big\}.$$

Implicit in equation (78) is the existence of unobservable location-specific utility terms that are irrelevant with respect to the utility of a choice once the group is formed. A standard result is that[63]

$$E\Big\{\max_{l}(h_{ilg}+Hp^{e}_{ilg}+\varepsilon_{ilg})\Big|(h_{ilg},p^{e}_{ilg})_{l=0}^{L-1}\Big\}=$$
$$\beta^{-1}\log\Big(\sum_{i}\exp\beta(h_{ilg}+Jp^{e}_{ilg}\Big). \quad (79)$$

Equation (79), together with equations (77) and (78) produce a joint probability description of group memberships and behaviors

$$\mu\Big(\omega_{i\bar{g}}=\bar{l},\delta_i=\bar{g}\Big|(h_{ilg},p^{e}_{ilg})_{l=0\ g=0}^{L-1\ G-1}\Big)=$$
$$\frac{\exp\Big(\beta_G\beta^{-1}\log\big(\sum_l\exp\beta(h_{il\bar{g}}+Jp^{e}_{il\bar{g}})\big)\Big)}{\sum_g\exp\Big(\beta_G\beta^{-1}\log\big(\sum_l\exp\beta(h_{ilg}+Jp^{e}_{ilg})\big)\Big)}\cdot\frac{\exp\beta(h_{i\bar{l}\bar{g}}+Jp^{e}_{i\bar{l}\bar{g}})}{\sum_l\exp\beta(h_{il\bar{g}}+Jp^{e}_{il\bar{g}})}. \quad (80)$$

As is well known, the compatibility of the nested logit structure (80) with an explicit utility maximization problem requires conditions on the parameters.[64] One condition that ensures compatibility with a well posed maximization problem is $\beta_G \leq \beta$ (McFadden, 1978, pp.86–7). This condition requires that the dispersion of random payoff terms across groups is greater than the dispersion in random payoff terms across behavioral choices within a group. We close the model with the equilibrium condition: For all i, l and g, $p^e_{ilg} = p_{lg}$, which links the choices at the two levels.

[63]See, for example, Anderson, de Palma, and Thisse (1992, p. 46).

[64]See McFadden (1978) and Börsch-Supan (1990).

v. transition dynamics versus steady-state behavior

A final research direction we believe can prove to be important concerns the use of transitional behavior to uncover social interactions. The linear in means and discrete choice models look at steady state behaviors in the sense that these systems, including their dynamic analogs, conceptualize the data as drawn from their associated invariant measures. While the duration models we describe, especially that of de Paula (2009), focus on transitions in a population, there has yet to be much systematic exploration of the evidence on social interactions that may be found in transitional dynamics versus steady state behavior. By analogy, the steady state distribution of disease rates across locations in a region will not speak to the contagion mechanism for the disease in the way that would data on the transition of the disease across locations.

Brock and Durlauf (2010) provide an example of how transition dynamics can produce evidence of social interactions. They consider a population of perfect foresight actors who, in continuous time, are deciding whether to adopt a new technology. The cost of the new technology is falling over time. The payoff to adoption is the present discounted value of payoff from the time of adoption. The payoff to adoption is increasing in the fraction of the population that has adopted. Agents are indexed by a scalar x, in which the payoff function is strictly increasing. Data are restricted to $q(t)$, the adoption curve for the technology, i.e. the fraction of the population that has adopted as of time t, and f_x, the cross sectional density of x . Brock and Durlauf consider the case where the distribution of types among adopters at each t is unknown as well as the case where the distribution of types among adopters is known at each t.

From the perspective of steady state behavior, there is nothing that can be learned about social interactions; the steady state data will consist of an adoption rate for the population as well as a cutoff value x such that agents with $x_i < x$ have not adopted while other agents have adopted. Such an observation is fully consistent with individual payoffs being independent of the adoption decisions of others. However, the full adoption curve, which represents the transition dynamics for the steady state adoption rate, can be informative about social interactions. For example, even if f_x is unobservable, $q(t)$ can be informative about social interactions. Brock and Durlauf show that if one is willing to assume that f_x contains some mass points, discontinuities in $q(t)$ can only occur because of endogenous social interactions. This does not follow because of multiple equilibria as occurred in the partial identification results under primary choice. Rather it follows from the fact that since higher x types who have not yet adopted always have a greater incentive to adopt than lower x types, even though the population fraction of the lower x types is larger, self-consistent bunching can occur at particular t values; intuitively, at these jump points the lowest x will meet the first order condition for adoption with equality whereas all others do not.[65]

[65] The possibility that social interactions could induce discontinuity in adoption curves was first recognized in a relatively neglected paper by Cabral (1990). Differences in the economic environments studied by Cabral and Brock and Durlauf are discussed in the latter. That said Cabral has priority in discovering the qualitative finding.

Brock and Durlauf (2010) also consider a case where x is a vector consisting of an observed variable x_1 and an unobserved variable x_2. For conditional adoption curves $q(t|x_1)$, social interactions can produce pattern reversals where lower x_1 types adopt before higher x_1 types. A condition such as stochastic dominance of the conditional density of x_2 given x_1 is needed for this type of observation to represent evidence of social interactions. As such, the Brock and Durlauf results are another example of how delineation of a complete economic environment can allow for partial identification of social interactions under what appear to be modest assumptions. Arguments of their type can be taken further, as is done in Young (2010), which we discuss next.

vi. microfoundations

A final area that warrants far more research is the microfoundations of social interactions. In the econometrics literature, contextual and endogenous social interactions are defined in terms of types of variables rather than via particular mechanisms. This can delimit the utility of the models we have, for example, if the particular mechanisms have different policy implications. Put differently, the current generation of social interactions models focuses on a relatively crude division of social interactions between factors that are predetermined and those that are contemporaneous; while one can rationalize this division as structural, this is only true by assumption; work in evolutionary game theory, for example, has a much more subtle view of how endogenous interactions arise. Young (2010) is an important next step in the social interactions research program as it explicitly studies the different empirical implications of alternate social interactions mechanisms. Young derives implications for aggregate behavior by considering where the social interaction comes from. Behavioral economists may be interested in individual behaviors for their own sake, but Young demonstrates here that particular features of the process generating the social interaction determine aggregate behavior, and it raises the prospect that microeconomic and behavioral hypotheses about where social interactions come from may be identifiable from aggregate data.

Young writes on identification of types of social interactions in diffusion processes from a theoretical perspective. He examines different diffusion models in a rather general large-population setting. The different explanations, inertia, contagion, social influence and social learning, are sets of assumptions about individual behaviors. The outcome of the analysis is a set of distinct properties of the diffusion curve, a system aggregate, an emergent property of the system.[66] Inertia is the hypothesis that individuals learn privately, but delay in making their decision. He supposes that each individual i in the (continuum) population can be characterized by a switch rate, λ_i, which is independent of the numbers and identities of those who have already switched. Young shows that no matter the distribution of the λ_i, the adoption curve must be concave. Contagion is a process wherein a given individual adopts when she sees an instance of the innovation, or hears about it. Perhaps the most famous model of this kind is the Bass (1969) model of new product adoption. In these models the instantaneous rate at which an individual adopts will depend upon the size of the pool of current adopters. Adoption curves derived from contagion models will be S-shaped, and under some reasonable assumptions, it must decelerate when the pool of adopters exceeds $1/2$ the population. Social influence models are threshold models. Each individual has a threshold r_i. If the adopting pool contains fraction r_i of the population, then individual i will adopt. Under some mild assumptions, the adoption process either initially decelerates or it accelerates at a super-exponential rate over some time interval.

[66] Emergent properties of dynamical systems are properties or structural features that occur on scales of aggregation or temporal scales which are different from those of the rules defining the system. See Blume and Durlauf (2001) for discussion of emergence in socioeconomic environments

8 Conclusions

As this chapter has demonstrated, a wide range of identification strategies for uncovering empirical evidence of social interactions are available to empirical workers. These approaches range across linear and nonlinear models, cross-section and time series data, and involve a remarkably broad range of portfolios of assumptions. The existing set of identification results thus does not lend itself to any straightforward summary. Rather, the body of arguments we have described represent different approaches to producing evidence of social interactions at two levels. First, under "ideal" assumptions with respect to unobserved heterogeneity, identification questions revolve around the disentangling of types of social interaction effects: contextual versus endogenous. Second, under more realistic specifications of unobserved heterogeneity, i.e. grouped individual-level heterogeneity as emerges from endogenous group formation and group-level heterogeneity that is not related to social interactions, identification involves the question of whether any evidence may be adduced for social interactions, let alone whether the specific type of social interaction is recoverable from the observed data.

One way to understand the many methods we have described is that they represent points along an "assumptions/possibilities" frontier. As is true throughout economics, there is a tradeoff between the strength of assumptions made prior to empirical analysis and the precision of the empirical claims that follow. And the types of assumptions we have described, whether they represent restrictions on the probability structure of unobservable stochastic processes or substantive assumptions about individual behavior, can never be expected to hold literally. This should not jaundice the consumers of empirical work on social interactions any more than it should affect consumers of other types of empirical social science. Scientific progress, arises from the interaction of a priori beliefs, data and logical reasoning. We therefore regard the interplay of economic theory, econometrics and empirical work as all necessary ingredients in understanding the social determinants of individual behavior.

It is no disparagement, therefore, to the science of Human Nature, that those of its general propositions which descend sufficiently into detail to serve as a foundation for predicting phenomena in the concrete, are for the most part only approximately true. But in order to give a genuinely scientific character to the study, it is indispensable that these approximate generalizations, which in themselves would only amount to the lowest kind of empirical laws, should be connected deductively with the laws of nature from which they result; should be resolved into the properties of the causes on which the phenomena depend. In other words, the science of Human Nature may be said to exist, in proportion as the approximate truths, which compose a practical knowledge of mankind, can be exhibited as corollaries from the universal laws of human nature on which they rest; whereby the proper limits of those approximate truths would be shown, and we should be enabled to deduce others for any new state of circumstances, in anticipation of specific experiences.

John Stuart Mill, *A System of Logic* (1859)[67]

[67] *Collected Works of John Stuart Mill,* J. Robson ed. Indianapolis: Liberty Fund Press. Book VI, chapter iii, section 2, pp. 847-848.

A1 Derivation and analysis of equilibria in the linear in means model

i. structure

The linear-in-means model can be derived simply as the unique Bayes-Nash equilibrium of a game in which each individual's choice is determined by a private benefit and a conformity benefit. Not surprisingly, the utility functions are quadratic, and the conformity benefit is modeled as linearly decreasing in the quadratic deviation of an individual's choice from the average behavior of all other players. Individuals belong to a common group g of size n_g. Group membership is exogenous. An individual's realized utility depends upon his own choice and the choices of others. Preferences are expected utility, and are of the form

$$u_i(\omega_{ig}, \omega_{-ig}) = \theta_{ig}\omega_{ig} - \frac{\omega_{ig}^2}{2} - \frac{\phi}{2}\mathrm{E}((\omega_{ig} - \bar{\omega}_{-ig})^2) \tag{81}$$

where $\bar{\omega}_{-ig} = (n_g - 1)^{-1} \sum_{j \neq i} \omega_{jg}$ is the average choice of the others in g. The individual marginal benefit θ_{ig} can be linearly decomposed as follows:

$$\theta_{ig} = \chi_0 + \chi_1 x_i + \chi_2 y_g + \varepsilon_i + f_g \tag{82}$$

where x_i and e_i are observable and unobservable individual characteristics and y_g is a vector of observable group characteristics and f_g is a group characteristic observable to all individuals in the group but unobservable to the econometrician. The determination of individual choices is a game of incomplete information, since each individual, and only that individual, observes ε_i. (Group characteristics unobservable to individual group members are irrelevant to choices as this model exhibits certainty equivalence in individual choices.)

The ε_i elements are i.i.d. draws from a distribution on the real line $\mathbf{R}$ with mean 0. For expositional purposes it will be useful to write $\theta_{ig} = \gamma_i + \gamma_g + \varepsilon_i$ where $\gamma_g = \chi_0 + \chi_2 y_g + f_g$ is the internally (to the actors) observable group contribution to the marginal utility of ω_i, and $\gamma_i = \chi_1 x_i$, the externally (to the econometrician) observable contribution to marginal utility of an individual's characteristics.

ii. existence of equilibrium

In a Bayes-Nash equilibrium, each individual maximizes expected utility, taking the expectation on $\bar{\omega}_{-ig}$ with respect to his belief distribution, and all belief distributions will be correct. The first-order condition for individual i is

$$\gamma_g + \gamma_i + \varepsilon_i - \phi(\omega_{ig} - E\bar{\omega}_{-ig}) - \omega_{ig} = 0,$$

and so

$$\begin{aligned}
\omega_{ig} &= \frac{1}{\phi+1}\gamma_g + \frac{1}{\phi+1}\gamma_i + \frac{\phi}{\phi+1}E\bar{\omega}_{-ig} + \frac{1}{\phi+1}\varepsilon_i \\
&= \frac{\chi_0}{\phi+1} + \frac{\chi_1}{\phi+1}x_i + \frac{1}{\phi+1}f_g + \frac{\chi_2}{\phi+1}y_g + \\
&\qquad \frac{\phi}{\phi+1}\,\mathrm{E}(\bar{\omega}_{-ig}) + \frac{1}{\phi+1}\varepsilon_i
\end{aligned} \tag{83}$$

This equation justifies (6) when there is no group level unobservable and (30) when there is such an unobservable, since the coefficients in (6) and (30) are proportional to those in (83), assuming that an equilibrium exists. Notice that the shock in (83), $(\phi+1)^{-1}\varepsilon_i$, has a variance that is affected by the strength of the conformity parameter.

We find an equilibrium by positing a functional form with undetermined coefficients, and then solving for the coefficients to make the beliefs correct. It will be convenient to define $\bar{\gamma}_{-ig} = (n_g - 1)^{-1} \sum_{j \neq i} \gamma_j$ to be the mean observable type component in the population. This is simply a sample mean. We suppose that for each individual j,

$$\omega_{jg} = A\gamma_g + B\gamma_j + C\bar{\gamma}_{-jg} + D\varepsilon_j + F \tag{84}$$

We derive consistency of beliefs by assuming all individuals other than individual i are choosing according to this functional form, computing the best response for individual i, seeing that it is of this linear form, and then solving for the coefficient values such that A through F are common through the entire population. We compute the best response simply by deriving an expression for $\bar{\omega}_{-ig}$ by substituting from equation (84) into equation (83). After some algebra one can show that the coefficients in (84) must fulfill

$$A = \frac{1 + \phi A}{\phi + 1}, \quad B = \frac{1 + \frac{C}{n_g - 1}}{\phi + 1}, \quad C = \frac{\phi(B + \frac{n_g - 2}{n_g - 1} C)}{\phi + 1},$$

$$D = \frac{1}{\phi + 1}, \quad F = \frac{\phi F}{\phi + 1}.$$

Solving these equations gives the values of the undetermined coefficients. Thus

$$\omega_{ig} = \gamma_g + \frac{n_g - 1 + \phi}{(\phi+1)n_g - 1}\gamma_i + \frac{\phi(n_g - 1)}{(\phi+1)n_g - 1}\bar{\gamma}_{-ig} + \frac{1}{\phi - 1}\varepsilon_i \tag{85}$$

When the population size is large, this is approximately

$$\omega_{ig} = \gamma_g + \frac{1}{\phi+1}\gamma_i + \frac{\phi}{\phi+1}\bar{\gamma}_g + \frac{1}{\phi - 1}\varepsilon_i \tag{86}$$

where $\bar{\gamma}_g$ is the group-level average of γ_i.[68] Recalling the definitions of the γ terms,

$$\omega_{ig} = \chi_0 + \chi_2 y_g + f_g + \frac{\chi_1}{\phi+1}x_i + \frac{\phi\chi_1}{\phi+1}\bar{x}_g + \frac{1}{\phi+1}\varepsilon_i$$

where $\bar{x}_g$ is the group mean of the individual characteristics. This expression corresponds to the reduced form equation (11) in the text when there is no group-level unobservable. Extending the model in this and a variety of other ways to match the other specifications discussed in section 3 is straightforward.

[68]Note that $\bar{\gamma}_g$ is different from γ_g, which is the direct group marginal utility contribution for an individual's choice.

iii. uniqueness of Bayes-Nash equilibrium

A strategy for player i is a map $f_i(\gamma_g, \gamma_i, \gamma_{-ig}, \varepsilon_i) \mapsto \mathbf{R}$, where $\gamma_{-ig} = (\gamma_j)_{j \neq i}$. The preceding section demonstrates the existence of a symmetric Bayes-Nash equilibrium with linear strategies. Discrete-choice models of social interaction are replete with multiple equilibria, so one might believe that multiple equilibria may arise here as well. This is not the case.

Theorem A.1. Uniqueness of equilibrium in the linear in means model. *The Bayes Nash equilibrium strategy for the model* (81) *and* (82), *and defined by* (83), *is unique.*

Proof. Equation (83) implies that $(f_1^*, \ldots, f_{n_g}^*)$ is a symmetric Bayes Nash-equilibrium if and only if for all i,

$$f_i^*(\gamma_g, \gamma_i, \gamma_{-ig}, \varepsilon_i) = \frac{1}{\phi+1}(\gamma_g + \gamma_i + \varepsilon_i) + \frac{\phi}{\phi+1}\frac{1}{n_g-1}\sum_{j\neq i} \mathrm{E}\Big(f_j^*(\gamma_g, \gamma_j, \gamma_{-jg}, \varepsilon_j)\Big)$$

Let B_i denote the set of measurable functions $f_i : (\gamma_g, \gamma_i, \gamma_{-ig}, \varepsilon_i) \mapsto \omega_i$ and let B denote the product of the B_i. Define the operator $T : B \to B$ such that

$$T(f_1, \ldots, f_{n_g})_i(\gamma_g, \gamma_i, \gamma_{-ig}, \varepsilon_i) = \frac{1}{\phi+1}(\gamma_g + \gamma_i + \varepsilon_i) + \frac{\phi}{\phi+1}\frac{1}{n_g-1}\sum_{j\neq i} \mathrm{E}(f_j(\gamma_g, \gamma_j, \gamma_{-jg}, \varepsilon_j)) \, .$$

A strategy profile $(f_1^*, \ldots, f_n^*)$ is a (not necessarily symmetric) Bayes-Nash equilibrium if and only if it is a fixed point of T. A straightforward calculation shows that T is a contraction mapping. At any point $(\gamma_g, \gamma_{1g}, \ldots, \gamma_{n_g g}, \varepsilon_i)$,

$$
\begin{aligned}
|(Tf)_i - (Tg)_i| = \frac{\phi}{\phi+1}\frac{1}{n_g-1}\Bigg| \sum_{j \neq i} \mathrm{E}\Big(f_j(\gamma_g, \gamma_j, \gamma_{-jg}, \varepsilon_j)\Big) \\
- \mathrm{E}\Big(g_j(\gamma_g, \gamma_j, \gamma_{-jg}, \varepsilon_j)\Big)\Bigg| \\
\leq \frac{\phi}{\phi+1}\|f-g\|_\infty .
\end{aligned}
$$

Since T is a contraction, it has a unique fixed point, and so equilibrium is unique. □

A2 Proof of theorems 3, 4, 5 and 7 on social networks

For exclusive averaging Bramoullé, Djebbari, and Fortin (2009) have already proven that if the network is the union of groups, then I, A and A^2 are linearly dependent if and only if groups are all the same size. They also have shown that if the network is transitive and contains no groups, then $A^2 = 0$. All we need to show is that linear dependence implies transitivity, and that transitivity implies that the network is the union of weakly connected components each of which either has $A^2 = 0$ or is a group. For inclusive averaging we simply replicate the entire program.

We begin with an elaboration of theorem 4 which does not depend on how the weighted adjacency matrix is assembled. The proof of this theorem, when combined with theorem 2, implies theorem 4 in the text. Theorem 2 states that the failure of identification implies that I, A and A^2 are linearly dependent. Theorem A.2 implies that if these matrices are linearly dependent, then the network must be transitive. If the network is both transitive and undirected, it must consist of the union of groups, as stated in the theorem.

Theorem A.2. Characterization of networks admitting non-identification. *Let (V, E) be a network with a weighted adjacency matrix A such that $(I - JA)$ is invertible for all values of J. Suppose that $\lambda_0 I + \lambda_1 A + \lambda_2 A^2 = 0$ for some λ weights not all zero.*

i. If $\lambda_2 = 0$, the network is totally disconnected.

ii. If $\lambda_2 \neq 0$, and $\lambda_1 = 0$, with exclusive averaging, the network is the union of isolates and of pairs $\{i, j\}$ such that each i is connected only to his j and each j only to her i. With inclusive averaging, $A = I$ and the network is totally disconnected.

iii. If $\lambda_2 \neq 0$ and $\lambda_1 \neq 0$, the network is transitive.

In all three cases, the network is transitive.

Proof of theorem A.2. It follows from theorem 2.i that if the hypothesis of theorem 4 is true, there are scalars λ_0, λ_1, and λ_2, not all 0, such that $\lambda_0 I + \lambda_1 A + \lambda_2 A^2 = 0$. Now we turn to theorem A.2. If $\lambda_2 = 0$, then $\lambda_0 I + \lambda_1 A = 0$. If there are nodes i and $j \neq i$ such that $a_{ij} \neq 0$, then $\lambda_1 = 0$, and so $\lambda_0 = 0$, which is a contradiction. Thus if $\lambda_2 = 0$, the matrix A is diagonal. The network is totally disconnected (and, in particular, transitive).

If $\lambda_2 \neq 0$, there are scalars γ_0 and γ_1 such that $A^2 = \gamma_0 I + \gamma_1 A$. If $\gamma_1 = 0$, then $A^2 = \gamma_0 I$. Since the row sums of A and A^2 are both 1, $\gamma = 1$ and $A^2 = I$. If $(i,k) \in E$ and $(k,j) \in E$, then $j = i$. If not, $[A^2]_{ij} > 0$ for $j \neq i$. For exclusive averaging, $a_{ii} > 0$ if and only if there is a j such that a_{ij} and a_{ji} are both positive. There can be no isolates since for all i, $a_{ii} = 0$ while $[A^2]_{ii} > 0$. This social network is a collection of marriages; groups of size 2. Suppose now that averaging is inclusive. Since $A^2 = I$, $A^{-1} = A$. For general matrices, A need not be the identity matrix, but for these matrices it must be. To see this, take the k'th unit vector e_k. Then $Ae_k = a_{.k}$, the k'th column vector of A. Then $Aa_{.k} = A^2 e_k = e_k$. Thus $\sum_j a_{ij} a_{jk}$ is 1 for $i = k$ and 0 for all $i \neq j$. From the last claim it follows that $a_{ik} = 0$ since $a_{kk} > 0$ and all terms of the matrix are non-negative. Since $a_{ik} = 0$, $a_{ki} = 0$ or the first inequality fails to hold. This works for all $i \neq k$, so A is a diagonal matrix, and since the row sums are 1, it follows that $A = I$.

If $\lambda_2 \neq 0$ and $\gamma_1 \neq 0$, then for nodes $i \neq j$, $[A^2]_{ij} = \gamma_1 a_{ij}$. If there is path of length 2 from nodes i to j, then $[A^2]_{ij} > 0$, and hence $a_{ij} > 0$, so $(i, j) \in E$ and the network is transitive.

Suppose that the network is undirected. and suppose that there is a path from i to j. Transitivity implies that $(i, j) \in E$, so $(j, i) \in E$ and thus i and j are in the same strongly connected component. If j influences i, then i influences j. □

We now turn to the proof of theorem 3. The remainder of this appendix explores the case where $\lambda_2 \neq 0$ and $\lambda_1 \neq 0$. To proceed we need some facts about transitive graphs. (These can be found in many graph theory texts and the proofs are nearly immediate.) The vertex set V of any graph (V, E) can be written as the union of disjoint strongly connected components (V_g, E_g).

Lemma A.1. *If (V, E) is transitive, then*

i. *If $(i, j) \in E$ for some $i \in V_g$ and $j \in V_h$, then for all $i' \in V_g$ and $j' \in V_h$, $(i', j') \in E$.*

ii. *The relation $V_g > V_h$ iff $(i, j) \in E$ for some $i \in V_g$ and $j \in V_h$ and $V_g \neq V_h$ is transitive and asymmetric.*

Assume without loss of generality that the graph contains a single weakly connected component. We can do this because each weakly connected component corresponds to a block of the block-diagonal matrix A, and the powers of A are linearly dependent if and only if the powers of each diagonal block are too. The facts about transitive networks imply that the matrix A has the following structure:

$$A = \begin{pmatrix} A_{g_a,g_a} & A_{g_a,g_b} & \cdots & A_{g_a,g_c} \\ 0 & A_{g_b,g_b} & \cdots & A_{g_b,g_c} \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{g_c,g_c} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Matrix A_{g_x,g_y} is of size $|V_{g_x}| \times |V_{g_y}|$. With exclusive averaging, each matrix on the diagonal has 0s on its diagonal, and is strictly positive off it. With inclusive averaging, diagonal entries are not 0. If a block A_{g_x,g_x} has only 0 blocks to its right, there must be a non-zero block above. Finally, for each i, j and k, $a_{ij} = a_{ik}$, and the row sums are 1.

Proof of theorem 3.

Lemma A.2. *Assume exclusive averaging. If (V, E) is a group, then $[A^2]_{ii} = (|V|-1)/|V|^2$ for $i = j$, $[A^2]_{ii} = (|V|-2)/|V|^2$ for $j \neq i$, and*

$$A^2 = \frac{1}{|V|-1} I + \frac{|V|-2}{|V|-1} A. \tag{87}$$

Conversely, if γ_0 and γ_1 are both greater than 0, then (V, E) is a group and equation (87) holds.

Proof of lemma A.2. "If" is a calculation. For the other direction, suppose the two coefficients are positive. Then $[A^2]_{ii} = \gamma_0$ for all i since $a_{ii} = 0$. Suppose (V, E) is not a group. There must be a strongly connected component V_h which is minimal with respect to $>$, and another strongly connected component V_g such that $V_g > V_h$. Members of V_h connect only to themselves, and so the cardinality of V_h must exceed 1, or else $\gamma_0 = 0$. For $i \in V_g$ and $j \in V_h$,

$$[A^2]_{ii} = \frac{|V_g| - 1|}{(|V_g| - 1 + |V_h| + m)^2}$$
$$[A^2]_{jj} = \frac{1}{|V_h| - 1}$$

where m is the number of nodes outside of V_h members of V_g are connected to. Both of these numbers must equal γ_0, and so $(|V_g| - 1)(|V_h| - 1) = (|V_g| - 1 + |V_h| - 1 + m + 1)^2$, which is impossible. Thus (V, E) is a group, and equation (87) follows. □

Now we identify the structure for the remaining case, which has $\gamma_1 > 0$ and $\gamma_0 = 0$. Note that for a directed bipartite network, $A^2 = 0$.

Lemma A.3. *Assume exclusive averaging.* $\gamma_1 > 0$ *and* $\gamma_0 = 0$ *if and only if* (V, E) *is a directed bipartite network.*

Proof of Lemma A.3. The row sums of A are 1; hence so are the row sums of A^2. Thus $\gamma_1 = 1$ and $A^2 = A$. Since $A^2 = A$, $[A^2]_{ii} = 0$ for all i, and so no one is strongly connected to anyone else. That is, each component is a singleton. Next we show that there are no chains of length three or more. That is, if (i, j) is in E, there is no k such that $(j, k) \in E$. Suppose there is a such a k. Without loss of generality we can take k to be a minimum with respect to the ordering $>$. Let n_j denote the number of nodes influencing j, and let n_i denote the number of additional nodes which influence i. One can write down the relevant rows and columns of the matrix A and see that the relevant pieces are

$$\begin{pmatrix} \cdots & 0 & 1/(n_i+n_j) & 1/(n_i+n_j) & \cdots & \cdots & 0 & 0 & 1/n_j & \cdots \\ \cdots & 0 & 0 & 0 & & \cdots & & & & \end{pmatrix}$$

where the first column of 0's belongs to i, the next to j and the next to k, the three rows belong to i, j and k, respectively, and the k row is all 0's. Preceding entries of all three rows are 0. Without loss of generality we can take these to be the last three rows of the matrix. Multiplying,

$$[A^2]_{ik} = \frac{1}{n_j}\frac{1}{n_i+n_j} = \gamma_1 \frac{1}{n_i+n_j},$$

$$[A^2]_{jk} = 0 = \gamma_1 \frac{1}{n_j}$$

so $\gamma_1 = 1/n_j$ and $0 = 1/n_j^2$, a contradiction. □

Now we repeat the same exercise for inclusive averaging. Again, assume (V, E) is weakly connected.

Lemma A.4. *Assume inclusive averaging. If (V, E) is a group, then $A^2 = A$, $\gamma_1 = 1$ and $\gamma_0 = 0$. If $A^2 = A$ and $A \neq 0$, then (V, E) is a group.*

Proof of Lemma A.4. If (V, E) is a group, then all elements of A are identical and the row sum is 1. Thus $A^2 = A$, and so forth.

Suppose that $A^2 \neq 0$ and that (V, E) is not a group. Let V_b denote a strongly connected component which is minimal with respect to $\succ$, and let V_a denote a connected component such that $V_a \succ V_b$. Let m denote the number of nodes not in V_b to which members of V_a are connected. Then for $i \in V_a$ and $j \in V_b$,

$$\frac{1}{|V_a| + |V_b| + m} = a_{ii} = [A^2]_{ii} =$$

$$\frac{|V_a|}{(|V_a| + |V_b| + m)^2} + \frac{1}{|V_a| + |V_b| + m}$$

which implies $|V_a| = 0$, a contradiction. □

Lemma A.5. *If $\lambda_2 \neq 0$, (V, E) is a group.*

Proof of lemma A.5. If (V, E) is not a group, each strongly connected component is of size 1. If not, and both i and j are in the same component, then

$$\gamma_0 + \gamma_1 a_{ii} = [A^2]_{ii} = [A^2]_{jj} = \gamma_1 a_{jj} = \gamma_1 a_{ii}. \tag{88}$$

Since $a_{ii} \neq 0$, $\gamma_0 = 0$ and $\gamma_1 = 1$ and so A is a group, which is impossible. If so, suppose that $\{j\}$ is minimal with respect to $\succ$ and that $\{i\} \succ \{j\}$. Let m denote the number of other nodes that influence i. Then

$$[A^2]_{ii} = \frac{1}{(2+m)^2} = \gamma_0 + \gamma_1 \frac{1}{2+m}$$
$$[A^2]_{ij} = \frac{1}{(2+m)^2} + \frac{1}{2+m} = \gamma_0 + \gamma_1 \frac{1}{2+m}$$

and would imply $0 = 1/(2+m)$ which is impossible. □

To complete the proof of theorem 3, note that if (V, E) is the union of groups and the powers of A are dependent, then the groups must be the same size. Bramoullé *et al.* prove this for exclusive averaging, and the proof for inclusive averaging proceeds the same way. □

Proof of theorem 5. It suffices to prove the theorem for the open and dense set of matrices S_1 which are strictly positive. Then $\lambda_2 \neq 0$, so write $A^2 = \gamma_1 A + \gamma_0 I$. We need to prove the claim of the theorem for the set S_1 of matrices that can be written this way. The set S_1 is semi-algebraic and closed. It suffices to show that S / S_1 is dense.

Consider a matrix A in S, and denote its square by B. Consider matrices of the form $A(\varepsilon)$ whose i, j element is $a_{11} + \varepsilon$ for $i = j$ and a_{ij} otherwise. Then

$$B(\varepsilon)_{11} = b_{11} + 2a_{11}\varepsilon + \varepsilon^2,$$
$$B(\varepsilon)_{1j} = b_{1j} + a_{1j}\varepsilon \quad \text{for } j \neq 1,$$
$$B(\varepsilon)_{i1} = b_{i1} + a_{i1}\varepsilon \quad \text{for } i \neq 1,$$
$$B(\varepsilon)_{ij} = b_{ij} \quad \text{otherwise.}$$

Suppose that $B(\varepsilon)$ is in S for all small ε. Computing.

$$\gamma_1 = (b_{21} + a_{21}\varepsilon)/a_{21}$$
$$\gamma_0 = b_{22} - \gamma_1 a_{22}.$$

Then the equation $B(\varepsilon)_{11} = \gamma_1 A(\varepsilon)_{11} + \gamma_0$ is a linear (not quadratic) equation in ε. A necessary condition for linear dependence of the powers of $A(\varepsilon)$ for more than 1 value of ε is that the coefficient on ε is zero. This happens only for a set of A-matrices S_2 of codimension at least 1. Hence for all but at most one small enough ε, $A(\varepsilon) \notin S_1/S_2$. Since S_2 is nowhere dense, this proves the theorem. $\square$

Proof of theorem 7. Let $m = F(c, d, J, \gamma)$; M is the matrix of reduced form coefficients. Our goal is to see how they map back to the structural parameters. We will prove the theorem for n_v odd and equal to $2K + 1$. The proof for even n_v is similar.

By hypothesis, $I - JA(\gamma)$ is non-singular. Thus

$$\begin{aligned} M &= \big(I - JA(\gamma)\big)^{-1}\big(cI + dA(\gamma)\big) \\ &= \big(I - JA(\gamma)\big)^{-1}\big(cI + dI - dI + dA(\gamma)\big) \\ &= (c + d)\big(I - JA(\gamma)\big)^{-1} - dI \end{aligned} \tag{89}$$

and so

$$dI + M = (c + d)(I - JA(\gamma)\big)^{-1}, \tag{90}$$

which verifies that $dI + M$ is non-singular if $c + d \neq 0$.

In view of equation (57) which defines $A(\gamma)$,

$$\big(I - JA(\gamma)\big)_{11} = 1$$

and

$$-\frac{\big(I - JA(\gamma)\big)_{12}}{\big(I - JA(\gamma)\big)_{11}} = J\gamma,$$

$$\frac{\big(I - JA(\gamma)\big)_{13}}{\big(I - JA(\gamma)\big)_{12}} = \cdots = \frac{\big(I - JA(\gamma)\big)_{1\,K+1}}{\big(I - JA(\gamma)\big)_{1K}} = \gamma\,.$$

Now define

$$\mathcal{M} = \{M : \text{ for some } (c, d, J, \gamma) \in \mathcal{P},\ F(c, d, J, \gamma) = M\}$$

$$\mathcal{M}_{dJ} = \{M : \text{ for some } (c, \gamma) \in \mathbf{R}^2 \times [0, 1),\ F(c, d, J, \gamma) = M\}$$

These are, respectively, the sets of all possible reduced form matrices and those reduced forms consistent with a particular parameter pair of structural parameters (d, J).

Equation (90) then requires the following: If $M \in \mathcal{M}_{d,J}$, then

$$(dI + M)_{11} \neq 0$$

and

$$-\frac{\left(dI + M\right)^{-1}_{12}}{\left(dI + M\right)^{-1}_{11}} = J\gamma, \tag{91}$$

$$\frac{\left(dI + M\right)^{-1}_{13}}{\left(dI + M\right)^{-1}_{12}} = \cdots = \frac{\left(dI + M\right)^{-1}_{1\,K+1}}{\left(dI + M\right)^{-1}_{1K}} = \gamma\,. \tag{92}$$

We will use this fact to show that for a given reduced form matrix $M \in \mathcal{M}$ there are at most $2(n_V - 1)$ possible values of (d, J) pairs consistent with equation (92). We will show that each of these (d, J) pairs is consistent with a unique (c, γ) pair, which proves the theorem.

Under our assumed model specification, M is symmetric. Thus it has real eigenvalues $\lambda_1, \ldots, \lambda_{n_V}$, and is diagonalizable by a unitary matrix P. Furthermore, for any scalar d, $dI + M$ is diagonalized by the same matrix P, and has Eigenvalues $d + \lambda_1, \ldots, d + \lambda_{n_V}$. Consequently,

$$(dI + M)^{-1} = P^{-1} \begin{pmatrix} \frac{1}{d+\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d+\lambda_2} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \frac{1}{d+\lambda_{n_V}} \end{pmatrix} P .$$

The i, j'th entriy of the matrix product on the right is

$$(dI + M)^{-1}_{ij} = \frac{1}{\prod_j (d + \lambda_j)} \sum_k p_{ik} p_{kj} \prod_{l \neq k} (d + \lambda_l) .$$

for any d which is not the negative of an eigenvalue of M (which would make $dI + M$ singular). We now ask, for which values of d can equation (92) be satisfied? Define $\phi_{ij}(d) = \sum_k p_{ik} p_{kj} \prod_{l \neq k} (d + \lambda_l)$. Equation (92) implies that

$$\frac{\phi_{13}(d)}{\phi_{12}(d)} = \frac{\phi_{14}(d)}{\phi_{13}(d)}$$

and so

$$p(d) \equiv \phi_{13}(d)^2 - \phi_{14}(d)\phi_{12}(d) = 0 .$$

Then $p(d)$ is a polynomial of degree at most $2(n_V - 1)$. The dependence of the polynomial on the terms p_{ij} is the link between the reduced form coefficients and the structural parameters. To see that it is not identically 0, suppose that $M = F(c', d', J', \gamma')$. From equation (89), it follows that the value of the derivative of the matrix $dI + M$ at $d = d'$ is $(c' + d')^2(I - J'A(\gamma'))^2$. From this fact, a calculation shows that if neither γ' nor J' are 0, and $c' + d' \neq 0$, then $p'(d') = 0$ only for the solutions to a polynomial equation in γ and J which is not identically 0. Thus $p'(d') \neq 0$ only on a lower-dimensional set C of (c', d', J', γ'). That is, for $M \in \mathcal{M}' = \{M :$ for some $(c, d, J, \gamma) \in \mathcal{P}/C$, $F(c, d, J, \gamma) = M\}$,

Off of this set, for any d sufficiently near to but not equal to d', $p(d') \neq 0$. Thus for $M \in \mathcal{M}' = \{M :$ for some $(c, d, J, \gamma) \in \mathcal{P}/C$, $F(c, d, J, \gamma) = M\}$, $p(d) = 0$ has at most $2(n_v - 1)$ solutions. For each d which is a root of $p(d)$, $-(dI + M)^{-1}_{12} / -(dI + M)^{-1}_{11} = J_d$ (equation (91)). The ratio of any other pair of adjacent entries of the $(dI + M)^{-1}$ matrix determines γ_d. Finally, c_d solves $\big(I - A(\gamma_d)\big)(dI + M) - dI = cI$ for c. Suppose, then, that $M = F(c', d', J', \gamma')$ for $(c', d', J', \gamma') \in \mathcal{P}/C$. If a parameter vector $(c'', d'', J'', \gamma'')$ is not such that d'' is a root of $p(d)$, or that c'', J'', and γ'' do not equal the corresponding c_d, J_d and γ_d, then $(c'', d'', J'', \gamma'')$ is not observationally equivalent to (c', d', J', γ'). If $\gamma = 0$, then neither d nor J can be identified. In this case $M = cI$. Conversely, if $M = c'I$, from (90) either $J = 0$ and $d = 0$ or $\gamma = 0$. From equation (58), if $J = 0$ and $M = c'I$, then either $\gamma = 0$ or $d = 0$ as well, and $c' = c$. □

A3 Equilibrium Properties of Discrete Choice Models with Social Interactions

This appendix describes some aspects of the group-level equilibria for discrete choice models of social interactions. The models we discuss are similar in structure to the quantal response equilibria first developed by McKelvey and Palfrey (1995). The social interactions and quantal response equilibria literatures have evolved independently; as is true for other cases of parallel development that we have noted, each literature would benefit from integration with the other.

i. basic structure of the binary choice model with social interactions

We first outline the theoretical properties of the binary choice model with social interactions for a single group g, following Brock and Durlauf (2001a). As in the text, choices are coded so that $\omega_i \in \{-1, 1\}$. Define $h_i = k + cx_i + dy_g$. From the perspective of the equilibrium of the group, contextual effects act in a way analogous to a constant term, an observation that is used in the proof of theorem 4 on identification. The utility function for a given choice is

$$V_i(\omega_{ig}) = h_i \omega_{ig} - \frac{J}{2} \mathrm{E}((\omega_{ig} - \bar{\omega}_{-ig})^2) + \eta_i(\omega_{ig}) \tag{93}$$

where $\bar{\omega}_{-ig} = (I-1)^{-1} \sum_{j \neq i} \omega_{jg}$ and $\eta_i(\omega_{ig})$ is a choice-specific random utility term. In parallel to the linear in means model, there is a penalty for expected square deviations of i's choice from the mean choices of others. Since $\omega_{ig}^2 \equiv 1$,

$$-\frac{J}{2}(\omega_{ig} - \bar{\omega}_{-ig})^2 = J\omega_{ig}\bar{\omega}_{-ig} - \frac{J}{2}(1 + \bar{\omega}_{-ig}^2) \; .$$

The second term on the right is independent of ω_{ig}, and so the utility function of equation (93) yields the same behaviors as

$$V_i(\omega_{ig}) = h_i\omega_{ig} + J\omega_{ig}m^e_{ig} + \eta_i(\omega_{ig})$$

where $m^e_{ig} = (I-1)^{-1}\sum_{j\neq i} E\omega_{ig}|F_i$. It is immediate that

$$V_i(1) - V_i(-1) = 2h_i + 2Jm^e_{ig} - \varepsilon_i \tag{94}$$

where $\varepsilon_i = \eta_i(-1) - \eta_i(1)$. This justifies equation (59). As the group size grows large, m^e_{ig} will become independent of i.

ii. equilibria under logit models of social interactions

To illustrate the qualitative properties of the binary choice model with social interactions, following Brock and Durlauf (2001a), we maintain the i.i.d. error assumptions (60) and (61) and further assume a functional form for F_ε:

$$F_\varepsilon(z) = \frac{1}{1 + \exp(-\beta z)}$$

that is, the individual-specific utility terms errors are negative exponentially distributed. The parameter β indexes the degree of unobserved heterogeneity. A larger β implies less heterogeneity in the sense that the probability mass of $F_\varepsilon(z)$ is more concentrated towards the origin.

This functional form, when combined with equation (94), produces the canonical logistic density for equilibrium choices

$$\mu(\omega_{ig}|h_i, m_g) = \frac{\exp(\beta h_i \omega_{ig} + \beta J m_g \omega_{ig})}{\exp(\beta h_i \omega_{ig} + \beta J m_g \omega_{ig}) + \exp(-\beta h_i \omega_{ig} - \beta J m_g \omega_{ig})}. \quad (95)$$

From equation (95) it is immediate that the expected value of agent i's choice is

$$\begin{aligned} \mathrm{E}(\omega_{ig}|h_i, m_g) &= \frac{\exp\left(\beta h_i \omega_{ig} + \beta J m_g \omega_{ig}\right) - \exp\left(-\beta h_i \omega_{ig} - \beta J m_g \omega_{ig}\right)}{\exp\left(\beta h_i \omega_{ig} + \beta J m_g \omega_{ig}\right) + \exp\left(-\beta h_i \omega_{ig} - \beta J m_g \omega_{ig}\right)} \\ &= \tanh\left(\beta h_i + \beta J m_g\right). \end{aligned} \tag{96}$$

The expected group mean is simply the unweighted average of (96) across i. Letting $dF_{h|g}$ denote the empirical density of h_i within group g, m_g is implicitly defined by

$$m_g = \int \tanh\left(\beta h + \beta J m_g\right) dF_{h|g}.$$

To understand the properties of the equilibrium, we consider the baseline case in the literature in which h_i is constant, that is, for all i, $h_i \equiv h$, so that the equilibrium expected average choice levels are described by a functional equation. (No closed form solution exists.)

$$m_g = \tanh\left(\beta h + \beta J m_g\right). \tag{97}$$

Brock and Durlauf (2001a) characterize the properties of solutions to equation (97), which we summarize as

Theorem A.3. Equilibria in the logistic version of the binary choice model with social interactions.

i. *If $\beta J < 1$, equation* (97) *has a unique solution.*

ii. *If $\beta J > 1$, then there exists a nondecreasing and positive function $\bar{h}(\beta J)$ of βJ such that*

    i. *the equilibrium solution to equation* (97) *is unique if $|h| > \bar{h}(\beta J)$; and*

    ii. *there exist three equilibrium solutions to equation* (97) *if $|h| < \bar{h}(\beta J)$. One equilibrum has the same sign as h.*

The intuition for the theorem is straightforward. $\beta J < 1$ means that the endogenous social interaction effect is too weak to generate multiple equilibria. Notice that strength of the interaction effect is not determined by J, the endogenous effect parameter, in isolation, but is multiplied by the measure for heterogeneity. Why would a small value of β work against the existence of multiple equilibria? A small β implies fatter tails for the unobserved heterogeneity density. By symmetry of this density, fat tails means a relatively large fraction of the population will, in expectation, have their choices determined by their heterogeneity draws. This leaves too small a fraction whose behavior can exhibit multiple equilibria via self-consistent bunching; the utility differential between the choices is insufficiently affected by the range of possible m_g values once the tail draws are accounted for. In contrast, $\beta J > 1$ means that the endogenous social utility payoff is large enough relative to the symmetrically distributed heterogeneity, then multiple expected average choice levels are possible, if $|h| < \bar{h}(\beta J)$. Why is this second condition needed? If the common private incentive h has sufficient magnitude, it will determine a sufficiently large fraction of choices so that self-consistent bunching is not possible. Again, greater heterogeneity reinforces this effect. Notice that qualitative changes in the number of equilibria for this model occur in neighborhoods of the value 1 for βJ and $\bar{h}(\beta J)$ for h. These are bifurcation thresholds.

Blume and Durlauf (2003) extend this theorem by considering a dynamic analog of the binary choice model with social interactions. Their analysis focuses on the stability of the rational expectations equilibria associated with (97). For a dynamic analog of the model we have outlined, one can show the population spends most of its time in the vicinity of the equilibrium that maximizes average utility in the group, which is the equilibrium whose mean choice has the same sign as h.

iii. generalizations of the binary choice model

The properties of this model generalize to a number of interesting related structures. For example, one can analyze the general preference specification

$$V_i(1) - V_i(-1) = h_i + Jm_g - \beta^{-1}\varepsilon_i$$

where F_ε is is an arbitrary probability distribution function for the unobservable individual-level ε heterogeneity. Retaining the i.i.d. error assumptions (60) and (61) and focusing on the case where h is constant, Brock and Durlauf (2006) prove that a close analog to theorem A.3 holds for this general binary choice model. Two changes occur when the logit function form assumption is dropped. First, the necessary condition for multiple equilibria takes the form $\beta J > T$, where the threshold T cannot be determined without specification of F_ε. In other words, some threshold for βJ always exists that can produce multiple equilibria. Second, for part ii.b, the threshold result for multiple equilibria states that at least three equilibria exist. The more precise structure of theorem A.3 derives from the specific functional form found in equation (97). The qualitative features of the theorem do not.

The qualitative properties of the theorem also extend to local interactions environments, i.e. contexts where individuals are arrayed in some social space and only interact with their suitably defined neighbors. One version of a local interactions model is studied in Blume (1993). An expectational version of his model can be represented by

$$V_i(1) - V_i(-1) = h_i + J \sum_{|i-j|=1} \mathrm{E}(\omega_{jg}) - \varepsilon_i$$

where $\mathrm{E}(\omega_{jg})$ is the equilibrium expectation of ω_{jg} conditional on the values of h_i across the population. If we impose the assumption that for all i, $h_i = h$, then it is immediate that equation (97) continues to characterize the symmetric equilibrium average choice levels in the population. It is obvious that other interactions structures can do the same. The similar aggregate properties for different interactions specifications is itself known as the property of universality, which in social interactions contexts means that there exist dimensions along which the qualitative properties of the models do not depend on the details of the interaction structure. The reader should consult Ioannides (2006) for extensions of these types of models to more complex interactions structures.

iv. multinomial choice models with social interactions

Multiple equilibria and bifurcations are not unique to the binary choice context. Brock and Durlauf (2006) show that theorem A.3 is a special case of

Theorem A.4. Multiple equilibria in the multinomial logit model with social interactions. *Suppose that individual choices are characterized by equations* (71), (72), *and* (73). *Assume that* $h_{il} = k$ *for all* i *and* l. *If* $\beta J > L$, *there will exist at least three self-consistent choice probabilities.*

The dependence of the threshold on the number of choices L is intuitive. The larger the number of choices, under independence of ε_{il} across l, the greater the probability that one of the draws will dominate the agent's choice, which reduces the fraction of agents whose behavior can exhibit self-consistent bunching. Brock and Durlauf (2006) additionally provide analogous results for general density functions for ε_{il}. As in the binary choice case the results are less precise. This theorem and its generalization in Brock and Durlauf (2006) explain various simulation results in Bayer and Timmens (2005).

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