Interpreting IV
What Does IV Estimate?

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What Does IV Estimate?

- Consider a linear regression approximation of $E\left(Y \mid P(Z)=p\right)$:

$$E^{*}(Y \mid P(Z)=p)=a+bp,$$

$$b=\frac{\operatorname{Cov}(Y,P(Z))}{\operatorname{Var}(P(Z))}=\frac{\operatorname{Cov}(E(Y \mid P(Z)),P(Z))}{\operatorname{Var}(P(Z))}.$$

- b is the same as the IV estimate of "the effect" of D on Y using $P(Z)$ as an instrument since $\operatorname{Cov}(P(Z),D)=\operatorname{Var}(P(Z))$.

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$$b = \frac{\text{Cov}(Y, P(Z))}{\text{Var}(P(Z))} = \frac{\text{Cov}(S(P(Z)), P(Z))}{\text{Var}(P(Z))}$$

$$= \frac{\text{Cov}\left(\int_0^{P(Z)} \text{MTE}(u_D) du_D, P(Z)\right)}{\text{Var}(P(Z))}. \tag{1}$$

- When $\text{MTE}(u_D)$ is constant in u_D $(\text{MTE}(u_D) = \mu_1 - \mu_0 = \bar{\beta})$ β is independent of D, the numerator simplifies to

$$\begin{aligned} \text{Cov}\left(\int_0^{P(Z)} \text{MTE}(u_D)\, du_D, P(Z) \right) &= \text{Cov}\left(\bar{\beta} P(Z), P(Z)\right) \\ &= \bar{\beta}\, \text{Var}(P(Z)) \end{aligned}$$

so $b = \mu_1 - \mu_0 = \bar{\beta}$.

- Traditional result for IV.

- In this case, the marginal gross surplus is the same as the average gross surplus for all values of p.
- Expression (1) arises because D depends on β $(=Y_1 - Y_0)$, something assumed away in traditional applications of IV.
- As a consequence, in general, the marginal surplus is not the average surplus.

- An explicit expression for the numerator of (1) is

$$\text{Cov}(Y, P(Z)) = \int_0^1 \left[\int_0^p \text{MTE}(u_D)\, du_D \right] (p - E(P)) f_P(p)\, dp.$$

- Reversing the order of the integration of the terms on the right hand side and respecting the requirement that $0 < u_D < p < 1$, we obtain

$$b = \frac{\text{Cov}(Y, P(Z))}{\text{Var}(P(Z))} = \frac{\int_0^1 \text{MTE}(u_D)\left[\int_{u_D}^1 (p - E(P))f_P(p)dp\right] du_D}{\text{Var}(P(Z))}$$

$$= \int_0^1 \text{MTE}(u_D) h^{\text{IV}}_{P(Z)}(u_D) du_D$$

where

$$h^{\text{IV}}_{P(Z)}(u_D) = \frac{\int_{u_D}^1 (p - E(P))f_P(p)dp}{\text{Var}(P(Z))}.$$

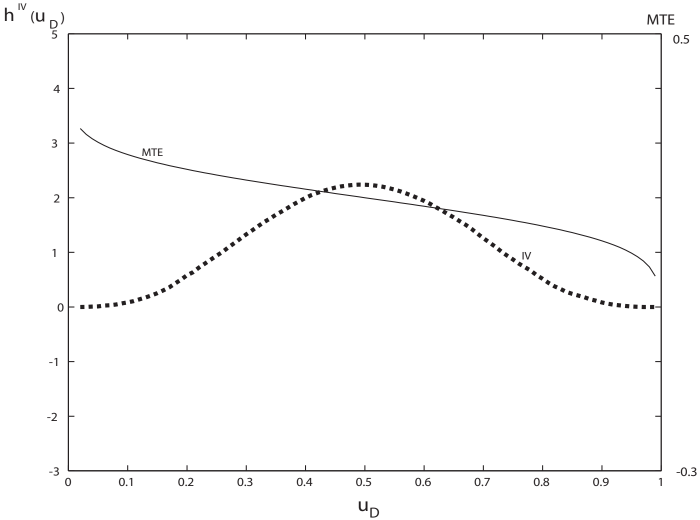
- An alternative expression for the weight is as the mean of left truncated $P(Z)$:

$$h^{\text{IV}}_{P(Z)}(u_D) = \frac{E(P(Z) - E(P(Z)) \mid P(Z) > u_D)\Pr(P(Z) > u_D)}{\text{Var}(P(Z))}$$

which shows that the weight on the $\text{MTE}(u_D)$ is non-negative for all u_D.

- Weights can be estimated from the sample distribution of $P(Z)$.
- Weights for $P(Z)$ as an instrument have a distinctive profile.

Figure 1: IV Weights as a Function of u_D.



Source: Heckman and Vytlacil (2005).

- For discrete valued instruments mapped into $P(z^1) = p_1 < P(z^2) = p_2 < \cdots < P(z^L) = p_L$,

$$\text{IV} = \sum_{\ell=1}^{L-1} \text{LATE}(p_{\ell+1}, p_\ell)\lambda_\ell$$

where $\lambda_\ell = \frac{1}{\text{Var}(P(Z))} \sum_{t>\ell}^{L} (p_t - E(P)) f_P(p_t)$ and $f_P(p_t)$ is the probability that $P(Z) = p_t$.

See Appendix for a more complete discussion of the derivation of the IV weights.

The Problem of Limited Support

- While the various treatment parameters can be defined from the generalized Roy model, they may not necessarily be identified from the data.
- $P(Z)$ may not be identified over the full unit interval.

- $P(Z)$ may only assume discrete values.
- This limits the identifiability of MTE.
- In this case, only LATE over intervals of $u_D \in [0, 1]$ can be identified from the values of $P(Z) = P(z)$ associated with the discrete instruments.

- One approach to this problem developed by Manski (1990, 1995, 2003) is to produce bounds on the treatment effects.
- Heckman and Vytlacil (1999, 2000, 2001a,b, 2007) developed specific bounds for the generalized Roy model that underlies the LATE model.
- The bounds developed in the literature are for conventional treatment effects and not for policy effects.

- Carneiro, Heckman, and Vytlacil (2010) consider an alternative approach based on marginal policy changes.
- Many proposed policy changes are incremental in nature, and a marginal version of the PRTE is all that is required to answer questions of economic interest.
- When some instruments are continuous, it is possible under the conditions in their paper to identify a marginal version of PRTE (MPRTE).

- MPRTE is in the form of a weighted average of MTE where the weights can be identified from the data and the support requirements are more limited than the conditions required to identify PRTE for large changes in policies.
- Application of these data sensitive nonparametric approaches enables analysts to avoid one source of instability of the estimates of policy effects that plagued 1980s econometrics.

More General Instruments

- Typically, economists use a variety of instruments one at a time and not just $P(Z)$ as an instrument and compare the resulting estimates (see, e.g, Card, 1999, 2001).
- When there is selection on the basis of gross gains ($\beta \not\perp\!\!\!\perp D$) so that the marginal gross surplus is not the same as the average gross surplus, different instruments identify different parameters.
- IV is a weighted average of MTEs where the weights integrate to 1 and can be estimated from sample data.
- However, in the case of general instruments, the weights can be negative over stretches of u_D.

- Consider using the first component of Z, Z_1, as an instrument for D.
- Suppose that Z contains two or more elements $(Z = (Z_1, \ldots, Z_K), K \geq 2)$.
- The economics implicit in LATE informs us that Z determines the distribution of Y through $P(Z)$.
- Any correlation between Y and Z_1 arises from the statistical dependence between Z_1 and $P(Z)$ operating to determine Y.

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- The IV estimator based on Z_1 is

$$IV_{Z_1} = \frac{\text{Cov}(Y, Z_1)}{\text{Cov}(D, Z_1)} = \frac{\text{Cov}(E(Y \mid Z_1), Z_1)}{\text{Cov}(D, Z_1)}.$$

- Note, however, that choices (and hence Y) are generated by the *full vector of Z* operating through $P(Z)$.
- The analyst may only use Z_1 as an instrument but the underlying economic model informs us that the full vector of Z determines observed Y.

- Conditioning only on Z_1 leaves uncontrolled the influence of the other elements of Z on Y.
- This is a new phenomenon in IV that would not be present if D did not depend on $\beta(= Y_1 - Y_0)$.
- An IV based on Z_1 identifies an effect of Z_1 on Y as it operates directly through Z_1 (Z_1 changing $P(Z_1, \ldots, Z_K)$) holding other elements in Z constant and indirectly through the effect of Z_1 as it covaries with $(Z_2, \ldots, Z_K)$, and how those variables affect Y through their effect on $P(Z)$.

- A linear regression analogy helps to fix ideas.
- Suppose that outcome Q can be expressed as a linear function of $W = (W_1, \ldots, W_L)$, an L-dimensional regressor:

$$Q = \sum_{\ell=1}^{L} \phi_\ell W_\ell + \varepsilon,$$

where $E(\varepsilon \mid W) = 0$.

- If we regress Q only on W_1, we obtain in the limit the standard omitted variable result that the estimated "effect" of W_1 on Q is

$$\frac{\text{Cov}(Q, W_1)}{\text{Var}(W_1)} = \phi_1 + \sum_{\ell=2}^{L} \phi_\ell \frac{\text{Cov}(W_\ell, W_1)}{\text{Var}(W_1)}, \tag{2}$$

  where ϕ_1 is the *ceteris paribus* direct effect of W_1 on Q and the summation captures the rest of the effect (the effect on Q of W_1 operating through covariation between W_1 and the other values W_ℓ, $\ell \neq 1$).
- (Note these L-1 coefficients are from a multiple regression.)
- An analogous problem arises in using one instrument at a time to identify "the effect" of Z_1.

- Thus if the analyst does not condition on the other elements of Z in using Z_1 as an instrument, the margin identified by variations of Z_1 does *not* in general correspond to variations arising solely from variations in Z_1, holding the other instruments constant.
- The margin of choice implicitly defined by the variation in Z_1 is difficult to interpret and depends on the parameters of the generalized Roy model generating outcomes as well as on the sample dependence between instrument Z_1 and $P(Z)$.
- Thus an IV based on Z_1 mixes causal effects with sample dependence effects among the correlated regressors.

- In a study of college going, if Z_1 and Z_2 are tuition and distance to college, respectively, the instrument Z_1 identifies the direct effect of variation in tuition on college attendance and the effect of distance to college on college attendance as it covaries with tuition in the sample used by the analyst.
- This is not the *ceteris paribus* effect of a variation in tuition.
- It does not correspond to the answer needed to predict the effects of a policy that operates solely through an effect on tuition.
- In models in which D depends on β, the traditional instrumental variable argument that analysts do not need a model for D and can ignore other possible determinants of D besides the instrument being used, breaks down.

- To interpret which margin is identified by different instruments requires that the analyst specify and account for all of the Z that form $P(Z)$.
- Since different economists may disagree on the contents of Z, different economists using Z_1 on the same data will obtain the same point estimate but will disagree about the interpretation of the margin identified by variation in Z_1.

- To establish these points, recall that as a consequence of Vytlacil's theorem, Z enters the distribution of Y only through $P(Z)$.
- Thus the conditional distribution of Y given $Z_1 = z_1$ operates through the effect of Z_1 as it affects $P(Z)$.
- That is a key insight from Vytlacil's theorem.
- Thus

$$E(Y \mid Z_1 = z_1) = \int_0^1 E(Y \mid P(Z) = p)\, g_{P(Z),Z_1}(p, z_1)\, dp$$

where $g_{P(Z),Z_1}(p, z_1)$ is the conditional density of $P(Z)$ given $Z_1 = z_1$.

- Putting all of these ingredients together, we obtain

$$\begin{aligned}
E(Y \mid Z_1 = z_1) \\
&= E(Y_0) + \int_0^1 S(p)\, g_{P(Z)|Z_1}(p, z_1)\, dp \\
&= E(Y_0) + \int_0^1 \underbrace{\left[\int_0^p \text{MTE}(u_D)\, du_D \right]}_{S(p)} g_{P(Z),Z_1}(p, z_1)\, dp.
\end{aligned}$$

- Using this expression to compute $\frac{\text{Cov}(Y,Z_1)}{\text{Cov}(D,Z_1)}$, we obtain

$$IV_{Z_1} = \frac{\int_{-\infty}^{\infty}(z_1 - E(Z_1)) \int_0^1 S(p) g_{P(Z),Z_1}(p, z_1)\, dp\, dz_1}{\text{Cov}(Z_1, D)}$$
$$= \frac{\int\limits_{-\infty}^{\infty}(z_1 - E(Z_1)) \int\limits_0^1 \left[\int\limits_0^p \text{MTE}(u_D)\, du_D\right] g_{P(Z),Z_1}(p, z_1)\, dp\, dz_1}{\text{Cov}(Z_1, D)}.$$

- This expression integrates the argument in the numerator with respect to u_D, p, and z_1 in that order.

- Reversing the order of integration to integrate with respect to p, z_1 and u_D in that order, we obtain

$$IV_{Z_1} = \int_0^1 \mathrm{MTE}(u_D) h^{IV}_{Z_1}(u_D)\, du_D$$

where

$$h^{IV}_{Z_1}(u_D) = \frac{\int_{-\infty}^{\infty} (z_1 - E(Z_1)) \int_{u_D}^{1} g_{P(Z),Z_1}(p, z_1)\, dp\, dz_1}{\mathrm{Cov}(Z_1, D)}.$$

- The weight integrates to 1 but can be negative over stretches of u_D.
- At the extremes ($u_D = 0, 1$), the weights are zero.

- An illuminating way to represent this weight is

$$h_{Z_1}^{IV}(u_D) = \frac{E(Z_1 - E(Z_1) \mid P(Z) > u_D)\Pr(P(Z) > u_D)}{\mathrm{Cov}(Z_1, D)}.$$

- As u_D is increased, the censored (by the condition $P(Z) > u_D$) mean of $(Z_1 - E(Z_1))$ may switch sign, and hence the weights may be negative over certain ranges.
- Thus the IV estimator may have a sign opposite to the true causal effect (defined by the MTE) for each value of u_D.

Example

Figure 2: Joint Distribution of Instruments $Z = (Z_1, Z_2)$

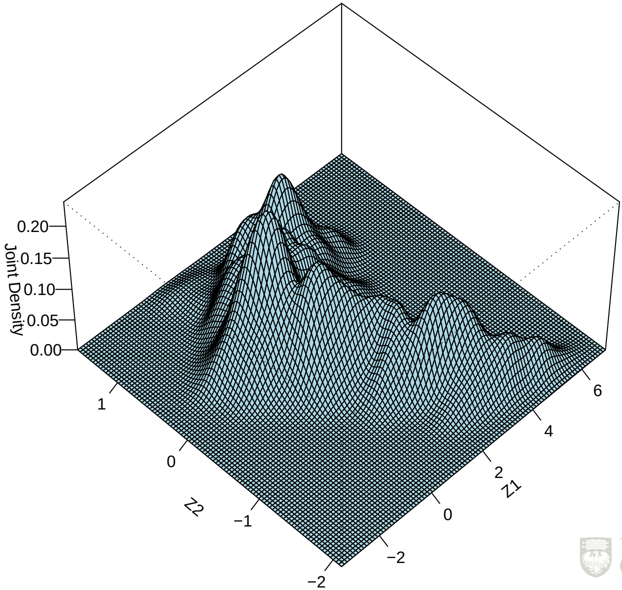
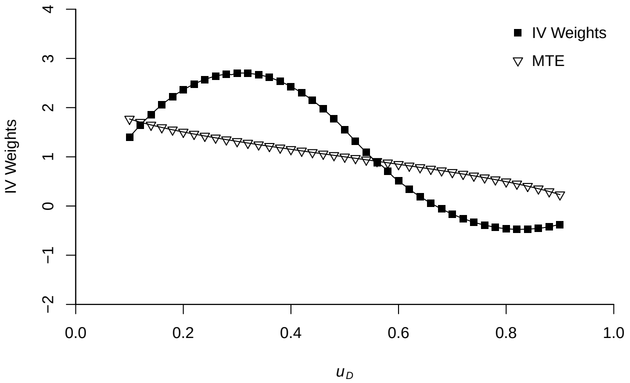


Figure 3: MTE and IV weights for a general instrument Z_1, a component of $Z = (Z_1, Z_2)$.



- Table 1, taken from Heckman, Urzúa, and Vytlacil (2006) shows how three different distributions of Z for the same underlying policy-invariant model with the same ATE can produce very different IV estimates.

Table 1: IV estimator for three different distributions of Z but the same generalized Roy model.

Data Distribution	IV	ATE
1	0.434	0.2
2	0.078	0.2
3	-2.261	0.2

Source: Heckman, Urzúa, and Vytlacil (2006, Table 3).

- *Note:* LATE is not invariant to choice of the distribution of $Z_i t$ is not a structural parameter.
- What is structural is MTE.

How Experiments Improve on LATE

- Experiments that manipulate Z_1 independently of other components of Z isolate the effects of Z_1 on outcomes in comparison with the effects obtained by sample variation in Z_1 correlated with other components of Z.
- Neither set of variations may identify the returns to any given policy unless the experimentally induced variation corresponds exactly to the variation induced by the policy.

- Economists can use experimental variation to identify the MTE.
- The features of a proposed policy are described by its effects on the PRTE weights as it affects the distribution of $P(Z)$.
- Proceeding in this way, one can use experiments to address a range of questions beyond those effects directly identified by the experiment.

- Using the implicit economic theory underlying LATE, economists can do better than just report an IV estimate.
- We can be data sensitive but not at the mercy of the data.
- We can determine the MTE (or LATEs) over the identified regions of u_D in the empirical support of $P(Z)$.
- We can also determine the weights over the empirical support of $P(Z)$ to determine whether they are negative or positive.
- We can bound estimates of the unidentified parameters.
- We can construct the effects of policy changes for new policies that stay within the support of $P(Z)$ (see Carneiro, Heckman, and Vytlacil, 2010).

Policy Effects, Treatment Effects, and IV

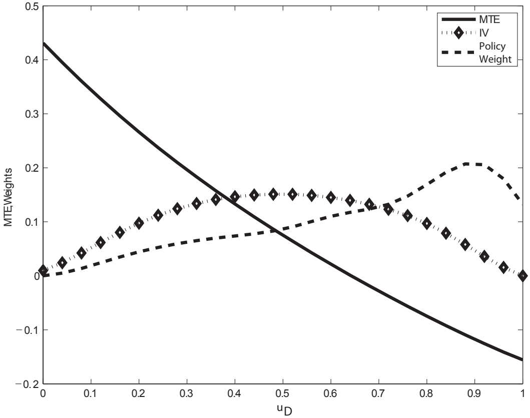
- A main lesson of this analysis is that policy effects are not generally the same as treatment effects and, in general, neither are produced from IV estimators.
- Since randomized assignments of components of Z are instruments, this analysis also applies to the output of randomized experiments.
- The economic approach to policy evaluation formulates policy questions using well-defined economic models.
- It then uses whatever statistical tools it takes to answer these questions.

- Policy questions and not statistical methods drive analyses.
- Well-posed economic models are scarce in the program evaluation approach.
- Thus in contrast to the structural approach, it features methods over economic content.
- "Credibility" in the program evaluation literature is assessed by statistical properties of estimators and not economic content or policy relevance.
- The "Credibility Revolution" demotes economic content and stresses statistical properties of the estimands.

- We can do better than hoping that an instrument or an estimator answers policy problems.
- By recovering economic primitives, we can distinguish the objects various estimators identify from the questions that arise in addressing policy problems.
- Constructing the PRTE is an example of this approach.
- An alternative approach developed in Heckman and Vytlacil (2005) constructs combinations of instruments using sample data on Z that address specific policy questions.

- Figure 4, taken from an analysis of the returns to attending college by Carneiro, Heckman, and Vytlacil (2009), plots the estimated weights for MTE from a marginal change in policy that proportionally expands the probability of attending college for everyone.
- The figure also plots the estimated MTE and the IV weight using $P(Z)$ as an instrument.
- The IV weights and the policy weights are very different.
- The policy weights oversample high values of u_D compared to the IV weights.

Figure 4: Weights for IV and MPRTE in the Carneiro-Heckman-Vytlacil (2009) Analysis of the Returns to College Going.



Source: Carneiro et al. (2009).
Notes: The scale of the *y*-axis is the scale of the MTE, not the scale of the weights, which are scaled to fit the picture. The IV is $P(Z)$.

Multiple Choices

- Imbens and Angrist analyze a two choice model.
- Heckman, Urzúa, and Vytlacil (2006, 2008) and Heckman and Vytlacil (2007) extend their analysis to an ordered choice model and to general unordered choice models.

- In the special case where the analyst seeks to estimate the mean return to those induced into a choice state by a change in an instrument compared to their next best option, the LATE framework remains useful (see Heckman, Urzúa, and Vytlacil, 2006, 2008; Heckman and Vytlacil, 2007).
- If, however, one is interested in identifying the mean returns to any pair of outcomes, unaided IV will not do the job.
- Structural methods are required.

- In general unordered choice models, agents attracted into a state by a change in an instrument come from many origin states, so there are many margins of choice.
- Structural models can identify the gains arising from choices at these separate margins.
- This is a difficult task for IV without invoking structural assumptions.
- Structural models can also identify the fraction of persons induced into a state coming from each origin state.
- IV alone cannot.

Conclusions

- This paper compares the structural approach to empirical policy analysis with the program evaluation approach.
- It offers a third way to do policy analysis that combines the best features of both approaches.
- This paper does not endorse or attack any particular statistical methodology.

- This paper advocates placing the economic and policy questions being addressed front and center.
- Economic theory helps to sharpen statements of policy questions.
- Modern advances in statistics can make the theory useful in addressing these questions.
- A better approach is to use the economics to frame the questions and the statistics to help address them.

- Both the program evaluation approach and the structural approach have desirable features.
- Program evaluation approaches are generally computationally simpler than structural approaches, and it is often easier to conduct sensitivity and replication analyses with them.
- Identification of program effects is often more transparent than identification of structural parameters.
- At the same time, the economic questions answered and the policy relevance of the treatment effects featured in the program evaluation approach are often very unclear.
- Structural approaches produce more interpretable parameters that are better suited to conduct counterfactual policy analyses.

- The third way advocated in this essay is to use Marschak's Maxim to identify the policy relevant *combinations* of structural parameters that answer well-posed policy and economic questions.
- This approach often simplifies the burden of computation, facilitates replication and sensitivity analyses, and makes identification more transparent.
- At the same time, application of this approach forces analysts to clearly state the goals of the policy analysis — something many economists (structural or program evaluation) have difficulty doing.
- That discipline is an added bonus of this approach.

- I have illustrated this approach by using the economics implicit in LATE to interpret the margins of choice identified by instrument variation and to extend the range of questions LATE can answer.
- This analysis is a prototype of the value of a closer integration of theory and robust statistical methods to evaluate public policy.

Appendix

Theorem 1

Assume (Y, X) *i.i.d.* $\quad E(|Y|) < \infty \quad E(|X|) < \infty$

$$\mu_Y = E(Y) \qquad \mu_X = E(X)$$

$$E(Y \mid X) = g(X)$$

Assume $g'(X)$ *exists and* $E\left(|g'(X)|\right) < \infty$.

Theorem 2 (cont.)

Then,

$$\frac{\text{Cov}(Y, X)}{Var(X)} = \int_{-\infty}^{\infty} g'(t)\,\omega(t)\,dt,$$

where

$$\begin{aligned}\omega(t) &= \frac{1}{Var(X)} \int_{t}^{\infty} (x - \mu_X)\, f_X(x)\, dx \\ &= \frac{1}{Var(X)} E\left(X - \mu_X \mid X > t\right) \Pr\left(X > t\right).\end{aligned}$$

$$\begin{aligned} Y &= \pi X + \eta, \\ \pi &= \frac{\text{Cov}(Y, X)}{Var(X)}.\end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{Cov}(Y,X) &= \mathrm{Cov}\left(E(Y \mid X), X\right) = \mathrm{Cov}\left(g(X), X\right) \\ &= \int_{-\infty}^{\infty} g(t)(t-\mu_X)\, f_X(t)\, dt\end{aligned}$$

where t is an argument of integration.

cont.

Integration by parts:

$$\begin{aligned}\mathrm{Cov}(Y, X) &= g(t) \int_{-\infty}^{t} (x - \mu_X)\, f_X(x)\, dx \Big|_{-\infty}^{\infty} \\ &\quad - \int_{-\infty}^{\infty} g'(t) \int_{-\infty}^{t} (x - \mu_X)\, f_X(x)\, dx\, dt \\ &= \int_{-\infty}^{\infty} g'(t) \int_{t}^{\infty} (x - \mu_X)\, f_X(x)\, dx\, dt,\end{aligned}$$

since $E(X - \mu_X) = 0$.

cont.

Therefore,

$$\text{Cov}(Y, X) = \int_{-\infty}^{\infty} g'(t)\, E\left(X - \mu_X \mid X > t\right) \Pr\left(X > t\right)\, dt.$$

∴ Result follows with

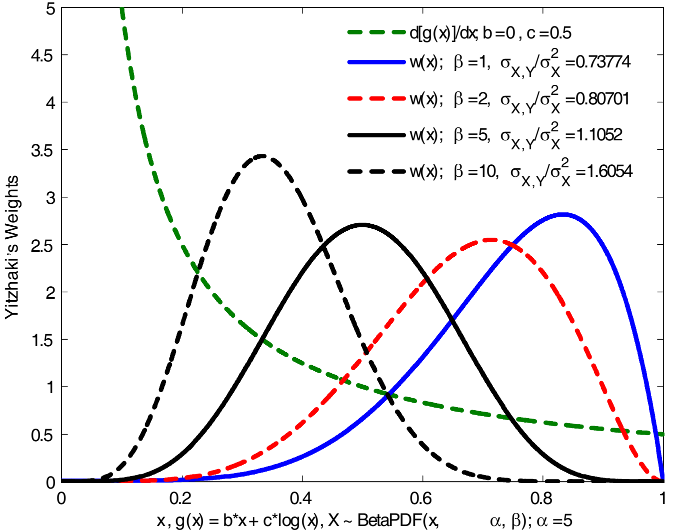
$$\omega(t) = \frac{1}{Var(X)} E\left(X - \mu_X \mid X > t\right) \Pr\left(X > t\right)$$

□

- Weights positive.
- Integrate to one (use integration by parts formula).
- $= 0$ when $t \to \infty$ and $t \to -\infty$.
- Weight reaches its peak at $t = \mu_X$, if f_X has density at $x = \mu_X$:

$$\begin{aligned}\frac{d}{dt}\int_t^{\infty}(x-\mu_X)\,f_X(x)\,dx\,dt &= -(t-\mu_X)f_X(t)\\ &= 0 \quad \text{at } t = \mu_X.\end{aligned}$$

Yitzhaki's weights for $X \sim \text{BetaPDF}(x, \alpha, \beta)$



Yitzhaki's weights for $X \sim \text{BetaPDF}(x, \alpha, \beta)$

$$E\left(Y|X=x\right)=g(x)\Rightarrow \frac{Cov(X,Y)}{Var(X)}=\int_{-\infty}^{\infty} g'(t)w(t)dx$$

$$w(t)=\frac{1}{Var(X)}E\left(X|X>t\right)\cdot \Pr\left(X>t\right)$$

$$\mathbf{X}\sim BetaPDF(x,\alpha,\beta)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)};\ \alpha=5;$$

$$\mathbf{g(x)}=\mathbf{0.5}\cdot\mathbf{x}+\mathbf{0.5}\cdot\log(\mathbf{X})$$

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- Can apply Yitzhaki's analysis to the treatment effect model

$$Y = \alpha + \beta D + \varepsilon$$

- $P(Z)$, the propensity score is the instrument:

$$E(Y \mid Z = z) = E(Y \mid P(Z) = p)$$

$$
\begin{aligned}
E\left(Y \mid P(Z)=p\right) &= \alpha + E\left(\beta D \mid P(Z)=p\right) \\
&= \alpha + E\left(\beta \mid D=1, P(Z)=p\right) p \\
&= \alpha + E\left(\beta \mid P(Z) > U_D, P(Z)=p\right) p \\
&= \alpha + E\left(\beta \mid p > U_D\right) p \\
&= \alpha + \underbrace{\int \beta \int_0^p f\left(\beta, u_D\right)\, du_D}_{g(p)}
\end{aligned}
$$

- Derivative with respect to p is MTE.
- $g'(p) =$MTE and weights as before.

- Under uniformity,

$$\frac{\partial E\left(Y \mid P\left(Z\right)=p\right)}{\partial p} = E\left(Y_1 - Y_0 \mid U_D = u_D\right)$$
$$= \Delta^{MTE}\left(u_D\right).$$

- More generally, it is LIV $= \frac{\partial E(Y|P(Z)=p)}{\partial p}$.
- Yitzhaki's result does not rely on uniformity; true of any regression of Y on P.
- Estimates a weighted net effect.
- The expression can be generalized.
- It produces Heckman-Vytlacil weights.

Proof.

$$\begin{aligned}\text{Cov}\left(J\left(Z\right),Y\right) &= E\left(Y\cdot\widetilde{J}\right) = E\left(E\left(Y\mid Z\right)\cdot\widetilde{J}(Z)\right)\\ &= E\left(E\left(Y\mid P\left(Z\right)\right)\cdot\widetilde{J}(Z)\right)\\ &= E\left(g\left(P\left(Z\right)\right)\cdot\widetilde{J}(Z)\right).\end{aligned}$$

$$\widetilde{J} = J\left(Z\right) - E\left(J\left(Z\right)\mid P\left(Z\right)\geq u_D\right),$$
$$E\left(Y\mid P\left(Z\right)\right) = g\left(P\left(Z\right)\right).$$

cont.

$$\begin{aligned}\mathrm{Cov}\left(J\left(Z\right), Y\right) &= \int_0^1 \int_{\underline{J}}^{\overline{J}} g\left(u_D\right) \widetilde{j} f_{P,J}\left(u_D, j\right) \mathrm{d}j \mathrm{d}u_D \\ &= \int_0^1 g\left(u_D\right) \int_{\underline{J}}^{\overline{J}} \widetilde{j} f_{P,J}\left(u_D, j\right) \mathrm{d}j \mathrm{d}u_D.\end{aligned}$$

cont.

Use integration by parts:

$$
\begin{aligned}
&\mathrm{Cov}\left(J\left(Z\right), Y\right) \\
&= g\left(u_D\right) \int_0^{u_D} \int_{\underline{J}}^{\overline{J}} \widetilde{j} f_{P,J}\left(p, j\right) \, \mathrm{d}j \mathrm{d}p \bigg|_0^1 \\
&\qquad - \int_0^1 g'\left(u_D\right) \int_0^{u_D} \int_{\underline{J}}^{\overline{J}} \widetilde{j} f_{P,J}\left(p, j\right) \, \mathrm{d}j \mathrm{d}p \mathrm{d}u_D \\
&= \int_0^1 g'\left(u_D\right) \int_{u_D}^1 \int_{\underline{J}}^{\overline{J}} \widetilde{j} f_{P,J}\left(p, j\right) \, \mathrm{d}j \mathrm{d}p \mathrm{d}u_D \\
&= \int_0^1 g'\left(u_D\right) E\left(\widetilde{J}(Z) \mid P\left(Z\right) \geq u_D\right) \Pr\left(P\left(Z\right) \geq u_D\right) \mathrm{d}u_D.
\end{aligned}
$$

The Heckman-Vytlacil weight as a Yitzhaki weight

cont.

$$g'(u_D) = \left. \frac{\partial E(Y \mid P(Z) = p)}{\partial P(Z)} \right|_{p=u_D} = \Delta^{\text{MTE}}(u_D).$$

□

- Under our assumptions the Yitzhaki weights and ours are equivalent.
-

$$\text{Cov}\left(J\left(Z\right),Y\right) \tag{3}$$
$$= \int_0^1 \Delta^{\text{MTE}}(u_D)E(J(Z) - E(J(Z)) \mid P(Z) \geq u_D) \Pr(P(Z) \geq u_D)du_D.$$

- Using (3),

$$\begin{aligned}\text{Cov}\left(J\left(Z\right),Y\right) &= E\left(Y \cdot \tilde{J}\right) = E\left(E\left(Y \mid Z\right) \cdot \tilde{J}\left(Z\right)\right)\\ &= E\left(E\left(Y \mid P\left(Z\right)\right) \cdot \tilde{J}\left(Z\right)\right)\\ &= E\left(g\left(P\left(Z\right)\right) \cdot \tilde{J}\left(Z\right)\right).\end{aligned}$$

- The third equality follows from index sufficiency and $\tilde{J} = J(Z) - E(J(Z) \mid P(Z) \geq u_D)$, where $E(Y \mid P(Z)) = g(P(Z))$.
- Writing out the expectation and assuming that $J(Z)$ and $P(Z)$ are continuous random variables with joint density $f_{P,J}$ and that $J(Z)$ has support $\left[\underline{J}, \overline{J}\right]$,

$$\begin{aligned}\operatorname{Cov}(J(Z), Y) &= \int_0^1 \int_{\underline{J}}^{\overline{J}} g(u_D)\tilde{j} f_{P,J}(u_D, j)\, \mathrm{d}j \mathrm{d}u_D \\ &= \int_0^1 g(u_D) \int_{\underline{J}}^{\overline{J}} \tilde{j} f_{P,J}(u_D, j)\, \mathrm{d}j \mathrm{d}u_D.\end{aligned}$$

- Using an integration by parts argument as in Yitzhaki (1989) and as summarized in Heckman, Urzua, Vytlacil (2006), we obtain

$$
\begin{aligned}
&\operatorname{Cov}(J(Z), Y) \\
&\quad = g(u_D) \int_0^{u_D} \int_{\underline{J}}^{\overline{J}} \tilde{j} f_{P,J}(p, j)\, \mathrm{d}j \mathrm{d}p \Big|_0^1 \\
&\qquad\qquad - \int_0^1 g'(u_D) \int_0^{u_D} \int_{\underline{J}}^{\overline{J}} \tilde{j} f_{P,J}(p, j)\, \mathrm{d}j \mathrm{d}p \mathrm{d}u_D \\
&\quad = \int_0^1 g'(u_D) \int_{u_D}^1 \int_{\underline{J}}^{\overline{J}} \tilde{j} f_{P,J}(p, j)\, \mathrm{d}j \mathrm{d}p \mathrm{d}u_D \\
&\quad = \int_0^1 g'(u_D)\, E\left(\tilde{J}(Z) \mid P(Z) \geq u_D\right) \Pr(P(Z) \geq u_D)\, \mathrm{d}u_D,
\end{aligned}
$$

which is then exactly the expression given in (3), where

$$
g'(u_D) = \frac{\partial E(Y \mid P(Z) = p)}{\partial P(Z)} \Big|_{p=u_D} = \Delta^{\mathrm{MTE}}(u_D).
$$

Under our conditions separable choice model

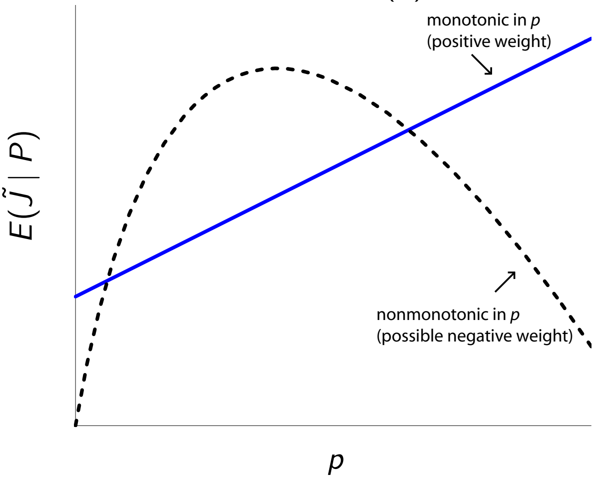
$$\Delta_J^{IV} = \int_0^1 \Delta^{MTE}(u_D)\, \omega_{IV}^J(u_D)\, du_D \tag{4}$$

$$\omega_{IV}^J(u_D) = \frac{E\left(J(Z) - \bar{J}(Z) \mid P(Z) > u_D\right) \Pr\left(P(Z) > u_D\right)}{\operatorname{Cov}(J(Z), D)}. \tag{5}$$

$J(Z)$ and $P(Z)$ do not have to be continuous random variables.Functional forms of $P(Z)$ and $J(Z)$ are general.

- Dependence between $J(Z)$ and $P(Z)$ gives shape and sign to the weights.
- If $J(Z) = P(Z)$, then weights obviously non-negative.
- If $E(J(Z) - \bar{J}(Z) \mid P(Z) \geq u_D)$ not monotonic in u_D, weights can be negative.

$$\tilde{J} = J - E(J)$$



Therefore, with positive (or negative) regression, can get negative IV weight.

When $J(Z) = P(Z)$, weight (5) follows from Yitzhaki (1989).

- He considers a regression function $E(Y \mid P(Z) = p)$.
- Linear regression of Y on P identifies

$$\beta_{Y,P} = \int_0^1 \left[\frac{\partial E(Y \mid P(Z) = p)}{\partial p} \right] \omega(p)\, dp,$$

$$\omega(p) = \frac{\int_p^1 (t - E(P))\, dF_P(t)}{Var(P)}.$$

- This is the weight (5) when P is the instrument.
- This expression **does not** require uniformity or monotonicity for the model; consistent with 2-way flows.

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Understanding the structure of the IV weights

Recapitulate:

$$\Delta_{\text{IV}}^{J} = \int \Delta^{\text{MTE}}(u_D)\,\omega_{\text{IV}}^{J}(u_D)\,du_D$$

$$\omega_{\text{IV}}^{J}(u_D) = \frac{\int (j - E(J(Z))) \int_{u_D}^{1} f_{J,P}(j,t)\ dt\,dj}{Cov(J(Z), D)} \tag{6}$$

- The weights are always positive if $J(Z)$ is monotonic in the scalar Z.
- In this case $J(Z)$ and $P(Z)$ have the same distribution and $f_{J,P}(j,t)$ collapses to a single distribution.

- The possibility of negative weights arises when $J(Z)$ is not a monotonic function of $P(Z)$.
- It can also arise when there are two or more instruments, and the analyst computes estimates with only one instrument or a combination of the Z instruments that is not a monotonic function of $P(Z)$ so that $J(Z)$ and $P(Z)$ are not perfectly dependent.

- The weights can be constructed from data on (J, P, D).
- Data on $(J(Z), P(Z))$ pairs and $(J(Z), D)$ pairs (for each X value) are all that is required.

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