Econometrics Lecture 7: Simultaneous Equations Models: Identification, Estimation and Testing

R. G. Pierse
National Institute of Economic and Social Research Fellow

James J. Heckman

THE UNIVERSITY OF CHICAGO

Econ 312, Spring 2021

1. Introduction

$$Q = \alpha_1 + \alpha_2 P + \alpha_3 Y + u_1 \tag{1.1}$$

$$P = \beta_1 + \beta_2 Q + \beta_3 W + u_2 \tag{1.2}$$

$$Q_t = \alpha_1 + \alpha_2 P_t + \alpha_3 Y_t + \alpha_4 Q_{t-1} + \alpha_5 P_{t-1} + u_{1t} \tag{1.3}$$

and

$$P_t = \beta_1 + \beta_2 Q_t + \beta_3 W_t + \beta_4 P_{t-1} + \beta_5 Q_{t-1} + u_{2t} \tag{1.4}$$

2. The Simultaneous Equations Model

The general linear simultaneous equations model with m equations can be written formally as

$$\mathbf{B}\mathbf{y}_t + \mathbf{\Gamma}\mathbf{z}_t = \mathbf{u}_t, \quad t = 1, \cdots, T \tag{2.1}$$

$$\begin{bmatrix} 1 & -\alpha_2 \\ -\beta_2 & 1 \end{bmatrix} \begin{bmatrix} Q_t \\ P_t \end{bmatrix} + \begin{bmatrix} -\alpha_1 & -\alpha_4 & -\alpha_5 & -\alpha_3 & 0 \\ -\beta_1 & -\beta_5 & -\beta_4 & 0 & -\beta_3 \end{bmatrix} \begin{bmatrix} 1 \\ Q_{t-1} \\ P_{t-1} \\ Y_t \\ W_t \end{bmatrix} = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}. \tag{2.2}$$

$$\mathbf{y}_t = -\mathbf{B}^{-1}\boldsymbol{\Gamma}\mathbf{z}_t + \mathbf{B}^{-1}\mathbf{u}_t \tag{2.3}$$

or

$$\mathbf{y}_t = \boldsymbol{\Pi}\mathbf{z}_t + \mathbf{v}_t. \tag{2.4}$$

3. Identification in Simultaneous Equations

Consider pre-multiplying the simultaneous equation system (2.1) by the $m \times m$ nonsingular matrix $\mathbf{F}$ to give

$$\mathbf{F}\mathbf{B}\mathbf{y}_t + \mathbf{F}\boldsymbol{\Gamma}\mathbf{z}_t = \mathbf{F}\mathbf{u}_t. \tag{3.1}$$

The reduced form of this transformed model is

$$\begin{aligned}
\mathbf{y}_t &= -(\mathbf{F}\mathbf{B})^{-1}\mathbf{F}\boldsymbol{\Gamma}\mathbf{z}_t + (\mathbf{F}\mathbf{B})^{-1}\mathbf{F}\mathbf{u}_t \\
&= -\mathbf{B}^{-1}\mathbf{F}^{-1}\mathbf{F}\boldsymbol{\Gamma}z_t + \mathbf{B}^{-1}\mathbf{F}^{-1}\mathbf{F}u_t \\
&= -\mathbf{B}^{-1}\boldsymbol{\Gamma}\mathbf{z}_t + \mathbf{B}^{-1}\mathbf{u}_t
\end{aligned}$$

4. Estimation: Single Equation Methods

5. Indirect Least Squares

$$\widetilde{\boldsymbol{\beta}} = (\mathbf{W}'\mathbf{X})^{-1}\mathbf{W}'\mathbf{y} \tag{5.1}$$

is called an *instrumental variables* or *IV* estimator.

$$\begin{aligned}\widetilde{\boldsymbol{\beta}} &= \left(\widehat{\mathbf{X}}'\mathbf{X}\right)^{-1}\widehat{\mathbf{X}}'\mathbf{y} = \left(\widehat{\mathbf{X}}'\widehat{\mathbf{X}}\right)^{-1}\widehat{\mathbf{X}}'\mathbf{y} \\ &= (\mathbf{X}'\mathbf{W}(\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{y} \qquad (5.2)\end{aligned}$$

In general, $q > k_j$ so that there are more instruments than regressors. Hence the application of *IV* leads to the *2SLS* estimator

$$\widetilde{\boldsymbol{\beta}}_j = (\mathbf{X}_j'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}_j)^{-1}\mathbf{X}_j'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_j \tag{5.3}$$

The variance of the 2SLS estimator is given by

$$\text{var}(\widetilde{\boldsymbol{\beta}}_j) = \sigma_{jj}(\mathbf{X}_j'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}_j)^{-1}$$

and a consistent estimator of σ_{jj} can be obtained from the *2SLS* residuals

$$\mathbf{e} = \mathbf{y}_j - \mathbf{X}_j\widetilde{\boldsymbol{\beta}}_j \tag{5.4}$$

using the expression

$$\widetilde{\sigma}_{jj} = \frac{\mathbf{e}'\mathbf{e}}{T-k}.$$

$$\sqrt{T}(\widetilde{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j) \sim_a N(0, \sigma_{jj}(\mathbf{Q}_{XZ}\mathbf{Q}_{ZZ}^{-1}\mathbf{Q}_{ZX})^{-1}). \qquad (5.5)$$

Consider the equation

$$\mathbf{y}_j = \mathbf{X}_j\boldsymbol{\beta}_j + \mathbf{u}_j = \mathbf{Y}_1\alpha_j + \mathbf{Z}_1\gamma_j + \mathbf{u}_j \tag{5.6}$$

6. Estimation: System Methods

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_m \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_m \end{bmatrix} + \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} \tag{6.1}$$

or

$$\overline{\mathbf{y}} = \overline{\mathbf{X}}\boldsymbol{\beta} + \overline{\mathbf{u}}$$

where

$$\text{var}(\overline{\mathbf{u}}) = \begin{bmatrix} \sigma_{11}\mathbf{I}_T & \cdots & \sigma_{1m}\mathbf{I}_T \\ \vdots & \ddots & \vdots \\ \sigma_{m1}\mathbf{I}_T & & \sigma_{mm}\mathbf{I}_T \end{bmatrix} = \boldsymbol{\Sigma} \otimes \mathbf{I}_T \tag{6.2}$$

$$\begin{bmatrix} \widehat{\mathbf{X}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_m \end{bmatrix} = \begin{bmatrix} \mathbf{P}_z\mathbf{X}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_z\mathbf{X}_m \end{bmatrix} = (\mathbf{I}_m \otimes \mathbf{P}_z)\overline{\mathbf{X}} \tag{6.3}$$

where $\mathbf{P}_z = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$ is the instrument projection matrix.

Applying both *GLS* using (6.2) and *IV* using instrument set (6.3) results in the *Three Stage Least Squares (3SLS) Estimator* of Zellner and Theil (1962)

$$\begin{aligned} \widetilde{\boldsymbol{\beta}} &= \left(\overline{\mathbf{X}}'(\mathbf{I}_m \otimes \mathbf{P}_z)'(\boldsymbol{\Sigma} \otimes \mathbf{I}_T)^{-1}(\mathbf{I}_m \otimes \mathbf{P}_z)\overline{\mathbf{X}}\right)^{-1} \overline{\mathbf{X}}'(\mathbf{I}_m \otimes \mathbf{P}_z)'(\boldsymbol{\Sigma} \otimes \mathbf{I}_T)^{-1}\overline{\mathbf{y}} \\ &= \left(\overline{\mathbf{X}}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{P}_z)\overline{\mathbf{X}}\right)^{-1} \overline{\mathbf{X}}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{P}_z)\overline{\mathbf{y}}. \end{aligned} \tag{6.4}$$

$$\sqrt{T}(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) \sim_a N\left(0, \left(\text{plim}\,\frac{1}{T}\overline{\mathbf{X}}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{P}_z)\overline{\mathbf{X}}\right)^{-1}\right). \qquad (6.5)$$