Simultaneous Causality: Part IV on Causality

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- A system of linear simultaneous equations captures interdependence among outcomes Y.

- Linear model in terms of parameters (Γ, B), observables (Y, X) and unobservables U:

$$\Gamma Y + BX = U, \qquad E(U) = 0, \tag{1}$$

- Y is now a vector of internal and interdependent variables
- X is external and exogenous $(E(U \mid X) = 0)$
- Γ is a full rank matrix ("completeness").
- $Y = \Gamma^{-1}BX + \Gamma^{-1}U$ (reduced form)
- Completeness is totally different from another concept of completeness:
- In nonparametric IV, $\int_{\text{Supp}X} \Phi_\theta(X)dF(X) = 0 \Leftrightarrow \Phi_\theta(X) = 0$

- This is a linear-in-the-parameters "all causes" model for vector Y, where the causes are X and $\mathcal{E}$.
- The "structure" is (Γ, B), Σ_U, where Σ_U is the variance-covariance matrix of U.
- In the Cowles Commission analysis it is assumed that Γ, B, Σ_U are invariant to general changes in X and translations of U.
- Autonomy (Frisch, 1938) later called one component of "SUTVA" (see Holland, 1986).
- X, U **external variables**.
- Y **internal variables**.

- Consider a two-agent model of social interactions.
- Y_1 is the outcome for agent 1; Y_2 is the outcome for agent 2.

- $$\begin{aligned} Y_1 &= \alpha_1 + \gamma_{12} Y_2 + \beta_{11} X_1 + \beta_{12} X_2 + U_1, \qquad (2a) \\ Y_2 &= \alpha_2 + \gamma_{21} Y_1 + \beta_{21} X_1 + \beta_{22} X_2 + U_2. \qquad (2b) \end{aligned}$$

- Social interactions model is a standard version of the simultaneous equations problem.
- This model is sufficiently flexible to capture the notion that the consumption of 1 (Y_1) depends on the consumption of 2 if $\gamma_{12} \neq 0$, as well as 1's value of X if $\beta_{11} \neq 0$, X_1 (assumed to be observed), 2's value of X , X_2 if $\beta_{12} \neq 0$ and unobservable factors that affect 1 (U_1).
- The determinants of 2's consumption are defined symmetrically.
- Allow U_1 and U_2 to be freely correlated.
- Captures essence of "reflection problem."

- Assume
$$E\left(U_1 \mid X_1, X_2\right) = 0 \tag{3a}$$
and
$$E\left(U_2 \mid X_1, X_2\right) = 0. \tag{3b}$$
- Completeness guarantees that (2a) and (2b) have a determinate solution for (Y_1, Y_2).
- Applying Haavelmo's (1943) analysis to (2a) and (2b), the causal effect of Y_2 on Y_1 is γ_{12}.
- This is the effect on Y_1 of fixing Y_2 at different values, holding constant the other variables in the equation.

- Symmetrically, the causal effect of Y_1 on Y_2 is γ_{21}.
- Conditioning, i.e., using least squares, in general, fails to identify these causal effects because U_1 and U_2 are correlated with Y_1 and Y_2.
- This is a traditional argument.
- It is based on the possibility of correlation between Y_2 and U_1 (Haavelmo, 1943).
- But even if $U_1 = 0$ and $U_2 = 0$, so that there are no unobservables, least squares breaks down because Y_2 is perfectly predictable by X_1 and X_2.

Question: Prove this.

- We cannot simultaneously vary Y_2, X_1 and X_2.
- **The error term is not the fundamental source of non-identifiability in these models**.

Reduced Form

- Under completeness, the reduced form outcomes of the model after social interactions are solved out can be written as

$$
\begin{aligned}
Y_1 &= \pi_{10} + \pi_{11} X_1 + \pi_{12} X_2 + \mathcal{E}_1, && \text{(4a)} \\
Y_2 &= \pi_{20} + \pi_{21} X_1 + \pi_{22} X_2 + \mathcal{E}_2. && \text{(4b)}
\end{aligned}
$$

$$
E(\mathcal{E}_1 | X) = 0
$$

$$
E(\mathcal{E}_2 | X) = 0
$$

- Least squares can identify the *ceteris paribus* effects of X_1 and X_2 on Y_1 and Y_2 because $E(\mathcal{E}_1 \mid X_1, X_2) = 0$ and $E(\mathcal{E}_2 \mid X_1, X_2) = 0$.
- Simple algebra:

$$\pi_{11} = \frac{\beta_{11} + \gamma_{12}\beta_{21}}{1 - \gamma_{12}\gamma_{21}}, \qquad \pi_{12} = \frac{\beta_{12} + \gamma_{12}\beta_{22}}{1 - \gamma_{12}\gamma_{21}},$$

$$\pi_{21} = \frac{\gamma_{21}\beta_{11} + \beta_{21}}{1 - \gamma_{12}\gamma_{21}},$$

$$\pi_{22} = \frac{\gamma_{21}\beta_{12} + \beta_{22}}{1 - \gamma_{12}\gamma_{21}}$$

$$\mathcal{E}_1 = \frac{U_1 + \gamma_{12} U_2}{1 - \gamma_{12}\gamma_{21}},$$

$$\mathcal{E}_2 = \frac{\gamma_{21} U_1 + U_2}{1 - \gamma_{12}\gamma_{21}}.$$

- **Without any further information on the variances of (U_1, U_2) and their relationship to the causal parameters, we cannot identify the causal effects γ_{12} and γ_{21} from the reduced form regression coefficients.**
- This is so because holding X_1, X_2, U_1 and U_2 fixed in (2a) or (2b), it is not possible to vary Y_2 or Y_1, respectively, because they are exact functions of X_1, X_2, U_1 and U_2.
- This exact dependence holds true even if $U_1 = 0$ and $U_2 = 0$ so that there are no unobservables.

- There is no mechanism yet specified within the model to independently vary the right hand sides of Equations (2a) and (2b).
- The mere fact that we can write (2a) and (2b) means that we "can imagine" independent variation.
- Causality is in the mind.

Question: Can we still define the causal effect of Y_2 on Y_1 and Y_1 on Y_2, even if we cannot identify them?

- We "can imagine" a model

$$Y = \varphi_0 + \varphi_1 X_1 + \varphi_2 X_2,$$

  but if part of the model is $(*)\, X_1 = X_2$, no causal effect of X_1 holding X_2 constant is possible in principle within the rules of the model.
- If we break restriction $(*)$ and permit independent variation in X_1 and X_2, we can define the causal effect of X_1 holding X_2 constant.
- But we can imagine such variation.

- In some conceptualizations, no causality is possible; in others it is.
- Distinguish identification from causation.
- The X effects on Y_1 and Y_2, identified through the reduced forms, combine the direct effects (through β_{ij}) and the indirect effects (as they operate through Y_1 and Y_2, respectively).
- If we assume exclusions ($\beta_{12} = 0$) or ($\beta_{21} = 0$) or both, we can identify the *ceteris paribus* causal effects of Y_2 on Y_1 and of Y_1 on Y_2, respectively, if $\beta_{22} \neq 0$ or $\beta_{11} \neq 0$, respectively.

Consider Standard Identification Analyses

- Suppose

$$\beta_{12} = 0 \text{ and } \beta_{21} = 0$$

$$\pi_{11} = \frac{\beta_{11}}{1 - \gamma_{12}\gamma_{21}} \qquad \pi_{12} = \frac{\gamma_{12}\beta_{22}}{1 - \gamma_{12}\gamma_{21}}$$

$$\pi_{21} = \frac{\gamma_{21}\beta_{11}}{1 - \gamma_{12}\gamma_{21}} \qquad \pi_{22} = \frac{\beta_{22}}{1 - \gamma_{12}\gamma_{21}}$$

$$\frac{\pi_{12}}{\pi_{22}} = \gamma_{12}$$
$$\frac{\pi_{21}}{\pi_{11}} = \gamma_{21}$$

- $\therefore$ we identify β_{11} and β_{22}.

- Suppose instead only $\beta_{12} = 0$

$$\pi_{22} = \frac{\beta_{22}}{1 - \gamma_{12}\gamma_{21}}$$
$$\pi_{12} = \frac{\gamma_{12}\beta_{22}}{1 - \gamma_{12}\gamma_{21}}$$
$$\frac{\pi_{12}}{\pi_{22}} = \gamma_{12}$$

- Then can form left-hand side of $y_1 - \gamma_{12} y_2 = \beta_{11} X_1 + \beta_{12} X_2 + U_1$.
- $\therefore$ can identify $\beta_{11} = 0$ from OLS (recall $E(U_1|X) = 0$).
- Can identify $\sigma_1^2 = Var(U_1)$.

- Symmetrically if $\beta_{21} = 0$ can identify β_{22}, σ_2^2.
- Suppose $Cov(U_1, U_2) = 0$.

$$
\begin{aligned}
Cov(\mathcal{E}_1, \mathcal{E}_2) &= Cov\left[\left(\frac{U_1 + \gamma_{12}U_2}{1 - \gamma_{12}\gamma_{21}}\right)\left(\frac{\gamma_{21}U_1 + U_2}{1 - \gamma_{12}\gamma_{21}}\right)\right] \\
&= \frac{\gamma_{21}\sigma_1^2 + \gamma_{12}\sigma_2^2}{(1 - \gamma_{12}\gamma_{21})^2} \\
Var(\mathcal{E}_1) &= \frac{\sigma_2^1 + \gamma_{12}^2\sigma_2^2}{(1 - \gamma_{12}\gamma_{21})^2} \\
Var(\mathcal{E}_2) &= \frac{\gamma_{21}^2\sigma_1^2 + \sigma_2^2}{(1 - \gamma_{12}\gamma_{21})^2}
\end{aligned}
$$

- Suppose we add this to $\beta_{12} = 0$.
- By previously analysis, we know γ_{12}, σ_1^2.

- Then we know

$$a = \frac{Cov(\mathcal{E}_1, \mathcal{E}_2)}{Var(\mathcal{E}_1)} = \frac{\gamma_{21}\sigma_1^2 + \gamma_{12}\sigma_2^2}{\sigma_1^2 + \gamma_{12}^2\sigma_2^2}$$

$$b = \frac{Cov(\mathcal{E}_1, \mathcal{E}_2)}{Var(\mathcal{E}_2)} = \frac{\gamma_{21}\sigma_1^2 + \gamma_{12}\sigma_2^2}{\gamma_1^2 + \sigma_1^2 + \sigma_2^2}$$

- 2 equations in 2 unknowns
- Can solve: σ_2^2, γ_{21} (in principle) letting " ^ " denote estimate

$$(\hat{a})(\sigma_1^2 + \gamma_{12}^2\sigma_2^2) = \gamma_{21}\sigma_1^2 + \gamma_{12}\sigma_2^2$$

$$(\hat{b})(\gamma_{21}^2\sigma_1^2 + \sigma_2^2) = \gamma_{21}\sigma_1^2 + \gamma_{12}\sigma_2^2$$

$$\hat{a}(\hat{\sigma}_1^2 + \hat{\gamma}_{12}^2\sigma_2^2) = \hat{b}(\gamma_{21}^2\hat{\sigma}_1^2 + \sigma_2^2)$$

$$\hat{c} = \frac{Var(\mathcal{E}_1)}{Var(\mathcal{E}_2)} = \frac{\widehat{\sigma_1^2} + \hat{\gamma}_{12}^2\sigma_2^2}{\gamma_{21}^2\hat{\sigma}_1^2 + \sigma_2^2}$$

- Alternatively, we could assume $\beta_{11} = \beta_{22} = 0$ and $\beta_{12} \neq 0$, $\beta_{21} \neq 0$ to identify γ_{12} and γ_{21}.
- These exclusions say that the social interactions only operate through the Y's.
- Agent 1's consumption depends only on agent 2's consumption and not on his value of X_2.
- Agent 2 is modeled symmetrically versus agent 1.
- Observe that we have *not* ruled out correlation between U_1 and U_2.

- When the procedure for identifying causal effects is applied to samples, it is called **indirect least squares** (Tinbergen, 1930).
- The analysis for social interactions in this section is of independent interest.
- It can be generalized to the analysis of N person interactions if the outcomes are continuous variables.

Nonlinear Systems Possible

- Thus we can postulate a system of equations $G(Y, X, U) = 0$ and develop conditions for unique solution of reduced forms $Y = K(X, U)$ requiring that certain Jacobian terms be nonvanishing (Matzkin, "Nonparametric Identification of Simultaneous Equations," 2007).
- The structural form (1) is an all causes model that relates in a deterministic way outcomes (internal variables) to other outcomes (internal variables) and external variables (the X and U).
- **Question:** Are *ceteris paribus* manipulations associated with the effect of some components of Y on other components of Y possible within the model?
- Yes.

- The intuition for these results is that if $\beta_{12} = 0$, we can vary Y_2 in Equation (2a) by varying the X_2 that does not directly affect Y_1 in the structural equation.
- Since X_2 does not appear in the equation, under exclusion, we can keep U_1, X_1 fixed and vary Y_2 using X_2 in (4b) if $\beta_{22} \neq 0$.
- Notice that we could also use U_2 as a source of variation in (4b) to shift Y_2.
- The roles of U_2 and X_2 are symmetric.
- However, if U_1 and U_2 are correlated, shifting U_2 shifts U_1 unless we control for it.
- The component of U_2 uncorrelated with U_1 plays the role of X_2.

- Symmetrically, by excluding X_1 from(2b), we can vary Y_1, holding X_2 and U_2 constant.
- These results are more clearly seen when $U_1 = 0$ and $U_2 = 0$.

- A hypothetical thought experiment justifies these exclusions.
- If agents do not know or act on the other agent's X, these exclusions are plausible.
- An implicit assumption in using (2a) and (2b) for causal analysis is invariance of the parameters $(\Gamma, \beta, \Sigma_U)$ to manipulations of the external variables.

- This definition of causal effects in an interdependent system generalizes the recursive definitions of causality featured in the statistical treatment effect literature (Holland, 1988, and Pearl, 2009).
- The key to this definition is manipulation of external inputs and exclusion, not randomization or matching.

Control Function Principle

$$E(U_1|\mathcal{E}_2) = \left(\frac{\sigma_{11}\gamma_{21} + \sigma_{12}}{1 - \gamma_{21}\gamma_{21}}\right)\mathcal{E}_2 + 0$$

$$U_1 = \underbrace{\left(\frac{\sigma_{11}\gamma_{12} + \sigma_{12}}{1 - \gamma_{21}\gamma_{21}}\right)\hat{\mathcal{E}}_2}_{\text{control function}} + V_1$$

- V_1: portion of U_1 not correlated with Y_2.
- If no exclusions in first equation, perfect multicollinearity, i.e., $Y_1 = \gamma_{12}(\hat{Y}_2) + \beta_{11}X_1 + \beta_{12}X_2 + \gamma_{12}\hat{\mathcal{E}}_2 + U_1$ controls for covariance between Y_2 and U_1.

In a General Nonlinear Model

$$\begin{aligned} Y_1 &= g_1(Y_2, X_1, X_2, U_1) \\ Y_2 &= g_2(Y_1, X_1, X_2, U_2), \end{aligned}$$

exclusion is defined as $\frac{\partial g_1}{\partial X_1} = 0$ for all (Y_2, X_1, X_2, U_1) and $\frac{\partial g_2}{\partial X_2} = 0$ for all (Y_1, X_1, X_2, U_2).

- Assuming the existence of local solutions, we can solve these equations to obtain

$$\begin{aligned} Y_1 &= \varphi_1(X_1, X_2, U_1, U_2) \\ Y_2 &= \varphi_2(X_1, X_2, U_1, U_2) \end{aligned}$$

- By the chain rule we can write

$$\frac{\partial g_1}{\partial Y_2} = \frac{\partial Y_1}{\partial X_1} \Big/ \frac{\partial Y_2}{\partial X_1} = \frac{\partial \varphi_1}{\partial X_1} \Big/ \frac{\partial \varphi_2}{\partial X_1}.$$

- We may define causal effects for Y_1 on Y_2 using partials with respect to X_2 in an analogous fashion.