Comparing IV With Explicitly Formulated Economic Structural Models: What Simple IV Can and Cannot Identify

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The Choice Model and Assumptions

- Following Heckman, Urzúa, and Vytlacil (2006, 2008) and Heckman and Vytlacil (2007b), consider the following model with multiple choices and associated multiple outcome states.
- Let $\mathcal{J}$ denote the agent's choice set, where $\mathcal{J}$ contains a finite number of elements.
- For example, $\mathcal{J}$ enumerates possible schooling states (e.g., GED, high school dropout, high school graduate). The value to the agent of choosing $j \in \mathcal{J}$ is

$$R_j(Z_j) = \vartheta_j(Z_j) - V_j, \tag{1.1}$$

where Z_j are the agent's observed characteristics that affect the utility from choosing j, and V_j is the unobserved shock to the agent's utility from choice j.

- We sometimes write R_j for $R_j(Z_j)$ to simplify notation.
- Let Z denote the random vector containing all unique elements of $\{Z_j\}_{j \in \mathcal{J}}$.
- We write $R_j(Z)$ for $R_j(Z_j)$, leaving implicit the condition that $R_j(\cdot)$ only depends on the elements of Z that are contained in Z_j.
- Let D_j be a variable indicating whether the agent would choose j if confronted with choice set $\mathcal{J}$:

$$D_j = \begin{cases} 1 & \text{if } R_j \geq R_k \quad \forall\ k \in \mathcal{J} \\ 0 & \text{otherwise.} \end{cases}$$

  Array the D_j into a vector D.

- Let Y be the outcome that would be observed if the agent faced choice set $\mathcal{J}$, defined as

$$Y = \sum_{j \in \mathcal{J}} D_j Y_j,$$

where Y_j is a potential outcome observed only if option j is chosen.

- Y_j is determined by

$$Y_j = \mu_j(X_j, U_j),$$

where X_j is a vector of the agent's observed characteristics and U_j is an unobserved random vector.

- Let X denote the random vector containing all unique elements of $\{X_j\}_{j \in \mathcal{J}}$.
- (Z, X, D, Y) is assumed to be observed by the analyst.

- Define $R_{\mathcal{J}}$ as the maximum obtainable value given choice set $\mathcal{J}$:

$$\begin{aligned} R_{\mathcal{J}} &= \max_{j \in \mathcal{J}} \{R_j\} \\ &= \sum_{j \in \mathcal{J}} D_j R_j. \end{aligned} \quad (1.2)$$

- This is the traditional representation of the decision process that if choice j is optimal, choice j is better than the "next best" option:

$$D_j = 1 \Longleftrightarrow R_j \geq R_{\mathcal{J} \setminus j}.$$

- Heckman, Urzúa, and Vytlacil (2006, 2008) and Heckman and Vytlacil (2007b) show that this simple, well-known, representation is the key intuition for understanding how instrumental variables estimate the effect of a given choice versus the "next best" alternative.

- IV is a weighted average of the effects for people induced into a choice from different margins.
- Analogous to the definition of $R_{\mathcal{J}}$, we define $R_{\mathcal{J}}(z)$ to be the maximum obtainable value given choice set $\mathcal{J}$ when instruments are fixed at $Z = z$,

$$R_{\mathcal{J}}(z) = \max_{j \in \mathcal{J}} \{R_j(z)\}.$$

Proofs and Derivations

Link

Interpreting Local Instrumental Variables in the Unordered Case

- We define local instrumental variables (LIV) using a variable that shifts people toward (or against) choice j by operating only on $R_j(Z_j)$.
- LIV identifies an average marginal return to j vs. the next best alternative across persons.
- However, without further assumptions, LIV will not decompose the average marginal return into its component parts corresponding to the effects for persons induced into j from each of the possible origin states.

- To see this, consider a three outcome case, $\mathcal{J} = \{1, 2, 3\}$.
- For concreteness, we pursue the education example previously stated and let 1 be GED, 2 be high school dropout, and 3 be high school graduate.
- Our results are more general but the three outcome case is easy to exposit.

- In this section, we assume that Z_1, Z_2, Z_3 are disjoint sets of regressors so $Z = (Z_1, Z_2, Z_3)$ but they are not necessarily statistically independent.
- We can easily relax this assumption but it simplifies the notation.
- We condition on X and keep it implicit throughout the analysis of this paper.

- In this notation,

$$
\begin{aligned}
E(Y \mid Z) &= E\left[\sum_{j=1}^{3} Y_j D_j \middle| Z\right] \quad (2.1) \\
&= E(Y_1 D_1 \mid Z) + E(Y_2 D_2 \mid Z) + E(Y_3 D_3 \mid Z).
\end{aligned}
$$

  $E(Y|Z)$ and its components can be estimated from data on (Y, Z).

- IV is based on (2.1). From (1.2), choices are generated by the following inequalities:

$$
\begin{aligned}
D_1 &= \mathbf{1}(R_1 \geq R_2, R_1 \geq R_3) \\
D_2 &= \mathbf{1}(R_2 \geq R_1, R_2 \geq R_3) \\
D_3 &= \mathbf{1}(R_3 \geq R_1, R_3 \geq R_2).
\end{aligned}
$$

- We define the marginal change in Y with respect to Z_1.
- IV methods are based on such types of variation.
- The local instrumental variable estimator using Z_1 as an instrument is the sample analogue of

$$\left.\frac{\frac{\partial E(Y|Z)}{\partial Z_1}}{\frac{\partial \Pr(D_1=1|Z)}{\partial Z_1}}\right|_{Z=z} = \text{LIV}(z),$$

  where LIV is a function of z.
- In the case of three choices, there are two margins from which persons can be attracted into or out of choice 1 by Z_1.

- From local variations in Z_1, one can recover the following combinations of parameters from the data on $Y_1 D_1$:

$$\frac{\partial E\left(Y_1 D_1 \mid Z = z\right)}{\partial Z_1}$$
$$= \frac{\partial}{\partial Z_1} \int \int_{-\infty}^{\vartheta_1(Z_1)-\vartheta_2(Z_2)} \int_{-\infty}^{\vartheta_1(Z_1)-\vartheta_3(Z_3)} y_1 f_{Y_1, V_1-V_2, V_1-V_3}\left(y_1, v_1 - v_2, v_1 - v_3\right) d\left(v_1 - v_3\right) d\left(v_1 - v_2\right) dy_1 \Bigg|_{Z=z}$$
$$= \frac{\partial \vartheta_1\left(Z_1\right)}{\partial Z_1}\Bigg|_{Z_1=z_1} \left[\begin{array}{c} \int y_1 \int_{-\infty}^{\vartheta_1(z_1)-\vartheta_3(z_3)} f_{Y_1, V_1-V_2, V_1-V_3}\left(y_1, \vartheta_1\left(z_1\right) - \vartheta_2\left(z_2\right), v_1 - v_3\right) d\left(v_1 - v_3\right) dy_1 \\ + \int y_1 \int_{-\infty}^{\vartheta_1(z_1)-\vartheta_2(z_2)} f_{Y_1, V_1-V_2, V_1-V_3}\left(y_1, v_1 - v_2, \vartheta_1\left(z_1\right) - \vartheta_3\left(z_3\right)\right) d\left(v_1 - v_2\right) dy_1 \end{array} \right]. \tag{3.2}$$

- By similar reasoning, we can recover the following combination of parameters from the data on $Y_2 D_2$:

$$\frac{\partial E\left(Y_2 D_2 \mid Z=z\right)}{\partial Z_1}$$
$$= \frac{\partial}{\partial Z_1} \int y_2 \int_{-\infty}^{\vartheta_2(Z_2)-\vartheta_1(Z_1)} \int_{-\infty}^{\vartheta_2(Z_2)-\vartheta_3(Z_3)} f_{Y_2, V_2-V_1, V_2-V_3}(y_2, v_2-v_1, v_2-v_3)\, d\left(v_2-v_3\right)\, d\left(v_2-v_1\right) dy_2 \Big|_{Z=z}$$
$$= \frac{-\partial \vartheta_1\left(Z_1\right)}{\partial Z_1}\Big|_{Z_1=z_1} \left[\int y_2 \int_{-\infty}^{\vartheta_2(z_2)-\vartheta_3(z_3)} f_{Y_2, V_2-V_1, V_2-V_3}(y_2, \vartheta_2(z_2)-\vartheta_3(z_1), v_2-v_3) d\left(v_2-v_3\right) dy_2\right]. \tag{3.3}$$

- From data on $Y_3 D_3$, we obtain the following combination of parameters:

$$\frac{\partial E\left(Y_3 D_3 \mid Z_1 = z\right)}{\partial Z_1} = \left.\frac{-\partial \vartheta_1\left(Z_1\right)}{\partial Z_1}\right|_{Z_1 = z_1} \int y_3 \int_{-\infty}^{\vartheta_3(z_3)-\vartheta_2(z_2)} f_{Y_3, V_3 - V_1, V_3 - V_2}\left(y_3, \vartheta_3\left(z_3\right) - \vartheta_1\left(z_1\right), v_3 - v_2\right) d\left(v_3 - v_2\right) dy_3 \,. \tag{3.4}$$

- Agents induced into 1 come from 2 and 3.
- There are two margins:

$$(R_1 = R_2) \text{ and } (R_1 \geq R_3) \quad \text{(margin of indifference between 1 and 2),}$$

and

$$(R_1 = R_3) \text{ and } (R_1 \geq R_2) \quad \text{(margin of indifference between 1 and 3).}$$

Unaided, IV does not enable analysts to identify the returns at each of the different margins.

- Instead, it identifies a weighted average of returns.
- It does not identify the density of persons at the various margins, i.e., the proportion of people induced into (or out of) 1 from each possible alternative state by a change in the instrument.

- Collecting terms and rewriting in more easily interpretable components, which generalize the MTE developed for a two choice model to a multiple choice unordered model:

$$\left.\frac{\left(\frac{\partial E(Y|Z)}{\partial Z_1}\right)}{\left(\frac{\partial \vartheta_1}{\partial Z_1}\right)}\right|_{Z=z} =$$

$$\left[\begin{array}{c} \overbrace{[E(Y_1 - Y_2 \mid R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3))]}^{\substack{\text{Generalization of MTE for persons indifferent} \\ \text{between 1 and 2, where choice 3 is dominated}}} \Pr(R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)) \\ + \underbrace{[E(Y_1 - Y_3 \mid R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2))]}_{\substack{\text{Generalization of MTE for persons indifferent} \\ \text{between 1 and 3, where choice 2 is dominated}}} \Pr(R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)) \end{array}\right].$$

- This is a weighted return to alternative 1 for persons coming from two separate margins: alternative 1 *versus* alternative 2, and alternative 1 *versus* alternative 3, i.e., the return to people induced into 1 from their next best choice.
- The weights are the proportion of people induced into 1 from each margin.
- This *combination* of parameters can be identified from IV.
- The components of the sum cannot be identified by IV without further assumptions.
- Note that it is possible that a group at one margin gains while a group at another margin loses.
- IV only estimates a net effect, which might be zero.

- Notice that from representation (2.1) and the assumption that the Z_j $(j \epsilon \mathcal{J})$ are distinct, *pairwise monotonicity*, an extension of the monotonicity assumption invoked by Imbens and Angrist (1994) for the binary choice case, is satisfied.
- In the context of a model with multiple choices, pairwise monotonicity means the same pattern of flow between any two states is experienced by everyone.
- Thus, as Z_j increases, there is a flow from i to j but not from j to i (or vice versa). From (1.1), changing Z_1 induces all persons to move in the same direction (*i.e.*, from 1 to 2 or 2 to 1 but not both, and from 1 to 3 or 3 to 1 but not both). Pairwise monotonicity does not rule out the possibility that a change in an instrument causes people to move in the direction from j to i but to move away from the direction from k to i for $j \neq k$, and $j, k \neq i$.

- By the chain rule, the derivative of $\Pr(D_1 = 1 \mid Z)$ is:

$$\frac{\partial \Pr(D_1 = 1 \mid Z = z)}{\partial Z_1} = \left.\frac{\partial \vartheta_1}{\partial Z_1}\right|_{Z_1 = z_1} \left[\begin{array}{c} \Pr(R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)) \\ + \Pr(R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)) \end{array} \right].$$

We can define LIV in terms of the preceding ingredients as

$$\mathrm{LIV}(z) = \left.\frac{\left(\frac{\partial E(Y|Z)}{\partial Z_1}\right)}{\left(\frac{\partial \Pr(D_1=1|Z)}{\partial Z_1}\right)}\right|_{Z=z} = \left[\begin{array}{c} E(Y_1 - Y_2 \mid R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3))\,\omega_{12} \\ + E(Y_1 - Y_3 \mid R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2))\,\omega_{13} \end{array} \right]. \quad (2.5)$$

The *combination* of terms can be identified by LIV from the data on (Y, D, Z).

- The IV weights are:

$$\omega_{12} = \frac{\Pr\left(R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)\right)}{\left[\begin{array}{c}\Pr\left(R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)\right) \\ +\Pr\left(R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)\right)\end{array}\right]} \quad (2.6)$$

(2.7)

and

$$\omega_{13} = \frac{\Pr\left(R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)\right)}{\left[\begin{array}{c}\Pr\left(R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)\right) \\ +\Pr\left(R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)\right)\end{array}\right]} \quad (2.8)$$

- The weights can be identified from a structural discrete choice analysis.
- They cannot be identified by an unaided instrumental variable analysis.
- Thus it is not possible to identify the component parts of (3.3) by LIV alone, i.e., one cannot separately identify the generalized MTEs:

$$E\left(Y_1 - Y_2 \mid R_1(z_1) = R_2(z_2), R_1(z_1) \geq R_3(z_3)\right)$$

and

$$E\left(Y_1 - Y_3 \mid R_1(z_1) = R_3(z_3), R_1(z_1) \geq R_2(z_2)\right),$$

unless one invokes "identification at infinity" arguments.

- Using a structural model, one can estimate the components of (2.5) and determine the flow into (or out of) state 1 from all sources.
- We illustrate this point in Section 5. First we consider what standard IV estimates.

What does standard IV estimate?

- To see what standard IV estimates, consider the following linear-in-schooling model of earnings that receives much attention in the literature in labor economics.
- Let Y denote log earnings and write S as years of schooling.

- The model writes

$$Y = \alpha + \beta S + U \tag{3.1}$$

where

$$S = \sum_{j=1}^{3} jD_j, \tag{3.2}$$

and Y is defined as in Section 2. It is interpreted in this section as an approximation to the general model presented in Section 2.

- S is assumed to be correlated with U, and β is a random variable that may be statistically dependent on S.

- The discrete choice model of Section 2 does *not*, in general, imply (3.1).
- Indeed, there is much empirical evidence against model (3.1). An analysis of what IV estimates when linearity in S is imposed as an approximation, even though it may be inappropriate, is an interesting exercise because linearity is so often invoked.

- Suppose Z_1 is a valid instrument.
- We now interpret what

$$\Delta^{\text{IV}}_{Z_1} = \frac{\text{Cov}(Z_1, Y)}{\text{Cov}(Z_1, S)} \tag{3.3}$$

estimates.

- We do this by decomposing $\Delta^{\text{IV}}_{Z_1}$ into components analogous to the decomposition produced by Heckman et al. (2006, 2008) and Heckman and Vytlacil (2007b).

- The Appendix presents the derivation of the following decomposition of IV into our pairwise generalization of MTE for the unordered case:

$$\Delta_{Z_1}^{\text{IV}} = \frac{\operatorname{Cov}(Z_1, Y)}{\operatorname{Cov}(Z_1, S)} = \tag{4.4}$$

$$\frac{\left(\begin{array}{l} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overbrace{E\left(Y_1 - Y_2 \mid V_2 - V_1 = v_2 - v_1, \vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right) \geq V_2 - V_3\right)}^{\text{Generalized MTE } (2 \to 1) \text{ not identified from LIV}} \\ \quad \times \underbrace{\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3), V_2 - V_1}\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right), v_2 - v_1\right)}_{\substack{\text{weight identified from discrete} \\ \text{choice analysis}}} d\left(v_2 - v_1\right) \, d\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right) \end{array}\right) + \left(\begin{array}{l} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overbrace{E\left(Y_1 - Y_3 \mid V_3 - V_1 = v_3 - v_1, \vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right) \geq V_3 - V_2\right)}^{\text{Generalized MTE}(3 \to 1) \text{ not identified from LIV}} \\ \quad \times \underbrace{\eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2), V_3 - V_1}\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right), v_3 - v_1\right)}_{\substack{\text{weight identified from discrete} \\ \text{choice analysis}}} d\left(v_3 - v_1\right) \, d\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right)\right) \end{array}\right)}{\begin{array}{l} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left[-\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3), V_2 - V_1}\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right), v_2 - v_1\right)\right]}_{\substack{\text{weight identified from discrete} \\ \text{choice analysis}}} d\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right) \, d\left(v_2 - v_1\right) \\ +2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left[-\eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2), V_3 - V_1}\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right), v_3 - v_1\right)\right]}_{\substack{\text{weight identified from discrete} \\ \text{choice analysis}}} d\left(v_3 - v_1\right) \, d\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right)\right). \end{array}}$$

- IV identifies a weighted average of gains to state 1 compared to the next best alternative which may be 2 or 3.
- The two terms of the decomposition are defined as generalized MTEs and are weighted averages of the gain of moving from state 2 to state 1 for persons on the margin of indifference between 1 and 2 and for whom 2 is a better choice than 3 (the first term) and the gain of moving from 3 to 1 for persons on the margin of indifference between 1 and 3 and for whom 3 is a better choice than 2 (the second term).

- In the Appendix, we derive the weights on the generalized MTEs and show that they do not sum to 1 even when normalized by the denominator.
- The mathematical reason for this result is simple.
- The weights in the numerator do not sum to the weights in the denominator.
- The second term in the denominator receives twice as much weight as the corresponding term in the numerator.

- This is a consequence of the definition of S (3.2), which plays no role in the numerator term.
- Thus, IV applied to the general model produces an arbitrarily weighted sum of generalized MTEs with weights that do not sum to 1, and which, in general, places more weight on the first generalized MTE term than on the second term, compared to the weights placed on the corresponding terms in the denominator.
- Using IV alone, we cannot decompose (4.4) into its component parts, even though the weights can be identified from discrete choice analysis.

The Mincer Model

- The Mincer (1974) model is a specialization of the general model discussed in Section 2 of this paper that justifies the precise functional form of equation (3.1). For this case, the weights in (4.4) in the numerator and denominator are the same.
- The Mincer model is formulated in terms of log earnings for Y_1, Y_2, and Y_3:

$$\begin{aligned} Y_2 &= \ln(1+g) + Y_1, \\ Y_3 &= \ln(1+g) + Y_2 = 2\ln(1+g) + Y_1, \end{aligned}$$

  where g is a growth factor for income that varies in the population.
- Earnings at each schooling level depend on two parameters: (g, Y_1).

- In this case, letting $\alpha = \ln(1+g)$,

$$
\begin{aligned}
\Delta^{\text{IV}}_{Z_1} &= \frac{\text{Cov}(Z_1, Y)}{\text{Cov}(Z_1, S)} \qquad (4.6)\\
&= \left[\left(\begin{array}{l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left(\alpha \mid V_2 - V_1 = v_2 - v_1, \vartheta_2(z_2) - \vartheta_3(z_3) \geq V_2 - V_3\right) \\ \qquad \times \eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3), V_2-V_1}\left(\vartheta_2(z_2) - \vartheta_3(z_3), v_2 - v_1\right) d\left(v_2 - v_1\right) d\left(\vartheta_2(z_2) - \vartheta_3(z_3)\right) \end{array} \right) \right. \\
&\qquad \left. + \left(\begin{array}{l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left(\alpha \mid V_3 - V_1 = v_3 - v_1, \vartheta_3(z_3) - \vartheta_2(z_2) \geq V_3 - V_2\right) \\ \qquad \times 2\eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2), V_3-V_1}\left(\vartheta_3(z_3) - \vartheta_2(z_2), v_3 - v_1\right) d\left(v_3 - v_1\right) d\left(\vartheta_3(z_3) - \vartheta_2(z_2)\right) \end{array} \right) \right] \\
&\times \left[\begin{array}{l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3), V_2-V_1}\left(\vartheta_2(z_2) - \vartheta_3(z_3), v_2 - v_1\right) d\left(v_2 - v_1\right) d\left(\vartheta_2(z_2) - \vartheta_3(z_3)\right) \\ \qquad + 2\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2), V_3-V_1}\left(\vartheta_3(z_3) - \vartheta_2(z_2), v_3 - v_1\right) d\left(v_3 - v_1\right) d\left(\vartheta_3(z_3) - \vartheta_2(z_2)\right) \end{array} \right]^{-1}.
\end{aligned}
$$

- In this case, the weights now sum to 1.
- The weights for the numerator term now are the same as the weights for the denominator term.
- But again, unaided IV does not identify the component parts of the term bundled in IV — the mean gains at each margin.

An Example

- It is instructive to summarize our analysis with an example.
- Consider a 3 choice model with associated outcomes.
- This corresponds to the GED, high school dropout and high school graduate example that we have used throughout the paper.
- Under conditions presented in Heckman and Vytlacil (2007a, Appendix B), the structural model is nonparametrically identified.
- A key assumption in their proof is the "identification at infinity" assumption previously discussed.

- This assumes the ability to vary (Z_1, Z_2, Z_3) freely and the existence of limit sets such that fixing any two of (Z_1, Z_2, Z_3), one makes the R_j associated with Z_j arbitrarily small.
- Heckman and Vytlacil (2007b) show that if one augments the IV assumptions with the same identification at infinity assumptions used in structural models, one can use IV in the limit to identify the components of (3.5). In the limit sets, one can identify

$$E(Y_1 - Y_2 | R_1(z_1) = R_2(z_2)) \tag{4.1}$$

and

$$E(Y_1 - Y_3 | R_1(z_1) = R_3(z_3)) \tag{4.2}$$

by setting Z_3 and Z_2 respectively to limit set values.

- Essentially one can use the limit sets to make a three choice model into a two choice model, and the standard results for the two choice model apply.
- Under these assumptions, and additional mild regularity assumptions, using structural methods, one can identify the distributions of (Y_1, Y_2) and (Y_1, Y_3) so that one can identify *distributions* of treatment effects, $Y_2 - Y_1$ and $Y_3 - Y_1$, in addition to the mean parameters identified by IV.
- One can also identify the proportion of people induced into 1 from each alternative state using variation in the instrument.

- Consider the model with the parameters presented in Table 1.
- This is a discrete choice model with associated outcome variables.
- The $Z_j, j = 1, \ldots, 3$, are assumed to be scalar and mutually independent.
- They are normally distributed so they satisfy large support ("identification at infinity") conditions.
- Table 2 shows how a change in Z_1, which increases it by .75 standard deviations, shifts people across categories.
- This corresponds to making GED attainment easier.

Table 1. Potential Outcomes, Choice Model and Parameterizations

Outcomes

$$Y_j = \alpha_j + U_j \quad \text{with} \quad j \in \mathcal{J} = \{1, 2, 3\}$$

$$Y = \sum_{j \in \mathcal{J}} Y_j D_j$$

Choice Model

$$D_j = \begin{cases} 1 & \text{if } R_j \geq R_k \; \forall \; j \in \mathcal{J} \\ 0 & \text{otherwise} \end{cases}$$

$$R_j = \gamma_j Z_j - V_j \quad \text{with} \quad j \in \mathcal{J}$$

Parameterization

$$(U_1, U_2, U_3, V_1, V_2, V_3) \sim N\left(\mathbf{0}, \mathbf{\Sigma}_{UV}\right) \quad , \quad (Z_1, Z_2, Z_3) \sim N\left(\mu_Z, \mathbf{\Sigma}_Z\right)$$

$$\mathbf{\Sigma}_{UV} = \begin{bmatrix} 0.64 & 0.16 & 0.16 & 0.024 & -0.32 & 0.016 \\ 0.16 & 1 & 0.20 & 0.020 & -0.30 & 0.010 \\ 0.16 & 0.20 & 1 & 0.020 & -0.40 & 0.040 \\ 0.024 & 0.020 & 0.020 & 1 & 0.6 & 0100 \\ -0.32 & -0.30 & -0.40 & 0.6 & 1 & 0.2 \\ 0.016 & 0.01 & 0.040 & 0100 & 0.2 & 1 \end{bmatrix}, \; \mu_Z = (1.0, 0.5, 1.5) \text{ and } \mathbf{\Sigma}_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = [\; 0.3 \quad 0.1 \quad 0.7 \;], \; \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = [\; 0.2 \quad 0.3 \quad 0.1 \;]$$

- The estimates reported in Table 2 can be obtained from a structural discrete choice model.
- The percentage initially in 1 (GED) increases from 33.17% to 38.8%.
- The percentage in 2 (dropout) decreases from 29.11% to 25.91%.
- The percentage in 3 (graduating high school) declines from 37.72% to 35.29%.

Table 2. Transition Matrix Obtained from the Change in the Instrument Z_1

The Instrument Increases by 0.75 Standard Deviation

		New Value of Instrument ($\widetilde{Z} = Z_1 + 0.75$)			
		$D_1 = 1$	$D_2 = 1$	$D_3 = 1$	*Total*
Original Value of Instrument (Z_1)	$D_1 = 1$	33.17%	0%	0%	*33.17%*
	$D_2 = 1$	3.20%	25.91%	0%	*29.11%*
	$D_3 = 1$	2.43%	0%	35.29%	*37.72%*
	Total	*38.80%*	*25.91%*	*35.29%*	*100%*

- The IV estimate is -.032.
- (See the base of Table 3) This is the only number produced by an IV analysis using Z_1 as an instrument that changes within the specified range.
- The structural analysis in Table 3 shows that the net effect produced by the change in Z_1 is composed of 2 terms.
- It arises from a gain of .199 for the switchers $2 \to 1$ (dropout to GED) and a loss of .336 ($3 \to 1$) (graduate to GED).

Table 3. Marginal Gains Identified from the Change in the Instrument Z_1

The Instrument Increases by 0.75 Standard Deviation

	Gains to Switchers	Fraction of Population Switching
From 2 to 1	0.199	3.20%
From 3 to 1	-0.336	2.43%
Overall (IV estimate)	-0.032	5.63%

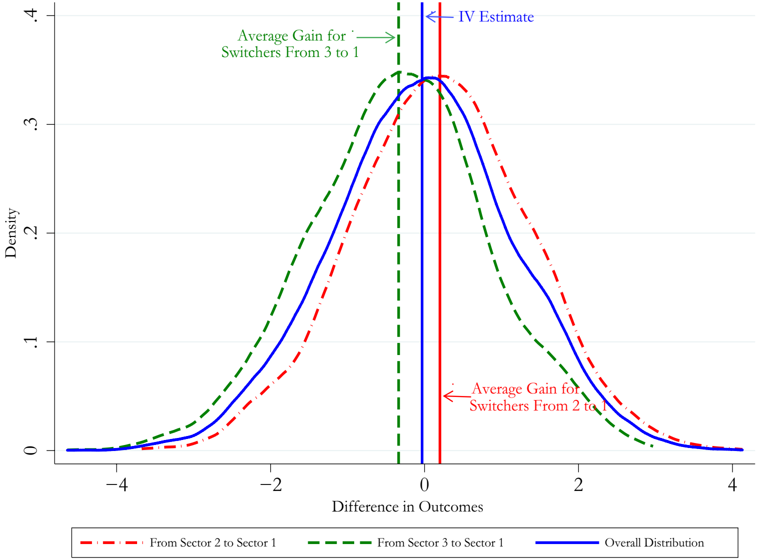
IV Estimate:

$$E\left[Y|\widetilde{Z}_1\right] - E\left[Y|Z_1\right] = \frac{3.20}{3.20+2.43} \times 0.199 - \frac{2.43}{3.20+2.43} \times 0.336 = -0.032$$

- Figure 1 shows what can be identified from the structural model.
- It plots the distributions of gains for persons going from 2 to 1 and from 3 to 1 as well as the overall distribution of gains to the switchers.
- Persons switching from 3 to 1 are harmed in gross terms by the policy that changes Z_1, while those who switch from 2 to 1 gain in gross terms.
- In utility terms, (R_j), people are better off.
- In terms of gross gains, about 56.8% of the people who switch from 2 to 1 are better off while 39.3% of the people who switch from 3 to 1 are better off.
- Overall, 49.2% are better off in gross terms even though the IV estimate is slightly negative.

Figure 1. Distribution of Gains in Outcomes Induced by the Change in the Instrument Z_1

The Instrument Increases by 0.75 Standard Deviation



- If one seeks to understand the distributional effects of the policy associated with a change Z_1, the structural analysis is clearly much more revealing.
- The IV estimate, which is a mean gross gain aggregating over origin states, does not capture the rich information about choices afforded by a structural analysis.
- However, it does identify the average gain to the program compared to the next-best alternatives.
- If that is the object of interest, linear IV is the right tool to use.

Appendix

- Following the analysis in Heckman et al. (2006, 2008) and Heckman and Vytlacil (2007b), we assume:

(A-1) The distribution of $(\{V_j\}_{j\in\mathcal{J}})$ is continuous.

(A-2) $\{(V_j, U_j)\}_{j\in\mathcal{J}}$ is independent of Z conditional on X.

(A-3) $E \mid Y_j| < \infty$ for all $j \in \mathcal{J}$.

(A-4) $\Pr(D_j = 1 \mid X) > 0$ for all $j \in \mathcal{J}$.

- In addition, we assume an exclusion restriction that requires some additional notation.
- Let $Z^{[-l]}$ denote all elements of Z except for the lth component.
- We assume

(A-5) For each $j \in \mathcal{J}$, their exists at least one element of Z, say $Z^{[l]}$, such that the distribution of $\vartheta_j(Z_j)$ conditional on $(X, Z^{[-l]})$ is continuous.

- With these assumptions, one can generalize the analysis of Heckman and Vytlacil (1999, 2001, 2005) to the unordered case.
- Assumptions (A-1) and (A-2) imply that $R_j \neq R_k$ (with probability 1) for $j \neq k$, so that $\text{argmax}_{j \in \mathcal{J}}\{R_j\}$ is unique (with probability 1). Assumption (A-2) assures the existence of an instrument.
- Assumption (A-3) is required for mean treatment parameters to be well defined.
- It also allows one to integrate to the limit and to produce well-defined means.

- Assumption (A-4) requires that at least some individuals participate in each choice for all X.
- Assumption (A-5) imposes the requirement that one be able to independently vary the index for the given value function.
- It imposes a type of exclusion restriction, that for any $j \in \mathcal{J}$, Z contains an element such that (i) it is contained in Z_j; (ii) it is not contained in any Z_k for $k \neq j$, and (iii) $\vartheta_j(\cdot)$ is a nontrivial function of that element conditional on all other regressors.

- In a series of papers, Heckman and Vytlacil (1999, 2001, 2005, 2007b), develop the method of local instrumental variables (LIV) to estimate the marginal treatment effect (MTE) for the case of binary choices.
- We now define and interpret the MTE and LIV in the case of general unordered choices.

Return to Text

Summary and Discussion

- The choice between using IV or a more structural approach for a particular problem should be made on the basis of Marschak's Maxim: use minimal assumptions to answer well-posed economic questions.
- Most IV studies do not clearly formulate the economic question being answered by the IV analysis.
- The probability limit of the IV estimator is defined to be the object of interest.

- In the binary outcome case, even if Z is a valid instrument, if Z is a vector, and analysts use only one component of the vector as an instrument, and do not condition on the other components of Z, the weights on the MTE can be negative over certain ranges.
- The practice of not conditioning on the other instruments is common in the literature.
- IV can estimate the wrong sign for the true causal effect.
- Recent analyses show how to improve on this practice and to design functions of standard instrumental variables that answer classes of well-posed economic questions.

- We have discussed a model with three or more choices where there is no particular order among the choices.
- Such examples arise routinely in applied economics.
- In this case, under conditions specified in this paper, IV estimates a weighted average of the mean gross gain to persons induced into a choice state by a change in the instrument (policy) compared to their next best alternative.
- It averages the returns to a destination state over all origin states.
- It does not produce the distribution of gains overall or by each origin state.

- Again, as in the binary choice case, for vector Z, using one component of Z as an instrument, and not conditioning on the other components can produce negative weights so that the sign of an IV can be opposite to that of the true causal effect which can be identified by a structural analysis.
- Structural methods provide a more complete description of the effect of the instrument or the policy associated with the instrument.
- They identify mean returns as well as distributions of returns for agents coming to a destination state from each margin.
- They also identify the proportion of people induced into a state from each origin state.

- Structural methods come at a cost.
- Unless distributional assumptions for unobservables are invoked, structural methods require some form of an "identification at infinity" assumption.
- However, in the general case in which responses to treatment are heterogeneous, IV requires the same assumption if one seeks to identify average treatment effects.
- An identification at infinity assumption can be checked in any sample so it does not require imposing *a priori* beliefs onto the data.
- Heckman, Stixrud, and Urzúa (2006) present an example of how to test an identification at infinity assumption.
- See also the discussion in Abbring and Heckman (2007).

- Many proponents of IV point to the strong distributional and functional form assumptions required to implement structural methods.
- They ignore recent progress in econometrics that identifies and empirically implements robust semiparametric and nonparametric approaches to structural analysis.
- Recent developments respond to arguments against the use of explicit econometric models made by a generation of applied economists that emerged in the 1980s.
- Those arguments are more properly directed against 1980s versions of structural models that were based on linearity and normality.
- Structural econometricians in the 21st century have listened to the critics and have perfected their tools to response to the criticism.

- The appeal to standard IV as a preferred estimator is sometimes made on the basis of "simplicity and robustness". Standard IV is certainly simple to compute although problems with weak instruments can make it empirically unstable.
- Since, in the general case, different instruments identify different parameters, IV is not robust to the choice of instrument.
- Since the sign of an IV can be different from the true causal effect, IV may even produce a misleading guide to policy or inference, so it is not robust.

- The meaning of "simplicity" is highly subjective.
- How simple is the economic interpretation of IV?
- Certainly decomposition (4.4) is not simple.
- The fact that simple IV can estimate wrong signs for true causal effects should give pause to those who claim that it is "robust". The weak instrument literature cautions us against uncritical claims about the sturdiness of IV estimators.
- The ability of different statistical estimators to answer questions of economic interest, or to show why they cannot be answered, should drive the choice of empirical techniques for analyzing data.

- Consider a worst case for structural estimation.
- Suppose that application of recently developed procedures for testing for structural identification reveal that a structural model is not identified or is only partially identified.
- Does this conclusion suggest that IV is a better choice for an estimator?
- That disguising identification problems by a statistical procedure is preferable to an honest discussion of the limits of the data?

- For underidentified structural models, it is possible to conduct sensitivity analyses guided by economic theory to explore the consequences of ignorance about features of the model.
- With IV, unaided by structural analysis, this type of exercise is not possible.
- Problems of identification and interpretation are swept under the rug and replaced by "an effect" identified by IV that is often very difficult to interpret as an answer to an interesting economic question.

A. Derivation of the Standard IV Estimator

- We first study the numerator of $\Delta^{\mathrm{IV}}_{Z_1}$ in the text.
- Recall that we keep the conditioning on X implicit.
- Using $\tilde{Z}_1 = Z_1 - \bar{Z}_1$,

$$\mathrm{Cov}\left(Y, Z_1\right) = E\left(\tilde{Z}_1\left(Y_1 D_1 + Y_2 D_2 + Y_3 D_3\right)\right).$$

  Using $D_1 = 1 - D_2 - D_3$, we obtain

$$\begin{aligned}\mathrm{Cov}\left(Y, Z_1\right) &= E\left(\tilde{Z}_1\left(Y_1 + \left(Y_2 - Y_1\right) D_2 + \left(Y_3 - Y_1\right) D_3\right)\right) \\ &= E\left(\tilde{Z}_1 Y_1\right) + E\left(\tilde{Z}_1\left(Y_2 - Y_1\right) D_2\right) + E\left(\tilde{Z}_1\left(Y_3 - Y_1\right) D_3\right),\end{aligned}$$

  where $E\left(\tilde{Z}_1 Y_1\right) = 0$.
- It is natural to decompose this expression using choice "1" as the base, because Z_1 only shifts $R_1(Z_1)$.

- The final two terms can be written as

$$
\begin{aligned}
&\mathrm{Cov}\,(Y, Z_1) \\
&= E\left(\tilde{Z}_1\,(Y_2 - Y_1)\,\mathbf{1}\,(R_2(Z_2) \geq R_1(Z_1), R_2(Z_2) \geq R_3(Z_3))\right) + E\left(\tilde{Z}_1\,(Y_3 - Y_1)\,\mathbf{1}\,(R_3(Z_3) \geq R_1(Z_1), R_3(Z_3) \geq R_2(Z_2))\right) \\
&= E\left[\tilde{Z}_1\,(Y_2 - Y_1)\,\mathbf{1}\left(\ (\vartheta_2\,(Z_2) - \vartheta_1\,(Z_1) \geq V_2 - V_1)\,, (\vartheta_2\,(Z_2) - \vartheta_3\,(Z_3) \geq V_2 - V_3)\ \right)\right] \\
&\quad + E\left[\tilde{Z}_1\,(Y_3 - Y_1)\,\mathbf{1}\left(\ (\vartheta_3\,(Z_3) - \vartheta_1\,(Z_1) \geq V_3 - V_1)\,, (\vartheta_3\,(Z_3) - \vartheta_2\,(Z_2) \geq V_3 - V_2)\ \right)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{z}_1\,(y_2 - y_1) \\
&\quad \times \left(\int_{-\infty}^{\vartheta_2(z_2)-\vartheta_1(z_1)}\int_{-\infty}^{\vartheta_2(z_2)-\vartheta_3(z_3)} f_{Y_2-Y_1, V_2-V_1, V_2-V_3}(y_2 - y_1, v_2 - v_1, v_2 - v_3)\, d\,(v_2 - v_3)\ d\,(v_2 - v_1)\ d\,(y_2 - y_1)\right) \\
&\quad \times f_{\tilde{Z}_1, \vartheta_2(Z_2)-\vartheta_1(Z_1), \vartheta_2(Z_2)-\vartheta_3(Z_3)}\,(\tilde{z}_1, \vartheta_2\,(z_2) - \vartheta_1\,(z_1)\,, \vartheta_2\,(z_2) - \vartheta_3\,(z_3))\ d\,(\vartheta_2\,(z_2) - \vartheta_3\,(z_3))\ d\,(\vartheta_2\,(z_2) - \vartheta_1\,(z_1))\ d\tilde{z}_1 \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{z}_1\,(y_3 - y_1) \\
&\quad \times \left(\int_{-\infty}^{\vartheta_3(z_3)-\vartheta_1(z_1)}\int_{-\infty}^{\vartheta_3(z_3)-\vartheta_2(z_2)} f_{Y_3-Y_1, V_3-V_1, V_3-V_2}\,(y_3 - y_1, v_3 - v_1, v_3 - v_2)\ d\,(v_3 - v_2)\ d\,(v_3 - v_1)\ d\,(y_3 - y_1)\right) \\
&\quad \times f_{\tilde{Z}_1, \vartheta_3(Z_3)-\vartheta_1(Z_1), \vartheta_3(Z_3)-\vartheta_2(Z_2)}\,(\tilde{z}_1, \vartheta_3\,(z_3) - \vartheta_1\,(z_1)\,, \vartheta_3\,(z_3) - \vartheta_2\,(z_2))\ d\,(\vartheta_3\,(z_3) - \vartheta_2\,(z_2))\ d\,(\vartheta_3\,(z_3) - \vartheta_1\,(z_1))\ d\tilde{z}_1.
\end{aligned}
$$

- By Fubini's Theorem, we can simplify the expressions and obtain for the first term:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left(Y_2 - Y_1 \mid V_2 - V_1 = v_2 - v_1, \vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right) \geq V_2 - V_3\right)$$
$$\times \left\{\int_{-\infty}^{\infty} \tilde{z}_1 \left[\left(\int_{-\infty}^{\vartheta_2(z_2)-\vartheta_3(z_3)} h_{V_2-V_1,V_2-V_3}\left(v_2 - v_1, v_2 - v_3\right) d\left(v_2 - v_3\right)\right)\right.\right.$$
$$\left.\left.\times \left(\int_{v_2-v_1}^{\infty} f_{\tilde{Z}_1,\vartheta_2(Z_2)-\vartheta_1(Z_1),\vartheta_2(Z_2)-\vartheta_3(Z_3)}\left(\tilde{z}_1, \vartheta_2\left(z_2\right) - \vartheta_1\left(z_1\right), \vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right) d\left(\vartheta_2\left(z_2\right) - \vartheta_1\left(z_1\right)\right)\right)\right] d\tilde{z}_1\right\}$$
$$\times d\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right) d\left(v_2 - v_1\right). \qquad \text{(A.1)}$$

- $h_{V_2-V_1,V_2-V_3}(.)$ is the joint density of $V_2 - V_1$, $V_2 - V_3$.
- Define the weighting term in braces in (A.1) as $\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3),V_2-V_1}\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right), v_2 - v_1\right)$.
- It is necessary to fix both $\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)$ and $v_2 - v_1$ in forming the weight.

- This weight can be estimated from a structural discrete choice analysis and the joint distribution of (Z, D_1, D_2, D_3).
- The terms multiplying the weight are marginal treatment effects generalized to the unordered case.
- (A.1) cannot be decomposed using IV.

- An alternative representation of the term in braces, $\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3),V_2-V_1}\left(\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right),v_2-v_1\right)$ is

$$\begin{aligned}
&\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3),V_2-V_1}\left(\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right),v_2-v_1\right)= \\
&\qquad E\left(Z_1-E\left(Z_1\right)\mid\vartheta_2\left(Z_2\right)-\vartheta_3\left(Z_3\right)=\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right),\vartheta_2\left(Z_2\right)-\vartheta_1\left(Z_1\right)\geq v_2-v_1\right) \\
&\qquad\qquad\times\Pr\left(\vartheta_2\left(Z_2\right)-\vartheta_3\left(Z_3\right)=\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right),\vartheta_2\left(Z_2\right)-\vartheta_1\left(Z_1\right)\geq v_2-v_1\right).
\end{aligned}$$

- An analysis parallel to the preceding one shows that the second term can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left(Y_3 - Y_1 \mid V_3 - V_1 = v_3 - v_1, \vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right) \geq V_3 - V_2\right)$$
$$\times\left\{\int_{-\infty}^{\infty} \tilde{z}_1 \left[\left(\int_{-\infty}^{\vartheta_3(z_3)-\vartheta_2(z_2)} h_{V_3-V_1,V_3-V_2}\left(v_3 - v_1, v_3 - v_2\right) d\left(v_3 - v_2\right)\right)\right.\right.$$
$$\left.\left.\times\left(\int_{v_3-v_1}^{\infty} f_{\tilde{Z}_1,\vartheta_3(Z_3)-\vartheta_1(Z_1),\vartheta_3(Z_3)-\vartheta_2(Z_2)}\left(\tilde{z}_1, \vartheta_3\left(z_3\right) - \vartheta_1\left(z_1\right), \vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right)\right) d\left(\vartheta_3\left(z_3\right) - \vartheta_1\left(z_1\right)\right)\right)\right] d\tilde{z}_1\right\}$$
$$\times d\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right)\right) d\left(v_3 - v_1\right). \tag{A.2}$$

- Define the term in braces in (A.2) as the weight $\eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2),V_3-V_1}\left(\vartheta_3\left(z_3\right) - \vartheta_2\left(z_2\right), v_3 - v_1\right)$.

- To obtain the denominator for the IV, recall that $S = \sum_{j=1}^{3} j\, D_j$.
- Substitute $D_1 = 1 - D_2 - D_3$,

$$\begin{aligned}\sum_{j=1}^{3} j\, D_j &= (1 - D_2 - D_3) + 2D_2 + 3D_3 \\ &= 1 + D_2 + 2D_3.\end{aligned}$$

- Then

$$\begin{aligned}\text{Cov}(S, \tilde{Z}_1) &= E\left(\tilde{Z}_1 D_2\right) + 2E\left(\tilde{Z}_1 D_3\right) \\ &= E\left(\tilde{Z}_1\left(\mathbf{1}\left(R_2 \geq R_1, R_2 \geq R_3\right)\right)\right) \\ &\quad +2E\left(\tilde{Z}_1\left(\mathbf{1}\left(R_3 \geq R_1, R_3 \geq R_2\right)\right)\right).\end{aligned} \quad \text{(B.3)}$$

- Using reasoning similar to that invoked for the analysis of the numerator terms, we obtain expressions for the terms corresponding to the two terms of (A.1) and (A.2).
- We obtain for the first term of (B.3)

$$\int_{-\infty}^{\infty} \tilde{z}_1 \Bigg[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\vartheta_2(z_2)-\vartheta_1(z_1)} f_{\tilde{Z}_1,\vartheta_2(Z_2)-\vartheta_1(Z_1),\vartheta_2(Z_2)-\vartheta_3(Z_3)} \left(\tilde{z}_1, \vartheta_2\left(z_2\right) - \vartheta_1\left(z_1\right), \vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right)$$
$$\times \left(\int_{-\infty}^{\vartheta_2(z_2)-\vartheta_3(z_3)} h_{V_2-V_1,V_2-V_3}\left(v_2 - v_1, v_2 - v_3\right) \, d\left(v_2 - v_3\right) \right) d\left(v_2 - v_1\right)$$
$$\times d\left(\vartheta_2\left(z_2\right) - \vartheta_3\left(z_3\right)\right) d\left(\vartheta_2\left(z_2\right) - \vartheta_1\left(z_1\right)\right) \Bigg] \, d\tilde{z}_1. \tag{A.4}$$

- By Fubini's Theorem, we obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{z}_1\Bigg[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\vartheta_2(z_2)-\vartheta_3(z_3)}h_{V_2-V_1,V_2-V_3}\left(v_2-v_1,v_2-v_3\right)d\left(v_2-v_3\right)\right)$$
$$\times\left(\int_{v_2-v_1}^{\infty}f_{\tilde{Z}_1,\vartheta_2(Z_2)-\vartheta_1(Z_1),\vartheta_2(Z_2)-\vartheta_3(Z_3)}\left(\tilde{z}_1,\vartheta_2\left(z_2\right)-\vartheta_1\left(z_1\right),\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right)\right)\right.$$
$$\left.\times d\left(\vartheta_2\left(z_2\right)-\vartheta_1\left(z_1\right)\right)\right)d\left(\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right)\right)\Bigg]d\left(v_2-v_1\right)d\tilde{z}_1 \quad \text{(A.5)}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta_{\vartheta_2(Z_2)-\vartheta_3(Z_3),V_2-V_1}\left(\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right),v_2-v_1\right)d\left(v_2-v_1\right)d\left(\vartheta_2\left(z_2\right)-\vartheta_3\left(z_3\right)\right).$$

- By parallel logic, we obtain for the second term in B.3:

$$
\begin{aligned}
& 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{z}_1\Bigg[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\vartheta_3(z_3)-\vartheta_2(z_2)} h_{V_3-V_1,V_3-V_2}\left(v_3-v_1,v_3-v_2\right)\,d\left(v_3-v_2\right)\right) \\
& \quad\times\left(\int_{v_3-v_1}^{\infty} f_{\tilde{Z}_1,\vartheta_3(Z_3)-\vartheta_1(Z_1),\vartheta_3(Z_3)-\vartheta_2(Z_2)}\left(\tilde{z}_1,\vartheta_3\left(z_3\right)-\vartheta_1\left(z_1\right),\vartheta_3\left(z_3\right)-\vartheta_2\left(z_2\right)\right)d\left(\vartheta_3\left(z_3\right)-\vartheta_1\left(z_1\right)\right)\right) \\
& \qquad\times d\left(\vartheta_3\left(z_3\right)-\vartheta_2\left(z_2\right)\right)\Bigg]\,d\left(v_3-v_1\right)\,d\tilde{z}_1 \\
= \; & 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta_{\vartheta_3(Z_3)-\vartheta_2(Z_2),V_3-V_1}\left(\vartheta_3\left(z_3\right)-\vartheta_2\left(z_2\right),v_3-v_1\right)\,d\left(v_3-v_1\right)\,d\left(\vartheta_3\left(z_3\right)-\vartheta_2\left(z_2\right)\right).
\end{aligned}
$$

- These terms can be identified from a structural analysis using the joint distribution of (Z, D_1, D_2, D_3).
- Collecting results, we obtain decomposition (4.4) in the text if we multiply both the numerator and denominator by -1.