Lecture 3: Choice under Uncertainty
Expected Utility

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10/21/19

Econ 21740, Fall 2019

Expected Utility Theory

The Space of Lotteries

The Space of Lotteries

- Note that

$$\mathbb{P}_i : C \to [0,1]$$

  is a function over a finite set of outcomes $C = \{x_1, x_2, ..., x_n\}$ ($n = 3$ outcomes in previous example).

- We can hence write down an outcome vector $(x_1, ..., x_n)$ and, given $\mathbb{P}_i$ a vector of probabilites corresponding to these outcomes:

$$\mathbf{L}_i := \big(\mathbb{P}_i(x_1), ..., \mathbb{P}_i(x_n)\big) = \big(p_1, ..., p_n\big)$$

- Saying (note that $\succeq$ is a **preference relation**)

$$\mathbb{P}_i \succeq \mathbb{P}_j$$

  therefore boils down to stating

$$\mathbf{L}_i \succeq \mathbf{L}_j.$$

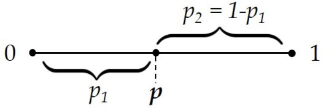
- $\mathbf{L}_i$ is a vector in $\mathbb{R}^n$ with the special property the $p_{is} \in [0,1], \forall s = 1, ..., n$ and $\sum_{s=1}^{n} p_{is} = 1$.
- The set of all such vectors is

$$\mathcal{L}^n = \left\{ \mathbf{L} = (p_1, ..., p_n) \in \mathbb{R}^n : \sum_{s=1}^{n} p_s = 1, p_s \in [0,1], \forall s = 1, ..., n \right\}, \quad (1)$$

  the **space of *n*-dimensional, discrete lotteries**.
- Mathematically, lottery spaces are called **simplexes**, which are *n*-dimensional generalizations of triangles.

Visualizing Lotteries

- $n = 1$. If only one outcome exists, $C = \{x_1\}$, so $\mathcal{L}^1 = \{(1)\}$ and x_1 realizes with certainty.
- $n = 2$. $C = \{x_1, x_2\}$, so $\mathcal{L}^1 = \{(p_1, 1 - p_1) : p_1 \in [0, 1]\}$. This is a line from 0 to 1, and any point **L** on it represents a lottery:

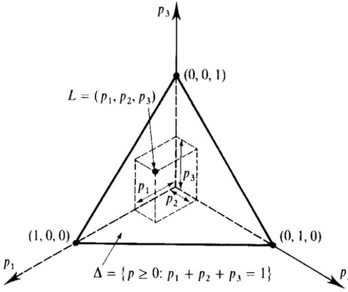


Visualizing Lotteries

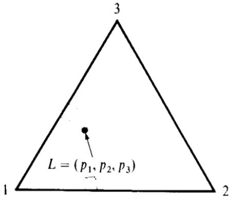
- $n = 3$. Every $\mathbf{L} \in \mathcal{L}^3$ can be written $(p_1, p_2 \geq 0, p_1 + p_2 \leq 1)$

$$\mathbf{L}^\top = p_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1 - p_1 - p_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

hence all these points lie in the triangle in $\mathbb{R}^3$ that is spanned by the points $(1,0,0), (0,1,0)$ and $(0,0,1)$. ((a): full view, (b): simplified view.)



(a)



(b)

Compound Lotteries

- Let there be two possible consequences $C = \{A, B\}$
- Possible lotteries over these consequences are $(\mathbb{P}(A), \mathbb{P}(B))$
- Someone has access to two lotteries

$$\mathbf{L}_1 = (0.4, 0.6) \text{ and } \mathbf{L}_2 = (0.6, 0.4)$$

and she suggests a game:

a. She **tosses a coin**. (The toss is independent of the lotteries.)
    - **Heads** $\Rightarrow$ You get to play $\mathbf{L}_1$
    - **Tails** $\Rightarrow$ You get to play $\mathbf{L}_2$
    - Summarize this offer as $(\alpha_1, \alpha_2; \mathbf{L}_1, \mathbf{L}_2)$, where α_i is the probability that you get to play lottery $i (i = 1, 2)$.
    - Coin toss implies $\alpha_1 = \alpha_2 = 0.5$

b. Alternatively, she will **not toss a coin** and you play a lottery

$$\mathbf{L}_3 = (0.5, 0.5)$$

- Playing the game, should you care whether she tosses the coin?

Compound Lotteries

- Say, you let her toss the coin
- The probability that consequence A realizes is then

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(Heads)\mathbb{P}_{\mathbf{L}_1}(A) + \mathbb{P}(Tails)\mathbb{P}_{\mathbf{L}_2}(A) \\ &= 0.5 \cdot 0.4 + 0.5 \cdot 0.6 \\ &= 0.5\end{aligned}$$

- Therefore, the probability that B realizes will also be 0.5.
- But that's just like playing lottery $\mathbf{L}_3$
- We say, the coin-toss lottery $(0.5, 0.5; \mathbf{L}_1, \mathbf{L}_2)$ which itself has lotteries as (intermediate) outcomes is a **compound lottery**.
- The lottery $\mathbf{L}_3 = 0.5(0.4, 0.6) + 0.5(0.6, 0.4)$ is the corresponding **reduced lottery**.

Compound Lotteries

- Let $\mathbf{L}_1, \mathbf{L}_2 \in \mathcal{L}^n$ be two lotteries.
- $\mathbf{L}_i = (p_{i1}, ..., p_{in}), i = 1, 2.$
- You buy a ticket that allows you participating in $\mathbf{L}_1$ (only) with probability α and in $\mathbf{L}_2$ (only) with probability $1 - \alpha$ ($\alpha \in [0, 1]$).
- For $C = \{x_1, ..., x_n\}$, consequence x_s will realize with probability $\alpha p_{1s} + (1 - \alpha) p_{2s}$ ($s \in \{1, ..., n\}$).
- The probability vector $(\alpha, 1 - \alpha)$ **compounds** lotteries $\mathbf{L}_1$ and $\mathbf{L}_2$.

Definition (Compound Lotteries)

Given K simple lotteries $\mathbf{L}_k \in \mathcal{L}^k$, $k = 1, ..., K$ and probabilities $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, the **compound lottery** $\mathbf{L}^{\mathrm{C}} = (\mathbf{L}_1, ..., \mathbf{L}_K; \alpha_1, ..., \alpha_K)$ is the risky alternative that yields the simple lottery $\mathbf{L}_k$ with probability α_k.

Compound Lotteries

Definition (Reduced Lotteries)

Given a compound lottery, $\mathbf{L}^{\mathrm{C}}$, the reduced lottery $\mathbf{L}$ is the lottery that yields outcome x_s with probability $\sum_k \alpha_k p_{sk}$. Hence, it generates the same outcome distribution as the compound lottery $\mathbf{L}^{\mathrm{C}}$.

Definition (Consequentialist Preferences)

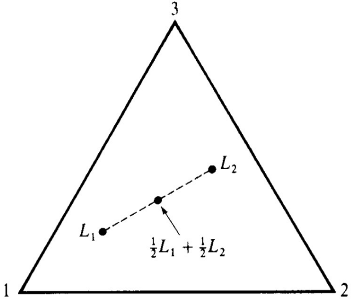
A decision maker is said to have **consequentialist preferences**, $\succeq$, if whenever $\mathbf{L}^{\mathrm{C}}$ is a compound lottery and $\mathbf{L}$ is the reduced lottery derived from it, then

$$\mathbf{L}^{\mathrm{C}} \sim \mathbf{L}.$$

- Consequentialists only care about the eventual outcome distribution.
- Note that a consequentialist needs **knowledge** that $\mathbf{L}^{\mathrm{C}}$ and $\mathbf{L}$ lead to the same outcome distributions
- Very complicated $\mathbf{L}^{\mathrm{C}}$ can obscure that fact.

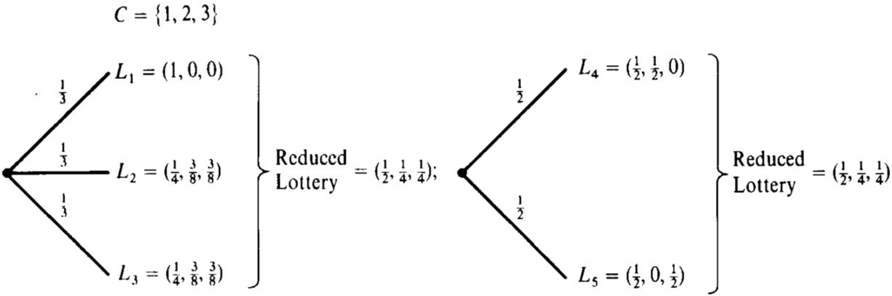
Visual Representation of Compounding

- Compounding lotteries $\mathbf{L}_1, \mathbf{L}_2$ with probability $\frac{1}{2}$ each, will yield a reduced lottery $\mathbf{L} \in \mathcal{L}^n$.
- Note that the fact that *reducing compound lotteries yields new lotteries* is equivalent to the fact that *the lottery space $\mathcal{L}^n$ is* **convex**.



Consequentialist Example

A consequentialist should be indifferent between following two compound lotteries and corresponding reduced lotteries.



We verify algebraically that this is true:

- First compound lottery:

$$\frac{1}{3}L_1 + \frac{1}{3}L_2 + \frac{1}{3}L_3 = \frac{1}{3}(1,0,0) + \frac{1}{3}\left(\frac{1}{4},\frac{3}{8},\frac{3}{8}\right) + \frac{1}{3}\left(\frac{1}{4},\frac{3}{8},\frac{3}{8}\right)$$
$$= \left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$$

- Second compound lottery:

$$\frac{1}{2}L_4 + \frac{1}{2}L_5 = \frac{1}{2}\left(\frac{1}{2},\frac{1}{2},0\right) + \frac{1}{2}\left(\frac{1}{2},0,\frac{1}{2}\right)$$
$$= \left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$$

Preferences and Utility on $\mathcal{L}^n$

Introductory Example

- You have preferences represented by *u over a set of outcomes* C, say $C = \{\text{diving}, \text{skiing}, \text{hiking}\}$.
- You are offered two lotteries $\mathbf{L}_1, \mathbf{L}_2$ with

$$\mathbf{L}_1 = (\mathbb{P}_1(\text{diving}), \mathbb{P}_1(\text{skiing}), \mathbb{P}_1(\text{hiking})) = (0.2, 0.3, 0.5)$$

and

$$\mathbf{L}_2 = (0.5, 0.4, 0.1)$$

- In this section we will show that, if your *preferences over lotteries* respect certain axioms, you will prefer $\mathbf{L}_1$ to $\mathbf{L}_2$ **if and only if**

$$\begin{aligned} &0.2u(\text{diving}) + 0.3u(\text{skiing}) + 0.5u(\text{hiking}) \\ \geq &0.5u(\text{diving}) + 0.4u(\text{skiing}) + 0.1u(\text{hiking}), \end{aligned}$$

so the expected utility of $\mathbf{L}_1$ exceeds that of $\mathbf{L}_1$.

Lotteries, Preferences and Utility

Natural Questions

Suppose, a decision maker has preferences $\succeq$ over lotteries, $\mathcal{L}^n$ defined on the set of n distinct outcomes.

Recall the definition of a utility function $U : \mathcal{L}^n \to \mathbb{R}$,

$$U(\mathbf{L}) \geq U(\mathbf{L}') \Leftrightarrow \mathbf{L} \succeq \mathbf{L}' \text{ for all } \mathbf{L}, \mathbf{L}' \in \mathcal{L}^n$$

- Can we represent $\succeq$ by some **utility function**, $U : \mathcal{L}^n \to \mathbb{R}$ over lotteries?

$\rightarrow$ **Yes**, given an appropriate definition of rationality.

- Will U have special properties that link it to the set-up under certainty?

$\rightarrow$ **Yes**. U will be the expected value of utility over all consequences.

- Under uncertainty, the axioms we impose on preferences to arrive at conclusions to these questions are stronger than in the certainty case.
- We have defined **consequentialist** preferences
- We also have defined **transitive** and **complete** preferences (first lecture). Recap:
  - **Transitivity**: $\mathbf{L} \succeq \mathbf{L}'$ and $\mathbf{L}' \succeq \mathbf{L}''$ imply $\mathbf{L} \succeq \mathbf{L}''$
  - **Completeness**: For any two $\mathbf{L}, \mathbf{L}' \in \mathcal{L}^n$, we have $\mathbf{L} \succeq \mathbf{L}'$ or $\mathbf{L}' \succeq \mathbf{L}$.
- The two new axioms we will impose are **continuity** and **independence**.

Independence

Definition (Independence on $\succeq$)

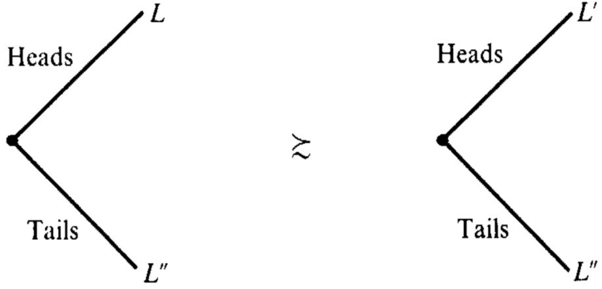
The preference relation $\succeq$ on the space of simple lotteries, $\mathcal{L}^n$, satisfies the **independence axiom** if for all $\mathbf{L}, \mathbf{L}', \mathbf{L}'' \in \mathcal{L}^n$ and $\alpha \in (0,1)$, we have that

$$\mathbf{L} \succeq \mathbf{L}' \text{ if and only if } \alpha \mathbf{L} + (1-\alpha)\mathbf{L}'' \succeq \alpha \mathbf{L}' + (1-\alpha)\mathbf{L}''.$$

- That is, if we mix two lotteries with a third one each, the preference over the resulting compound lotteries follows the preference of the two initial lotteries.
- Note that this axiom has no counterpart in the model of choice under certainty.
  - Might prefer (2 Soups, 0 Salami, 0 Bread) $\succ$ (0 Soups, 0 Salami, 2 Bread), but (0 Soups, 1 Salami, 1 Bread) $\succ$ (1 Soups, 1 Salami, 0 Bread) for dinner
  - Third option relevant under certainty, mixing changes preference ordering
  - But under uncertainty one **never mixes outcomes**, depending on the state of nature, the outcomes will realize **instead of** one another, **not together**.

Independence

Example: Consider **L**, **L**′ and **L**″. If one likes **L** better than **L**′, then the compound lottery which plays with 50% chance **L** and 50% **L**″ is also preferred to the compound lottery which plays with 50% chance **L**′ and 50% **L**″.



if and only if

$$L \succsim L'$$

Continuity

A **sequence of lotteries** $(\mathbf{L}_i)_{i=1}^{\infty}$ is understood as a sequence of vectors $(p_{i1}, ..., p_{in}), i = 1, 2, ...$ in the lottery simplex.

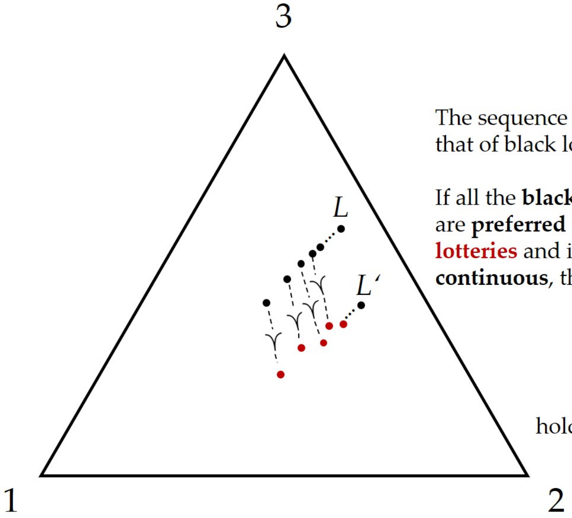
Definition (**Continuity**)

$\succeq$ is a **continuous preference** relation over $\mathcal{L}^n$ if for any two sequences of lotteries $(\mathbf{L}_i)_{i=1}^{\infty}$, $(\tilde{\mathbf{L}}_j)_{j=1}^{\infty}$.

$$\mathbf{L}_i \succeq \tilde{\mathbf{L}}_i \ \forall i \in \mathbb{N} \Rightarrow \lim_{i\to\infty} \mathbf{L}_i \succeq \lim_{i\to\infty} \tilde{\mathbf{L}}_i.$$

Continuity rules out sudden changes of preferences when we vary probabilities just a little.

Graphical Example Continuous Preferences



The sequence of red lotteries converges to L', that of black lotteries to L.

If all the **black lotteries** in the red sequence are **preferred to** the corresponding **red lotteries** and if the preference relation is **continuous**, then

$$L \succ L'$$

holds.

Numerical Example Continuous Preferences

Consider sequences of lotteries

$$(\mathbf{L}_n)_{n\in\mathbb{N}} = \left(0.3 - \frac{1}{10n}, 0.3 + \frac{1}{10n}, 0.4\right)$$

and

$$(\mathbf{L}'_n)_{n\in\mathbb{N}} = \left(0.6 - \frac{1}{10n}, 0.2 + \frac{1}{10n}, 0.2\right).$$

If

$$\left(0.3 - \frac{1}{10n}, 0.3 + \frac{1}{10n}, 0.4\right) \succeq \left(0.6 - \frac{1}{10n}, 0.2 + \frac{1}{10n}, 0.2\right)$$

holds for all finite n, then, for continuous preferences, it will also hold that, as $n \to \infty$,

$$(0.3, 0.3, 0.4) \succeq (0.6, 0.2, 0.2).$$

Saying that $\succeq$ is continuous is equivalent to saying that, if one strictly prefers

$$(p_1, ..., p_n) \succ (p_1', ..., p_n'),$$

then we can **change the probabilities in** $(p_1, ..., p_n)$ **by sufficiently small amounts** $(p_1 + \epsilon_1, ..., p_n + \epsilon_1)$ **with** $\sum \epsilon_i = 0$**, so that**

$$(p_1 + \epsilon_1, ..., p_n + \epsilon_1) \succ (p_1', ..., p_n')$$

still holds, and $(p_1 + \epsilon_1, ..., p_n + \epsilon_1) \neq (p_1, ..., p_n)$.

Example Continuous Preferences

- It is early morning and at the end of your day, there are three possible outcomes:

$$C = \{x_1 = \text{Had great Day Trip}, x_2 = \text{Stayed at Home}, x_3 = \text{Crashed Car}\}$$

- Two actions available: "Stay home" and "Go on Trip"
- Action "Go on Trip" means choosing some lottery $\mathbf{L}_1$ and "Stay home" means choosing some lottery $\mathbf{L}_2$ over outcomes.
- Say you value the outcome x_1 higher than x_2, so

$$(1, 0, 0) \succ (0, 1, 0)$$

  and if $\mathbf{L}_1 = (1, 0, 0), \mathbf{L}_2 = (0, 1, 0)$ you will make the trip.
- What if making the trip actually exposes you to a small risk of crashing our car, so $\mathbf{L}_1 = (1 - \varepsilon, 0, \varepsilon)$?
- If your preferences are **continuous**, then some very small probability $\varepsilon > 0$ of crashing your car will not change your choice:

$$\mathbf{L}_1 = (1 - \varepsilon, 0, \varepsilon) \succ (0, 1, 0) = \mathbf{L}_2$$

- Continuity is not as radical as example might suggest.
- Could argue against continuity that people are infinitely averse towards death risks.
- So, once a lottery assigns $\mathbb{P}(\text{Death}) > 0$, people would always avoid it.
- But if true, how do we explain:
    - People crossing roads
    - People doing manual labor
    - People engaging in sports
    - People signing up for the Army
    - ...

Savage Axioms

Rationality under Uncertainty

Assume, The **consequentialist approach** holds. Let $\succeq$ be a preference order over $\mathcal{L}^n$. We say that $\succeq$ satisfies the **Savage Axioms** if and only if

1. **Separability** holds: Actions, states and preferences over outcomes are independent of one another.[1]
2. $\succeq$ is **complete** and **transitive**,
3. $\succeq$ is **continuous** and
4. $\succeq$ satisfies **independence**.

(1)-(3) are basic and guarantee that some $U : \mathcal{L}^n \to \mathbb{R}$ exists and represents $\succeq$. (4) is new and assigns U a particular form (**expected utility form**).

[1] Note that we do not need to make states explicit primitives of this model; however, we still need to assume that lotteries are exogenously given.

The Role of Separability

- Recall the workhorse decision set-up $D : A \times B \to C$
- **Timing**: take an **action first**, then a **state realizes**.
- By **separability**, picking an action $a \in A$ **does not affect which state** $b \in B$ **will realize**
- States will realize with exogenous probabilities
- But contingent on what action the decision maker takes, **she can influence her (personal) outcome** $c \in C$ **in each state**.
- So, if "taking an action" is equivalent to "choosing a lottery" **L** over outcomes C.

Expected Utility (EU) Form

Definition (**Expected Utility (EU) Form**)

The utility function $U : \mathcal{L}^n \to \mathbb{R}$ is said to have **expected utility form** if there is an assignment of numbers $(u_1, ..., u_n)$ to the n outcomes such that for every simple lottery $\mathbf{L} \in \mathcal{L}^n$ we have

$$U(\mathbf{L}) = \sum_{i=1}^{n} p_i u_i.$$

More intuitively, define the function $u(x_i) := u_i, u : C \to \mathbb{R}$. Then, u is a utility function for the certain outcomes. If $X_\mathbf{L}$ is a random variable taking values in C with distribution $\mathbf{L}$, then

$$U(\mathbf{L}) = \mathbb{E}(u(X_\mathbf{L})).$$

This is the expectation of the utilities of all individual outcomes.

Properties of the EU Form

Linearity

Proposition (**EU Form** $\Leftrightarrow$ **Linearity**)

A utility function $U : \mathcal{L}^n \to \mathbb{R}$ has expected utility form if and only if it is linear, i.e. it holds

$$U\left(\sum_{s=1}^{S} \alpha_s \mathbf{L}_s\right) = \sum_{s=1}^{S} \alpha_s U(\mathbf{L}_s)$$

for any $s = 1, ..., S$ lotteries $\mathbf{L}_s \in \mathcal{L}^n$ and probabilities $\alpha_s \in [0, 1], \sum_s \alpha_s = 1$.

Uniqueness

Proposition (Unique Representation)

Suppose $U : \mathcal{L}^n \to \mathbb{R}$ is an expected utility function for the preference relation $\succeq$ over $\mathcal{L}^n$. Let $U' : \mathcal{L}^n \to \mathbb{R}$ be another expected utility function representing the same preferences. Then there exist constants $a \in \mathbb{R}$ and $b > 0$ such that for all $\mathbf{L} \in \mathcal{L}^n$,

$$U(\mathbf{L}) = a + b \cdot U'(\mathbf{L}).$$

Conversely, if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that for all $\mathbf{L} \in \mathcal{L}^n$, $U(\mathbf{L}) = a + b \cdot U'(\mathbf{L})$ holds, then U' has expected utility format and represents the same preferences.

Link to proof.

As a consequence of uniqueness, **differences in utility have meaning**. Example:

- Suppose there are four outcomes with certainty utility assignments u_1, u_2, u_3, u_4

"The difference in utility between outcomes 1 and 2 is greater than the difference in utility between outcomes 3 and 4."

$$\Leftrightarrow$$

$$u_1 - u_2 > u_3 - u_4$$

$$\Leftrightarrow$$

$$\frac{1}{2}u_1 + \frac{1}{2}u_4 > \frac{1}{2}u_3 + \frac{1}{2}u_2$$

$$\Leftrightarrow$$

$$U\Big(\underbrace{(\frac{1}{2}, 0, 0, \frac{1}{2})}_{\mathbf{L}}\Big) > U\Big(\underbrace{(0, \frac{1}{2}, \frac{1}{2}, 0)}_{\mathbf{L}'}\Big)$$

$$\Leftrightarrow$$

$$\mathbf{L} \succ \mathbf{L}'$$

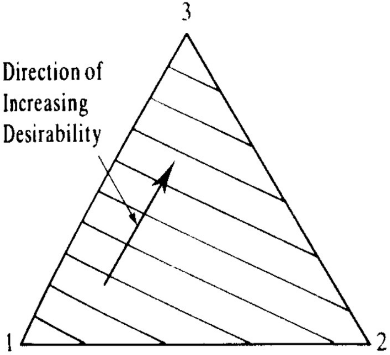
The Expected Utility Theorem

Theorem (The Expected Utility Theorem (EUT))

Suppose, that the preference relation $\succeq$ on the space of lotteries $\mathcal{L}^n$ satisfies **completeness, transitivity, continuity and independence***. Furthermore, let the consequentialist approach hold. Then,* **there exists a utility function** *$U : \mathcal{L}^n \to \mathbb{R}$ representing $\succeq$. Furthermore, U has* **expected utility form***.*

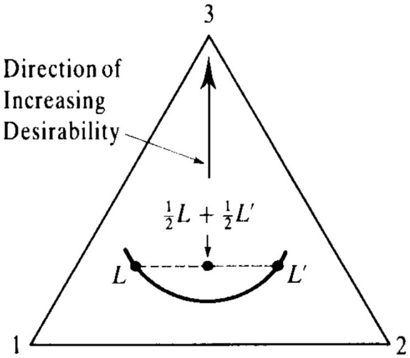
- Existence of **some** utility function is guaranteed by our assumptions without **independence**.
- The EUT crucially **hinges on the independence axiom**.
- It implies that **indifference curves** on the unit simplex are **linear** and **parallel**.
- With linear indifference curves over the space of lotteries (not: outcomes), the consquentialism assumption implies that convex combinations of equally preferred lotteries will again yield equally preferred lotteries.
- This is because a convex combination of lotteries (compound lottery) is worth just as much as its reduced counterpart
- It is easy to check that the **EU form also implies linear and parallel indifference curves**.
- Turns out that these are a defining feature of the EU form.

Implications Independence Axiom



Indifference curves are **straight**, **parallel** lines, if independence axiom holds.

Straight Lines

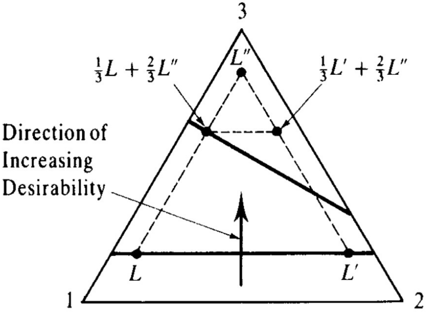


Suppose, $\mathbf{L} \sim \mathbf{L}'$. Invoking independence, we can take a combination with **any** $\mathbf{L}''$ and have

$$0.5\mathbf{L} + 0.5\mathbf{L}'' \sim 0.5\mathbf{L}' + 0.5\mathbf{L}''.$$

Letting $\mathbf{L}'' = \mathbf{L}$ yields $\mathbf{L} \sim 0.5\mathbf{L}' + 0.5\mathbf{L}$, contradicting the situation in the picture and thus nonlinear indifference curves.

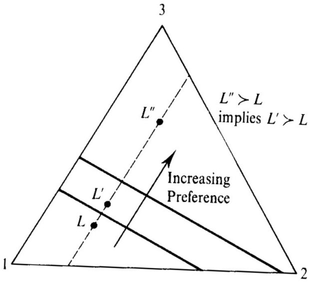
Parallel Lines



Here, independence is violated by assuming non-parallel indifference curves. The convex combinations of $\mathbf{L}$ and $\mathbf{L}''$ as well as $\mathbf{L}'$ and $\mathbf{L}''$ should be on the same indifference curve. Does such $\mathbf{L}''$ always exist? Yes, appendix for proof.
Link to Geometric Proof

Significance of the EUT

1. **Technical advantage**: Implies strong predictions from analytically simple calculations.
2. **Normative advantage**: Individuals who accept the axioms for their own decision making can use the EUT for introspection. **Example** (figure):
   - Three lotteries on a line, $\mathbf{L}'$ lies between (ie. is a convex combination of) $\mathbf{L}$ and $\mathbf{L}''$.
   - Decision maker is unsure whether $\mathbf{L}' \succ \mathbf{L}$ but knows that $\mathbf{L}'' \succ \mathbf{L}$
   - EUT $\Rightarrow$ $\mathbf{L}' \succ \mathbf{L}$ holds true.



Recap and Integration into Generic Decision Model

- Uncertainty decisions can, in the simplest example, be the act of literally buying a lottery ticket.
- $A \times B$ maps into consequences, C. We have modeled X as a random variable which (1) is determined by the states of nature in $b \in B$ and (2) realizes as a consequence in C.
- Hence, we can write $X(b) = c \in C$ and whether a outcome, c, realizes, **depends on the appropriate state b being true**.
- Thus, the distribution of consequences $\mathbb{P}_X$ depends on the distribution of states, $\mathbb{P}$.

- Choosing an **action**, a, can be thought of changing the dependency of X on b:
  - Write $X^a(b)$
  - For some a, the decision maker might be able to cancel out all risks, so that $X^a(b) = c$ for all $b \in B$.
  - For other a' the distribution of $X^{a'}(b)$ can be any lottery in $\mathcal{L}^n$
- Choosing $a \Rightarrow$ choosing $\mathbf{L}$ through a
  - If an action brings about a particular lottery, it is thus valid to write $\mathbf{L} = \mathbf{L}^a$
  - Two different actions might imply the same lottery (but not vice versa)
- The optimal choice of lottery is dictated by picking the maximizer $\mathbf{L}^*$ of $U : \mathcal{L}^n \to \mathbb{R}$
- U has expected utility from and relies on some utility function for consequences $u : C \to \mathbb{R}$
- Explicitly,

$$U(\mathbf{L}) = \mathbb{E}(u(X^a)) = \sum_{c \in C} u(c)\mathbb{P}(X^a = c) = \sum_{b \in B} u(X^a(b))\mathbb{P}(b)$$

Appendix

Proof: Unique Representation of Expected Utility

Link back to Proposition (main lecture).

Proof '$\Leftarrow$'

Suppose, some $a \in \mathbb{R}, \beta > 0$ exist such that $U(\mathbf{L}) = a + b \cdot U'(\mathbf{L})$. Since $ax + b$ is an increasing transformation, U' represents the same preferences. U' also must be of expected utility form, since

$$
\begin{aligned}
U'\left(\sum_k \alpha_k \mathbf{L}\right) &= b^{-1} U\left(\sum_k \alpha_k \mathbf{L}\right) - a/b && \text{by assumption} \\
&= b^{-1} \sum_k \alpha_k U(\mathbf{L}) - a/b && \text{use EU form of } U \\
&= \sum_k \alpha_k b^{-1} U(\mathbf{L}) - \sum_k (\alpha_k a/b) && \text{use } \sum_k \alpha_k = 1 \\
&= \sum_k \alpha_k (b^{-1} U(\mathbf{L}) - a/b) \\
&= \sum_k \alpha_k U'(\mathbf{L})
\end{aligned}
$$

Proof '⇒'

Suppose, U, U' both represent $\succeq$ and assume both have EU form. The lottery space is closed and bounded and U, U' are continuous functions. Then we can pick a most preferred, $\bar{\mathbf{L}}$, and a least preferred $\underline{\mathbf{L}}$ lottery. Choose

$$a = \frac{U(\bar{\mathbf{L}}) - U(\underline{\mathbf{L}})}{U'(\bar{\mathbf{L}}) - U'(\underline{\mathbf{L}})},\ b = U'(\bar{\mathbf{L}}) - U(\underline{\mathbf{L}})\frac{U(\bar{\mathbf{L}}) - U(\underline{\mathbf{L}})}{U'(\bar{\mathbf{L}}) - U'(\underline{\mathbf{L}})}.$$

Then, for given $\mathbf{L}$, choose $\lambda \in [0,1]$ such that $\lambda U(\bar{\mathbf{L}}) + (1-\lambda)U(\underline{\mathbf{L}}) = U(\mathbf{L})$. By EU form of U, we have

$$U(\lambda\bar{\mathbf{L}} + (1-\lambda) + \underline{\mathbf{L}}) = U(\mathbf{L})$$

and since U and U' represent the same preferences and U' also has EU form

$$U'(\mathbf{L}) = U'(\lambda\bar{\mathbf{L}} + (1-\lambda) + \underline{\mathbf{L}}) = \lambda U'(\bar{\mathbf{L}}) + (1-\lambda)U'(\underline{\mathbf{L}}).$$

Noting that $aU'(\bar{\mathbf{L}}) + b = U(\bar{\mathbf{L}})$ and $aU'(\underline{\mathbf{L}}) + b = U(\underline{\mathbf{L}})$, rearrange these and substitute into the right hand side of last equation to see

$$U'(\mathbf{L}) = a\left(\lambda U(\bar{\mathbf{L}}) + (1-\lambda)U(\underline{\mathbf{L}})\right) + b = aU(\mathbf{L}) + b.$$

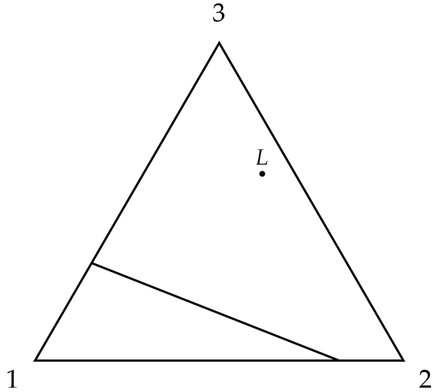
□

Link back to Proposition (main lecture).

Proof: Indifference Curves are Parallel

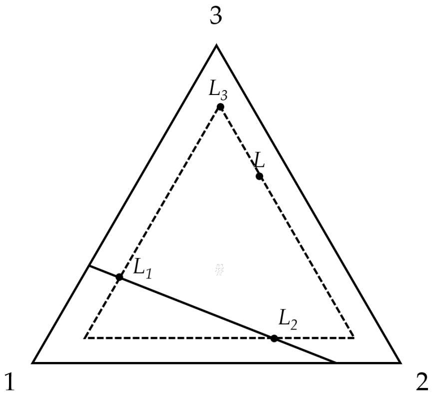
Link back to proposition (main lecture)

Constructive Proof: Parallel Lines



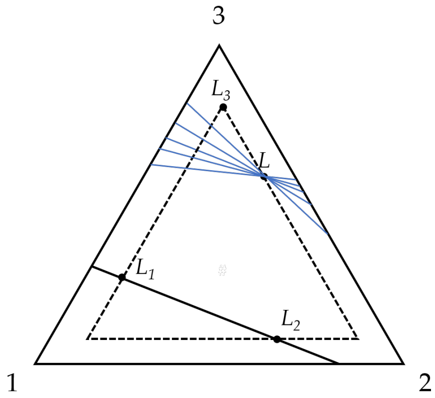
Let some indifference curve (IC) and some lottery **L** of different utility be given. Want to show that the IC through **L** is parallel to given IC. Assume, outcome 3 is most preferred, so we know the direction of increasing utility.

Constructive Proof: Parallel Lines



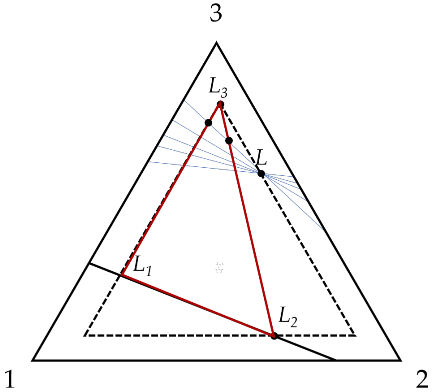
First, shrink the simplex so it passes through **L** and mark where the given IC is intersected ($\mathbf{L}_1, \mathbf{L}_2$) and mark the upper corner point, $\mathbf{L}_3$.

Constructive Proof: Parallel Lines



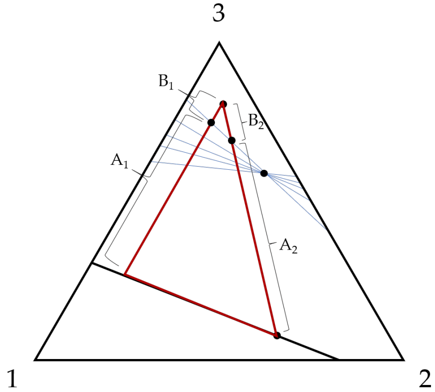
Consider all candidates that could possibly be indifference curves through **L**. Because we know that 3 is most preferred and because we are not indifferent between **L** and $\mathbf{L}_1$, the possible slopes are bounded.

Constructive Proof: Parallel Lines



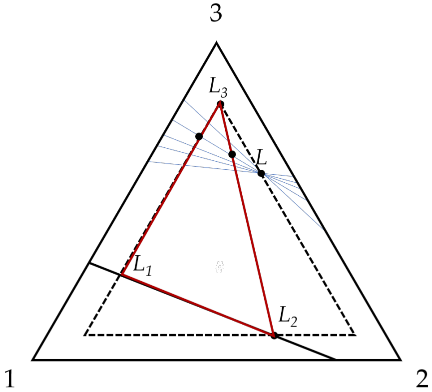
Independence axiom: Since $\mathbf{L}_1 \sim \mathbf{L}_2$, if we combine $\alpha\mathbf{L}_3 + (1-\alpha)\mathbf{L}_1$ and $\beta\mathbf{L}_3 + (1-\beta)\mathbf{L}_2$, we are indifferent between the two combinations if and **only if** $\alpha = \beta$. Go through IC candidates to see whether $\alpha = \beta$ is true.

Constructive Proof: Parallel Lines



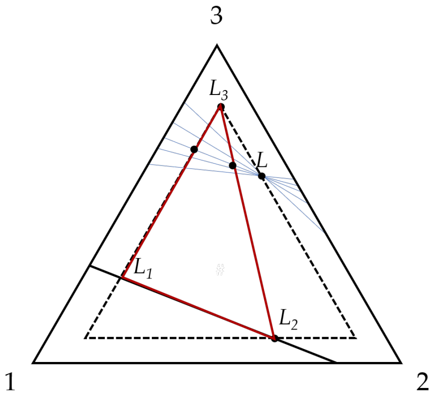
This holds precisely if the ratios A_1/B_1 and A_2/B_2 are equalized.

Constructive Proof: Parallel Lines



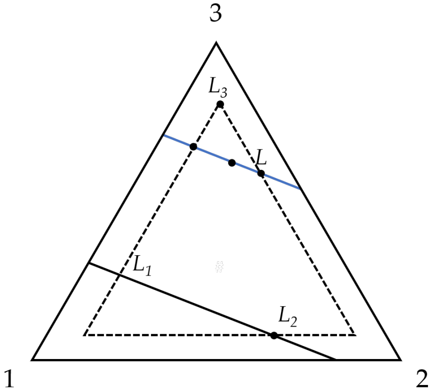
This holds precisely is the ratios A_1/B_1 and A_2/B_2 are equalized.

Constructive Proof: Parallel Lines



This holds precisely is the ratios A_1/B_1 and A_2/B_2 are equalized. **We have a winner**.

Constructive Proof: Parallel Lines



This holds precisely is the ratios A_1/B_1 and A_2/B_2 are equalized. **We have a winner**.

Link back to Proposition (main lecture)