# What is a causal effect? How to express it? And why it matters.

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Lecture 1 : Causality

# Topics to be Covered

- **A. Basic Concepts**
  - What are the key concepts in Causality?
  - What is a causal Model?
  - What is a Causal Operation versus a statistical Operation?
  - Fixing/Setting, Conditioning, Counterfactuals, Causal Effects
  - Some common misconceptions
  - Sequential Tasks of Causal Analysis

Theory $\Rightarrow$ Causal Model $\Rightarrow$ Identification $\Rightarrow$ Estimation $\Rightarrow$ Inference

# Topics to be Covered

- **B. Causal Frameworks**
  - How to express causality?
  - Discuss three distinct and widely used causal frameworks
    1. Potential Outcomes Framework (Neyman-Rubin-Holland causal model)
    2. Causal Model based on Structural/Autonomous Equations
    3. Frameworks for Causal Calculus (Do-calculus, Hypothetical Model Framework, Settable Systems)
  - Clarify properties and differences
  - Discuss nomenclature and applicability
  - Illustrate advantages and disadvantages through selected examples that are well-known in economics

## Related Literature on Causality

1. Pearl (2009)
   Causal Inference in Statistics: An Overview
2. Freedman (2010)
   Statistical Models and Causal Inference: A Dialogue with the Social Sciences
3. Heckman (2008)
   Econometric Causality
4. Heckman (2005)
   The Scientific Model of Causality
5. Heckman and Pinto (2015)
   Causal Analysis after Haavelmo

# Related Literature on:

## Language of Potential Outcomes (LPO)

1. Holland (1986)
   Statistics and Causal Inference

2. Angrist, Guido and Rubin (1996)
   Identification of Causal Effects Using Instrumental Variables

## Identification Theory

1. Arthur Lewbel (2019)
   The Identification Zoo - Meanings of Identification in Econometrics

2. Matzkin (2005, 2007, 2013)
   Identification of Consumers' Preferences When Their Choices Are Unobservable. Nonparametric Identification
   Nonparametric Identification in Structural Economic Models

3. Newey and McFadden (1994) - extremum-based identification
   Large Sample Estimation and Hypothesis Testing

# Related Literature on:

## Evaluation Approaches in Applied Economics

1. Blundell and Costa Dias (2008)
   Alternative Approaches to Evaluation in Empirical Microeconomics
2. Abadie and Cattaneo (2018)
   Econometric Methods for Program Evaluation
3. Athey and Imbens (2017)
   The State of Applied Econometrics: Causality and Policy Evaluation

## Causal Calculus

1. Pearl (1995)
   Causal Diagrams for Empirical Research
2. Jaber, Zhang, Bareinboin (2018) Causal Identification under Markov Equivalence
3. Heckman and Pinto (2020)
   Causal Calculus for the Hypothetical Model Framework

# 1. Introduction

## Frisch: "Causality is in the Mind "

"*. . . we think of a cause as something imperative which exists in the* **exterior world.** *In my opinion this is fundamentally* **wrong**. *If we strip the word cause of its animistic mystery, and leave only the part that science can accept, nothing is left except* a certain way of thinking, [T]he scientific . . . problem of **causality** is essentially a problem regarding our **way of thinking,** not a problem regarding the nature of the exterior world." (Frisch 1930, p. 36, published 2011)

## Haavelmo's (1943, 1944) Insights:

1. What are Causal Effects?
   - **Not** empirical descriptions of **actual worlds**,
   - **But** descriptions of **hypothetical worlds**.
2. How are they obtained?
   - **Through** Models – idealized thought experiments.
   - **By** varying–**hypothetically**–the inputs causing outcomes.
3. But what are models?
   - Frameworks defining **causal relations** among variables.
   - Based on **scientific knowledge**.

## Haavemo's Contributions to Causality are Many:

Haavelmo's two seminal papers (1943, 1944):

1. **Laid** the foundations for *counterfactual* policy analysis.
2. **Distinguished** *fixing* (causal operation) from *conditioning* (statistical operation).
3. **Formalized** Yule's credo: *Correlation is not causation.* (1895 paper on pauperism written when Yule was at UCL)
4. **Developed** Marshall's notion of *ceteris paribus* (Marshal, 1890).

### Most Important

Causal effects are determined by the impact of **hypothetical** manipulations of an input on an output.

## Regression: Conditional Expectation or Thought Experiment?

- Simple question: regression linear equation

$$Y = X\beta + U \tag{1}$$

- Source of confusion: relationships like (1) defined by statisticians as conditional expectations
- For $Y = X\beta + U$,

$$E(Y|X) = X\beta \text{ if } E(U|X) = 0.$$

- $E(Y|X) = X\beta + E(U|X)$ if $U \not\perp\!\!\!\perp X$.

## Thought Experiment

- Another way to define $Y = X\beta + U$.
- Hypothetically vary $X$ and $U$.
- $(X, U) \rightarrow Y$ via $Y = X\beta + U$
- This is *not* a statistical operation.
- This is not mysterious; it's what you learned in high school algebra; what is mysterious is why economists throw their basic training in mathematics to the wind when they enter the world of "causal analysis."
- A whole literature has emerged to justify $Y = X\beta + U$ as a causal model.
- **It involves operations outside statistics.**
- Algebra much older than statistics.
- Economists (and other scientists) use hypothetical models (thought experiments) to capture phenomena.
- These are not *defined* by statistical operations, although they may be *estimated* by statistical methods.

# Intuition of Ceteris Paribus (holding variables)

- Consider a simple linear model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + U$
- $Y$ is a function of observed $X_1, X_2$ and unobserved $U$.
- This is called an "all causes" model in the literature.
- Let $Y(x_1, x_2, u) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$ be the counterfactual outcome $Y$ when variables $(X_1, X_2, U)$ are set at $(x_1, x_2, u)$.
- What is the causal effect of an unit increase in input $X_1$ on outcome $Y$ *Ceteris Paribus* (holding $X_2, U$ fixed at $u$)?

$$
\begin{aligned}
& Y(x_1+1, x_2, u) - Y(x_1, x_2, u) \\
& \quad = \beta_0 + \beta_1(x_1+1) + \beta_2 x_2 - (\beta_0 + \beta_1 x_1 + \beta_2 x_2) \\
& \quad = \beta_1(x_1 + 1 - x_1) = \beta_1
\end{aligned}
$$

- A variety of potential outcomes can be obtained by varying $X_1, X_2$ and $U$ in different ways.
- All potential outcomes are outputs of such relationships.

## Ceteris Paribus and Conditional Expectation

- Now let $U$ have mean zero and be (mean) independent of $(X_1, X_2)$.
- If we evaluate $Y$ for the fixed values $(X_1, X_2, U) = (x_1, x_2, 0)$ we obtain:

$$Y(x_1, x_2, 0) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

- Which is mathematically equal to the conditional expectation:

$$E(Y|X_1 = x_1, X_2 = x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

- However these equations are conceptually very different.

## Ceteris Paribus versus Conditioning

- $Y(x_1, x_2, 0)$ is a thought experiment that hypothetically assigns values to the inputs of outcome $Y$.
- $E(Y|X_1 = x_1, X_2 = x_2)$ is the conditional expectation of a random variable that is believed to describe the data.
- $Y(x_1, x_2, 0)$ is useful to characterise causal parameters.
- $E(Y|X_1 = x_1, X_2 = x_2)$ is useful to estimate parameters from data using statistical methods.
- $Y(x_1, x_2, 0)$ is a causal ingredient. It generates an outcome value when input variables are fixed.
- $E(Y|X_1 = x_1, X_2 = x_2)$ is a statistical operation and can be used to estimate model parameters.

# Two sources of Confusion

1. The concept of Ceteris Paribus is based on the causal operation of *fixing* variables to values.
   - Fixing differs from statistical conditioning
   - *Fixing* is a causal operation outside probability/statistical theory
2. Identification is often conflated with estimation
   - Identification logically precedes estimation and is not dependent on any estimation procedure (RCT, IV, etc.)
   - However, identification and estimation are often describe as they were the same action (Granger Causality)
   - An example of this fact is the Diff-in-Diff estimator
   - In statistics, it is common to merge identification and estimation while seeking to prove that an estimator is consistent

**The econometric approach to causality was developed to address questions that arise in policy problems.**

## Three Distinct Policy Questions Reviewed

- P1 *Evaluating the Impact of Historical Interventions on Outcomes of the Treated Society at Large*
- P2 *Forecasting the Impacts (Constructing Counterfactual States) of Interventions Implemented in one Environment in Other Environments (External Validity)*
- P3 *Forecasting the Impacts of Interventions (Constructing Counterfactual States Associated with Interventions) Never Historically Experienced to Various Environments*

# Econometric Approach to Causality

- To study causality, it is necessary to disentangle causal models from particular estimation procedures
- Econometric approach to causality uses structural equation models do describe causal models
- Identification is a mathematical/probability analysis that study if counterfactuals have counterparts in observed data
- Estimation is an statistical exercise that employs observed data and considers properties of estimators (limits, means, variances, etc.)

# Steps for Building An Empirical Causal Model

- A causal *framework* is a selection of mathematical and statistical tools that are suitable to perform three distinct tasks of causal inference:

| Task | Description | Requirements |
|---|---|---|
| **1** | Defining Causal Models | A Scientific Theory<br>A Mathematical Framework |
| **2** | Identifying Causal Parameters from Known Population Distribution Functions of Data | Mathematical Analysis<br>Connect Hypothetical Variation with Data Generating Process<br>(Identification in the Population) |
| **3** | Estimating Parameters from Real Data | Statistical Analysis<br>Estimation and Testing Theory |

1. Task 1 uses scientific theory outside Probability/Statistics
2. Task 2 relates causal concepts to hypothetical samples using probability theory

# Section 2: Basic Tools/Causal Languages

# Defining Causal Models

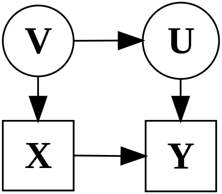
**Causal Model:** defined by a **4** components:

1. **Random Variables** that are observed and/or unobserved by the analyst: $\mathcal{T} = \{Y, U, X, V\}$. $Y$ outcomes, $U, X, V$ inputs.
2. **Error Terms**: mutually independent: $\epsilon_Y, \epsilon_U, \epsilon_X, \epsilon_V$.
3. **Structural Equations** that are autonomous : $f_Y, f_U, f_X, f_V$.
   - **Autonomy** means deterministic functions that are "invariant" to changes in their arguments (Frisch. 1938). (Outputs may change but functions do not.)
   - Also known as "structural" (Hurwicz, 1962).
   - Warning: various literature use different meanings of "structural." Recently, term has been applied to highly parameterized econometric models.
   - That is a misuse of traditional terminology.
4. **Causal Relationships** that map the inputs causing each variable: $Y = f_Y(X, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V)$

## Structural Relationships / Autonomous Functions

| | |
|---|---|
| $Y = f_Y(X, U, \epsilon_Y),$ | $Y$ observed |
| $X = f_X(V, \epsilon_X),$ | $X$ observed |
| $U = f_U(V, \epsilon_U),$ | $U$ unobserved |
| $V = f_V(\epsilon_V),$ | $V$ unobserved |

## Directed Acyclic Graph (DAG) representation

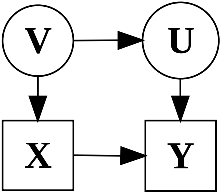


## Some Questions

- What statistical relationships are generated by this (or any) causal model?
- Is there an equivalence between statistical relationships and causal relationships?

## A Useful Tool: Local Markov Condition (LMC): (Kiiveri, 1984, Lauritzen, 1996)

**LMC:** A variable is independent of its non-descendants conditional on its parents




- For example: $Y \perp\!\!\!\perp \underbrace{V}_{\text{non-descendants}} |\underbrace{(X, U)}_{\text{parents}}$
- A fully non-parametric causal model can be equivalently described by its LMCs.

## Additional Tool: Graphoid Axioms (GA)
(Dawid, 1979)

**Primary** GA rules:

Weak Union: $X \perp\!\!\!\perp (W, Y)|Z \Rightarrow X \perp\!\!\!\perp Y|(W, Z)$.

Contraction: $X \perp\!\!\!\perp W|(Y, Z)$ and $X \perp\!\!\!\perp Y|Z \Rightarrow X \perp\!\!\!\perp (W, Y)|Z$.

Intersection: $X \perp\!\!\!\perp W|(Y, Z)$ and $X \perp\!\!\!\perp Y|(W, Z) \Rightarrow X \perp\!\!\!\perp (W, Y)|Z$

Remaining GA rules:

Symmetry: $X \perp\!\!\!\perp Y|Z \Rightarrow Y \perp\!\!\!\perp X|Z$.

Decomposition: $X \perp\!\!\!\perp (W, Y)|Z \Rightarrow X \perp\!\!\!\perp Y|Z$.

Redundancy: $X \perp\!\!\!\perp Y|X$.

## Analysis of Counterfactuals – The Fixing Operator

- **Fixing:** causal operation sets $X$-inputs of structural equations to $x$.

| Standard Model | Model under Fixing |
|---|---|
| $V = f_V(\epsilon_V)$ | $V = f_V(\epsilon_V)$ |
| $U = f_U(V, \epsilon_U)$ | $U = f_U(V, \epsilon_U)$ |
| $X = f_X(V, \epsilon_X)$ | $\boldsymbol{X = x}$ |
| $Y = f_Y(X, U, \epsilon_Y)$ | $Y = f_Y(\boldsymbol{x}, U, \epsilon_Y)$ |

- **Importance:** Establishes the framework for counterfactuals.
- **Counterfactual:** $Y(x)$ represents outcome $Y$ when $X$ is fixed at $x$.
- **Linear Case:** $Y = X\beta + U + \epsilon_Y$ and $Y(x) = x\beta + U + \epsilon_Y$;

## Fixing Properties

**Fixing:** *causal* exercise that *hypothetically* assigns values to inputs of the autonomous equation we analyze.

- Fixing determines counterfactual outcomes: $Y(x) = f_Y(x, U, \epsilon_Y)$
- Counterfactual outcomes are used to define causal effects
- The average Causal Effects of $X$ on $Y$ when $X$ is *fixed* at $x, x'$ is:

$$ATE = \mathrm{E}(Y(x) - Y(x'))$$

- Fixing $X$ does not affect the distribution of random variables not caused by $X$, namely $V, U$.

## Fixing Properties $\neq$ Conditioning

**Fixing:** *causal* exercise that *hypothetically* assigns values to inputs of the autonomous equation we analyze.

$$Y \text{ when } X \text{ is fixed at } x \Rightarrow Y(x) = f_Y(x, U, \epsilon_Y)$$

$$\text{Linear Case: } E(Y(x)) = x\beta + E(U); E(\epsilon_Y) = 0.$$

**Conditioning:** *Statistical* exercise that considers the dependence structure of the data generating process.

$$Y \text{ Conditioned on } X = x : E(Y|X = x) = E(f_Y(X, U, \epsilon_Y)|X = x)$$

$$\text{Linear Case: } E(Y|X = x) = x\beta + E(U|X = x)$$

$$E(\epsilon_Y|X = x) = 0$$

## Joint Distributions

**Model:** $Y = f_Y(x, U, \epsilon_Y); X = x; U = f_U(V, \epsilon_U); V = f_V(\epsilon_V)$.

1. **Standard Joint Distribution Factorization:**

$$P(Y, V, U|X = x) = P(Y|U, V, X = x)P(U|V, X = x)P(V|X = x).$$
$$= P(Y|U, V, X = x)P(U|V)\mathbf{P}(\mathbf{V}|\mathbf{X} = \mathbf{x})$$

because $U \perp\!\!\!\perp X|V$ by LMC.

2. **Factorization under Fixing $X$ at $x$:**

$$P(Y, V, U|X \text{ fixed at } x) = P(Y|U, V, X = x)P(U|V)\mathbf{P}(\mathbf{V}).$$

- **Conditioning** on $X$: affects the distribution of all variables in the system (including $V$)
- **Fixing**: $X$ does **not** affect the distribution of $V$ because $X$ does not cause $V$ (and $U$)

## Fixing Cannot Be Defined by Standard Probability Theory

- Fixing is a **causal operator**, not a statistical operator.
- Fixing does not affect the distribution of ancestors variables (including parents)
- Conditioning is a statistical operator that affects all variables.

## Problem: Causal Concepts are not Well-defined in Traditional Statistics

| Causal Inference | Statistical Models |
|---|---|
| Directional | Lacks directionality |
| Counterfactual | Correlational |
| Fixing | Conditioning |

1. **Fixing:** *Causal* operation that assigns values to the inputs of structural equations associated with the variable we fix.
2. **Conditioning:** *Statistical* exercise that encompasses the dependence structure of the entire data generating process.

## A Causal Model – Theoretical Benefits

A Causal model:

- Clearly defines causal relations among variables
- Allows one to clearly define the operation of fixing
- Allows analyst to clearly define counterfactuals and causal effects
- Allows for the definition and investigation of unobserved confounding variables.
- Allows for the precise assumptions regarding the interaction between unobserved confounding variables and observed variables.
- This is missing in many rival approaches

# Section 3: Causal Languages

## Causal Languages that Cope with Fixing

- The attempt to integrate fixing into practical statistic frameworks led to the development of several causal languages
- These languages append additional structure to standard probability theory to cope with the abstract operation of fixing

## Examples of Causal Frameworks

1. Neyman-Rubin model (Potential Outcomes)
    - Does not use structural equations (no mechanisms).
    - Choice of input ($X$) not modeled.
    - No explicit link of inputs and outputs.
2. Hypothetical model (Heckman & Pinto, 2015)
    - Framework fully integrated into standard probability theory.
3. Do-Calculus (Pearl, 2009)
    - Defines new rules outside of standard probability and statistics.

# 3.1 The Language of Potential Outcomes

# The Language of Potential Outcomes
## Basic Definitions

- The primitive object of analysis in the potential-outcome framework is the Unit-based response variable, denoted $Y_\omega(t)$,
- Read: "the (potential) value that outcome $Y$ would obtain in experimental unit (individual) $\omega$, had treatment $T$ been $t$."
- $Y_\omega(t)$ is the counterfactual outcome when $T$ is fixed at a value $t \in \text{supp}(T)$
- **No equations** are available for guidance on the causal relation among variables
- Model properties are stated as **independence relations**
- And only among potential outcomes of **observed variables**

# The Language of Potential Outcomes
## A Simple Model

- The Neyman-Rubin-Holland causal framework of potential outcomes.
- Variables in common probability space $(\Omega, \mathcal{F}, P)$
  1. $T$ Treatment choice
  2. $Y$ Outcome
  3. $X$ Baseline Characteristics
- Potential outcome $Y$ of agent $\omega$ for fixed $T = t$ is $Y_\omega(t)$.
- Causal effects of $t'$ versus $t$ for $\omega$ is $Y_\omega(t) - Y_\omega(t')$.
- The observed outcome is given by:

$$Y_w = \sum_{t \in \text{supp}(T)} Y_w(t) \cdot \mathbf{1}[T_w = t] \equiv Y_w(T_w)$$

# The Language of Potential Outcomes (LPO)
# Tools and Goals

- Example of Observed Variables:
  Treatment $T$, Outcome $Y$, Instrument $Z$, Controls $X$, Mediators $M$
- Mediators describe channels of influence
- Unobserved Variables:
  Potential outcomes $Y(t)$
- Goal: Identification of causal Parameters
  - Counterfactual Outcome Mean $\mathbf{E}(Y(t))$
  - Average Treatment Effect $ATE = \mathbf{E}(Y(t_1) - Y(t_0))$
  - Counterfactual Dist. $\mathbf{P}(Y(t) \leq y) = \mathbf{E}(\mathbf{1}[Y(t) \leq y])$
- How? Assume Independence Relations on Potential Outcomes
  Ex.: IV Model $Y(t, z) = Y(t, z')$, and $(Y(t), T(z)) \perp\!\!\!\perp Z|X$

## Example: Randomized Controlled Trials (RCT)

$$\textbf{RCT Assumption}: \; Y(t) \perp\!\!\!\perp T$$

$Y(t) \perp\!\!\!\perp T \Rightarrow$ counterfactual outcomes identified:

$$\mathbf{E}(Y|T=t) = \mathbf{E}\left(\sum_{t\in \text{supp}(T)} Y(t)\cdot \mathbf{1}[T=t]|T=t\right)$$
$$= \mathbf{E}\left(Y(t)|X, T=t\right) = \mathbf{E}(Y(t)) \text{ due to } Y(t) \perp\!\!\!\perp T.$$

Average causal effects obtained as:

$$E(Y(t_1) - Y(t_0)) = \big(E(Y|T=t_1) - E(Y|T=t_0).$$

## Example: The Exogeneity (Matching) Assumption

*Statistical assumption* that $Y(t) \perp\!\!\!\perp T|X$ is also called **matching**.

- Agents $\omega$ are comparable when conditioned on observed values $X$,
- Causal effects are weighted average of treated and control participants
- Conditional on their pre-intervention variables $X$.

1. Matching $\Rightarrow$ exogenous variation of $T$ under $X$ *by assumption*
2. Randomization $\Rightarrow$ exogenous variation of $T$ under $X$ *by design* where $X$ in RCT are the variables used in the randomization protocol

# Example: The Exogeneity (Matching) Assumption

The identification relies on assuming independence *when* controlling for pre-program variables $X$

$$\textbf{Matching Assumption: } Y(t) \perp\!\!\!\perp T|X,$$

$Y(t) \perp\!\!\!\perp T|X \Rightarrow$ counterfactual outcomes identified:

$$\mathbf{E}(Y|T=t,X) = \mathbf{E}\left(\sum_{t\in \text{supp}(T)} Y(t)\cdot \mathbf{1}[T=t]|X, T=t\right)$$
$$= \mathbf{E}\left(Y(t)|X, T=t\right) = \mathbf{E}(Y(t)|X) \text{ due to } Y(t) \perp\!\!\!\perp T|X.$$

Average causal effects obtained as:

$$\mathbf{E}(Y(t_1)-Y(t_0)) = \int \Big(E(Y|T=t_1, X=x)-E(Y|T=t_0, X=x)\Big)dF_X(x).$$

- However, often want effects conditional on $X$

## Contrast: Potential Outcomes × Causal Model Grounded in Structural Equations

1. **In LPO (Language of Potential Outcomes),** statistical independence relations among variables are assumed.
2. **In a causal model** (that relies on structural equations) independence relations come as a consequence of the causal relations of the model.

## Why bother to define a structural causal model?

- A desired property of the PO is its **simplicity**:
  - **Why?** It circumvents the necessity of defining structural equations
  - **How?** It invokes the conditional independence conditions generated an implicit causal model
- But this strategy has some **limitations**:
  - Causal relations among variables are implicit, which complicates model interpretation
  - Ingredients never specified.
  - Unobserved variables are absent in the PO language, which often prohibit the investigation of assumptions that are based on unobserved variables.
  - The lack of tools to model unobserved variables impairs the advance of identification theory and application of an entire body of econometric tools designed to cope with unobservables

# Traditional Approach That Links Causal Analysis to Structural Models: Decomposing Unobserved Confounders

"Transmission Model"

- Marschak and Andrews (1944) decompose the unobservable:

$$U = \phi \underset{\substack{\uparrow \\ \text{Source of} \\ \text{Confounding}}}{V} + W \tag{2}$$

- $V \not\!\perp\!\!\!\perp X$ so $U \not\!\perp\!\!\!\perp X$ and $W \perp\!\!\!\perp (V, X)$.
- $E(Y \mid X) = X\beta + \phi E(V \mid X)$.
- All estimators for causal models control for the effects of $V$ (implicitly or explicitly).
- Factor measurements $M = \mu(V, \varepsilon)$ might be used to control for $V$.
- There are many other ways.

# Benefits of a Structural Causal Model versus LPO

- A proper Causal Model substantially **enhances the toolkit of causal analysts**.
- Structural (Autonomous) equations clearly define causal relations among variables
- Independence relations among counterfactual variables are not necessarily assumed.
- Instead they arise as a consequence of the assumed causal relations among the model variables
- The Structural Causal Model enables a **better interpretation** of the model properties
- Links more tightly with economic theory
- Also enables to insert/manipulate unobserved variables which render **more sophisticated analyses**
- Allows us to use toolkit of traditional econometrics

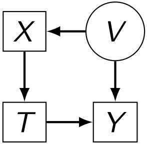
# Matching and RCT

1. Matching ⇒ exogenous variation of $T$ under $X$ *by assumption*
2. Randomization ⇒ exogenous variation of $T$ under $X$ *by design* (where $X$ are the variables used in the randomization protocol)

- LPO is simple and perfectly suitable to investigates these simple cases
- LPO limitations only become apparent for more complex models we encounter in encounter in doing economics everyday

## Revisiting the Matching Assumption: LPO x Structural Eq.

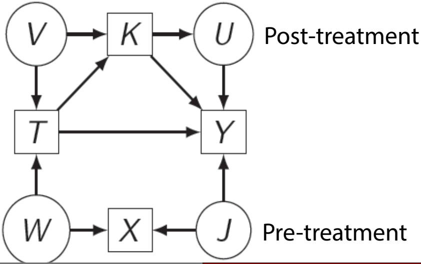
$$V = f_V(\epsilon_Z)$$
$$X = f_X(V, \epsilon_X)$$
$$T = f_T(X, \epsilon_T)$$
$$Y = f_Y(T, V, \epsilon_Y)$$




**The Matching Assumption:** $Y(t) \perp\!\!\!\perp T | X$

# Revisiting the Matching Assumption: Wrong Interpretation

- The $Y(t) \perp\!\!\!\perp T|X$ can be generated by causal models that differ from the original causal interpretation
- The common belief that matching is obtained by conditioning on a rich set of **pre-treatment** variables is misleading
- For example, consider the model below:
  1. $X$ are pre-program variables, but $Y(t) \not\!\perp\!\!\!\perp T|X$
  2. $K$ are post-treatment variables, but $Y(t) \perp\!\!\!\perp T|K$

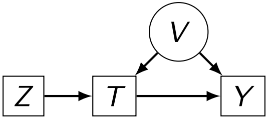


# Third Example: The IV Model

## The Instrumental Variable Model

The standard IV model is defined by the following causal model:

$$
\begin{aligned}
Z &= f_Z(\epsilon_Z) \\
V &= f_V(\epsilon_X) \\
T &= f_T(Z, V, \epsilon_T) \\
Y &= f_Y(T, V, \epsilon_Y)
\end{aligned}
$$




- $Z$ is exogenous, it is not caused by $V$, thus $Z \perp\!\!\!\perp V$
- IV relevance: $Z$ causes $T$
- Exclusion restriction: $Z$ does directly not cause $Y$
- Consequence is the exogeneity condition: $Z \perp\!\!\!\perp (Y(t), T(z))$

## Identification Requires Additional Assumptions

- The exogeneity condition $(Y(t), T(z)) \perp\!\!\!\perp Z$ is necessary
- But **not sufficient** to identify causal effects
- Must evoke additional assumptions to achieve identification
- A possibility is to invoke linearity $\Rightarrow$ standard Two-stage Least Squares on a constant coefficient model
- But it does not allow for unobserved heterogeneity in treatment effects

Instead...

# Examples of Additional Assumptions using LPO

## Most Famous Assumption in PO

- Binary/Ordered Monotonicity (Imbens and Angrist 1994, Imbens and Angrist 1995)

$$T_\omega(z) \leq T_\omega(z') \text{ for all } \omega \text{ such that } \operatorname{supp}(T) = \{1, ..., K\}$$

## Other PO Assumptions Exist (Examples)

- Partial Monotonicity (Mogstag, Torgovitsky, Walters, 2018)

$$T_\omega(z, \bar{z}) \leq T_\omega(z', \bar{z}) \text{ for all } \omega, \text{ and } T \in \{0, 1\}$$

- Unordered Monotonicity (Heckman and Pinto, 2018)

$$\mathbf{1}[T_\omega(z) = t] \leq \mathbf{1}[T_\omega(z') = t] \text{ for all } \omega \text{ and all } t \in \operatorname{supp}(T)$$

## Examples of Additional Assumptions using Structural Causal Model

- Separability/Monotonicity (Heckman and Vytlacil, 2005)

$$T = \mathbf{1}[P(T = 1|Z) \geq g(V)] \text{ such that } T \in \{0, 1\}$$

- Vytlacil's theorem shows that LATE makes a functional form assumption on choice equation and is based on an unobserved random variable; this came as a shock to the statisticians who thought they had no use for unobservables

- Unordered Monotonicity (Heckman and Pinto, 2018)

$$\mathbf{1}[T = t] = \mathbf{1}[P(T = t|Z) \geq g_t(V)] \text{ for all } t \in \{t_1, t_2, ...t_N\}$$

- Control Function Approach:
  1. $Y(t) \perp\!\!\!\perp T|V \Rightarrow V$ is a matching variable
  2. But $T = f_T(Z, V, \epsilon_t)$
  3. Invoke assumptions that enable analyst to estimate (or eliminate) $V$ as a function of $T, Z$

# Clarifying the Limitations of the LPO

Binary choice model $T \in \{0, 1\}$ under Monotonicity/Separability:

- IV model represented by LPO:
  1. $(Y(t), T(z)) \perp\!\!\!\perp T$
  2. $T_\omega(z) \geq T_\omega(z') \, \forall\omega$ or $T_\omega(z) \leq T_\omega(z')$ for any $z, z'$
- IV model represented by a Structural Causal Model:
  1. $T = \mathbf{1}[h(Z) \geq (V)]$, (separability = monotonicity, Vytlacil 2002)
  2. $Y = f_Y(T, V, \epsilon_Y)$
  3. $Z \perp\!\!\!\perp (V, \epsilon_Y)$
- $T = \mathbf{1}[h(Z) \geq g(V)]$ is equivalent to state: $T = \mathbf{1}[P(Z) \geq U]$; $U \sim unif[0, 1]$ and $P(Z) \equiv P(T = 1|Z)$
- Models are **causally equivalent** for certain questions, but differ in **power of analysis**

## Main Results of the IV Model using LPO

For $z, z' \in \text{supp}(Z)$, let the propensity score $P(z) > P(z')$ :

- The Two-Stage Least Squares:

$$2SLS = \frac{cov(Y,Z)}{cov(Z,T)} = \frac{E(Y|Z=z) - E(Y|Z=z')}{P(z) - P(z')}$$

- Identifies LATE, the causal effect for compliers:

$$LATE(z,z') = E(Y(1) - Y(0)|T(z) \neq T(z'))$$

# A Structural Causal Model enable us to Define MTE

- Causal Model enables an enhanced analysis.
- It explicitly defines/declares the unobserved variable $U$ implied by LATE
- Unobservable emerges from the axioms
- Which enable us to define the Marginal Treatment Effect (MTE):

$$\Delta^{MTE}(u) = E(Y(1) - Y(0)|U = u); u \in [0, 1]$$

- $\Delta^{MTE}(p)$ stands for the causal effect of $T$ on $Y$ for the share of the participants $\omega$ such that $U_\omega = p$.
- These agents whose unobserved variable $U$ takes value $u \in [0, 1]$
- $\Delta^{MTE}(p)$ can be identified by
  - Estimating a function of the $Y$ in terms of the propensity score $P(Z) \in [0, 1]$
  - Differentiating this function with respect to $P(Z)$ at value

## What are the benefits of the MTE ?

- MTE renders powerful tools of analyses.
- MTE is a primary concept that ties several causal parameters
- Example of causal parameters of interest:

$$
\begin{aligned}
ATE &= \mathbf{E}(Y(t_1) - Y(t_0)) \\
TT &= \mathbf{E}(Y(t_1) - Y(t_0)|T = t_1) \\
TUT &= \mathbf{E}(Y(t_1) - Y(t_0)|T = t_0) \\
PRTE &= \mathbf{E}(Y(t_1) - Y(t_0)|P(Z, X) = P^*) \\
IV &= \frac{Cov(Y, Z))}{Cov(T, Z)} \text{ (TSLS)} \\
OLS &= \mathbf{E}(Y|T = t_1) - \mathbf{E}(Y|T = t_0) \text{ not a causal parameter}
\end{aligned}
$$

All causal parameters can be expressed as a weighted average of the $\Delta^{MTE}(p)$ (Heckman and Vytlacil, 2005)!

# Causal Parameters as a function of the MTE

All causal parameters can be expressed as a weighed average of MTE:

$$ATE = \int_0^1 \Delta^{MTE}(p) W^{ATE}(p) dp; \qquad W^{ATE}(p) = 1$$

$$TT = \int_0^1 \Delta^{MTE}(p) W^{TT}(p) dp; \qquad W^{TT}(p) = \frac{1 - F_P(p)}{\int_0^1 \big(1 - F_P(t)\big) dt}$$

$$TUT = \int_0^1 \Delta^{MTE}(p) W^{TUT}(p) dp; \qquad W^{TUT}(p) = \frac{F_P(p)}{\int_0^1 \big(1 - F_P(t)\big) dt}$$

$$IV = \int_0^1 \Delta^{MTE}(p) W^{IV}(p) dp; \qquad W^{IV}(p) = \frac{\int_p^1 \big(t - E(P)\big) dF_P(t)}{\int_0^1 \big(t - E(P)\big)^2 dF_P(t)}$$

$$OLS = \int_0^1 \Delta^{MTE}(p) W^{OLS}(p) dp; \qquad W^{OLS}(p) = \frac{1 - F_P(p)}{\int_0^1 \big(1 - F_P(t)\big) dt}$$

$$LATE = \int_{P(z')}^{P(z)} \Delta^{MTE}(p) W^{LATE}(p) dp; \qquad W^{LATE}(p) = \frac{1}{P(z) - P(z')}$$

## Summary of IV Model: LPO versus Structural Equations

- PO does not allow for Variable U
- Nor the separability equation
- MTE cannot be defined in PO
- As a consequence, the researcher using PO would never develop the MTE analysis
- Nevertheless, the models at some level are equivalent

Link to Appendix A: Mediation Model

Link to Appendix B: Further Remarks on Causality

# What about a General Framework for Causal Calculus?

- The goal of a framework for causal calculus is to deliver a standard methodology that applies to any DAG.
- A set general of rules that can be used to assess counterfactual outcomes whenever those are identified.
- A methodology/algorithm that be coded, so the researcher does not need to investigate case by case.
- Such framework is useful to investigate which properties of DAGs are necessary/sufficient to render identification of causal parameters.
- Most important, a framework that facilitates to investigate the identification of causal effect is more complex DAGs.

## A the Risk of Being Too Repetitious, Fixing is Not Well-defined in Statistics

1. **Fixing:** *causal* operation that assigns values to the inputs of structural equations associated to the variable we fix upon.
2. **Conditioning:** *Statistical* exercise that considers the dependence structure of the data generating process.

- Fixing has direction while conditioning does not.
- **Question:** How can we make statistics converse with causality?
- **Answer:** The hypothetical model

# The Hypothetical Model Framework

## The Causal Calculus using The Hypothetical Framework
Merging Statical Theory and Causal Analysis

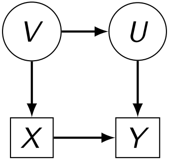
- The mismatch between statistical theory and causal inference motivated the study of the Hypothetical Model Framework
- The framework merges statical theory and causal analysis without the necessity of defining new tools of analysis

## Properties of the Hypothetical Model

- **Insight:** express causality through a *hypothetical model* assigning independent variation to inputs determining outcomes.
- **Data:** generated by an empirical model that shares some features with the hypothetical model.
- **Simplicity:** the method does not rely on additional tools of analysis beyond standard statistical theory
- **Identification:** relies on evaluating causal parameters defined in the *hypothetical model* using data generated by the *empirical model*.

## **Example of Data Generating Model** (DAG) Representation

**Model:** $Y = f_Y(X, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V)$.




- The Local Markov Condition (LMC) generates two independence conditions:
- $Y \perp\!\!\!\perp V | (U, X)$ and $U \perp\!\!\!\perp X | V$

## Defining The Hypothetical Model

The hypothetical model stems from the following properties:

1. **Same** set of structural equations as the empirical model.
2. **Appends** a hypothetical variable that we *fix*.
3. **Hypothetical variable** not caused by any other variable.
4. **Replaces** the input variables we seek to fix by the hypothetical variable.

Usage:

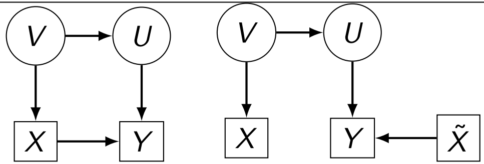
**Empirical Model:** Governs the data generating process.
**Hypothetical Model:** Abstract model used to examine causality.

## Example of the Hypothetical Model for fixing $X$

### The Associated Hypothetical Model

$$Y = f_Y(\tilde{X}, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V).$$

| Empirical Model | Hypothetical Model |
|---|---|




| LMC | LMC |
|---|---|
| $Y \perp\!\!\!\perp V \mid (U, X)$ | $Y \perp\!\!\!\perp (X, V) \mid (U, \tilde{X})$ |
| $U \perp\!\!\!\perp X \mid V$ | $U \perp\!\!\!\perp (X, \tilde{X}) \mid V$ |
| | $\tilde{X} \perp\!\!\!\perp (U, V, X)$ |
| | $X \perp\!\!\!\perp (U, Y, \tilde{X}) \mid V$ |

## Why the hypothetical variable is useful?

Properties the Hypothetical Model:

1. **Hypothetical Variable:** $\tilde{X}$ replaces the $X$-inputs of structural equations.
2. **Characteristic:** $\tilde{X}$ is an **external variable**, i.e., no parents.
3. **Thus:** Hypothetical variable has independent variation.
4. **Usage:** hypothetical variable $\tilde{X}$ enables analysts to examine fixing using standard tools of probability (conditioning).

## Main Benefit

- Fixing in the empirical model is translated to
- statistical conditioning in the hypothetical model

$$\underbrace{E_{\mathrm{E}}(Y(t))}_{\text{Causal Operation Empirical Model}} \quad = \quad \underbrace{E_{\mathrm{H}}(Y|\tilde{T}=t)}_{\text{Statistical Operation Hypothetical Model}}$$

- Causality is defined Within Statistics/Probability.
- No additional Tools Required.

## Identification

- **Hypothetical Model** allows analysts to define and examine causal parameters.
- **Empirical Model** generates observed/unobserved data;

### Clarity: What is Identification?

The capacity to express causal parameters of the hypothetical model through observed probabilities in the empirical model.

### Tools: What does Identification require?

Probability laws that connect *Hypothetical* and *Empirical* Models.

## Connecting Hypothetical and Empirical Models: Two Useful Conditions

**Only** two conditions **suffice** to investigate the identification of causal parameters!

For any disjoint set of variables $Y, W$ in $\mathcal{B}_e$, we have that:

**Rule 1:** $Y \perp\!\!\!\perp \widetilde{T} | (T, W) \Rightarrow$

$$\mathbf{P}_H(Y|\widetilde{T}, T = t', W) = \mathbf{P}_H(Y|T = t', W) = \mathbf{P}_E(Y|T = t', W)$$

**Rule 2:** $Y \perp\!\!\!\perp T | (\widetilde{T}, W) \Rightarrow$

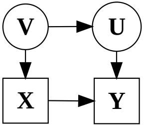
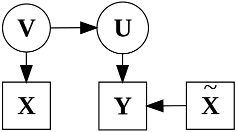
$$\mathbf{P}_H(Y|\widetilde{T} = t, X, W) = \mathbf{P}_H(Y|\widetilde{T} = t, W) = \mathbf{P}_E(Y|T = t, W)$$

If $Y \perp\!\!\!\perp \widetilde{T} | (T, W)$ or $Y \perp\!\!\!\perp T | (\widetilde{T}, W)$ occurs in the hypothetical model, then we are able to equate variable distributions of the hypothetical and empirical models!

## How to use this Causal Framework? Rules of Engagement

1. **Define** the empirical and associated hypothetical model.
2. **Hypothetical Model:** Generate statistical relationships (LMC, GA).
3. **Express** $\mathbf{P}_{\mathrm{H}}(Y|\widetilde{X})$ in terms of other variables.
4. **Connect** this expression to the empirical model using $Y \perp\!\!\!\perp \widetilde{T}|(T,W)$ or $Y \perp\!\!\!\perp T|(\widetilde{T},W)$

# Example of the Hypothetical Model for Fixing $X$

| Empirical Model | Hypothetical Model |
| --- | --- |
|   |   |
| Local Markov Condition | Local Markov Condition |
| $Y \perp\!\!\!\perp V \vert (U, X)$ | $Y \perp\!\!\!\perp (X, V) \vert (U, \tilde{X})$ |
| $U \perp\!\!\!\perp X \vert V$ | $X \perp\!\!\!\perp (U, Y, \tilde{X}) \vert V$ |

1. $E_H(Y|\widetilde{X} = x, V) = E_E(Y(x)|V)$ by the main property of the HM
2. $X \perp\!\!\!\perp (U, Y, \tilde{X})|V \Rightarrow X \perp\!\!\!\perp Y|(\tilde{X}, V)$ holds by LMC
3. $E_H(Y|\widetilde{X} = x, V) = E_E(Y|X = x, V)$ by rule 2

## Rule 2 is a Matching Property

If there exist $V$ such that, $X \perp\!\!\!\perp Y|V,\tilde{X}$, then $E_{\mathrm{H}}(Y|V,\tilde{X}=x)$ in hypothetical model is equal to $E_{\mathrm{E}}(Y(x)|X=x)$ in empirical model.

- Main Property of the Hypothetical Model implies that counterfactual outcome $E_{\mathrm{E}}(Y(x))$ can be expressed as

$$E_{\mathrm{E}}(Y(x)) = \int E_{\mathrm{H}}(Y|V=v,\tilde{X}=x)dF_V(v)$$

- LMC for the hypothetical model generates $Y \perp\!\!\!\perp X|(V,\tilde{X})$.
- By Rule 2, $E_{\mathrm{H}}(Y|V=v,\tilde{X}=x) = E_{\mathrm{E}}(Y|V=v,X=x)$
- Thus, the counterfactual outcome $E_{\mathrm{E}}(Y(x))$ can be obtained by:

$$E_{\mathrm{E}}(Y(x)) = \underbrace{\int E_{\mathrm{E}}(Y|V=v,X=x)dF_V(v)}_{\text{In Empirical Model by Rule 2}}$$

**CONCLUSION**

Link to Appendix C: Some Additional Examples

Link to Appendix D: The Do Calculus

# Appendix

# Appendix A: Mediation Model

# Fourth Example: The Mediation Model

## Fourth Example: The Mediation Model

Three observed variables:

1. $T$ is the causal treatment choice
2. $M$ is the mediator caused by $T$
3. $Y$ is the outcome caused by both $T$ and $M$

1. $Y(t)$ is the counterfactual outcome for $T$ fixed at $t$
2. $Y(t,m)$ for $T$ and $M$ fixed to $(t,m)$
3. $M(t)$ stands for the counterfactual mediator for $T$ fixed at $t$

## Part 1: The Language of Potential Outcomes
## Third Example – Mediation Model

Causal parameters of mediation analysis are:

$$\begin{array}{lcll} \text{Average Total Effect :} & ATE & = & E(Y(t_1) - Y(t_0)) \\ \text{Average Direct Effect :} & ADE(t) & = & E(Y(t_1, M(t)) - Y(t_0, M(t))) \\ \text{Average Indirect Effect :} & AIE(t) & = & E(Y(t, M(t_1)) - Y(t, M(t_0))) \end{array}$$

The total effect (TE or ATE) is the sum of direct and indirect effects (Robins & Greenland, 1992):

$$\begin{aligned} ATE &= E(Y(t_1, M(t_1)) - Y_i(t_0, M(t_0))) \\ &= DE(t_1) + IE(t_0) \\ &= IE(t_1) + DE(t_0). \end{aligned}$$

We seek to identify $E(Y(t, M(t'))$

## A PO Assumption for the Mediation Model

Statistical Assumption: **Sequential Ignorability** (Imai et al., 2010):

$$\big(Y(t', m), M(t)\big) \perp\!\!\!\perp T|X$$
$$Y(t', m) \perp\!\!\!\perp M(t)|(T, X),$$

For any r.v. $A, B, C, D$, the graphoid axiom of *Intersection* states that

$$A \perp\!\!\!\perp B|(C, D) \quad \& \quad A \perp\!\!\!\perp C|(B, D) \quad \Rightarrow \quad A \perp\!\!\!\perp (C, B)|D$$

Setting $A, B, C, D$ to $Y(t', m), T, M(t), X$, we obtain:

$$Y(t', m) \perp\!\!\!\perp T|(M(t), X) \quad \& \quad Y(t', m) \perp\!\!\!\perp M(t)|(T, X)$$
$$\Rightarrow \quad Y(t', m) \perp\!\!\!\perp (M(t), T)|X$$

## Identifying the Mediation Model

Identification:

$$(1) Y(m,t') \perp\!\!\!\perp (M(t),T) \quad \text{and (2)} \quad M(t) \perp\!\!\!\perp T$$

$$
\begin{aligned}
& E(Y(t,M(t'))) = \\
&= \int E\big(Y(t,m) | M(t') = m\big) dF_{M(t')}(m), && \text{L.I.E.} \\
&= \int E\big(Y(t,m)\big) dF_{M(t')}(m), && \text{by 1} \\
&= \int E\big(Y(t,m)\big) dF_{M|T=t'}(m), && \text{by 2} \\
&= \int E\big(Y(t,m) | T = t, M(t) = m\big) dF_{M|T=t'}(m), && \text{by 1} \\
&= \int E\big(Y(T,m) | T = t, M(T) = m\big) dF_{M|T=t'}(m), \\
&= \int E\big(Y(T,M(T)) | T = t, M(T) = m\big) dF_{M|T=t'}(m), \\
&= \int E\big(Y | T = t, M = m\big) dF_{M|T=t'}(m),
\end{aligned}
$$

## Identifying Mediation Effects

**Sequential Ignorability**

$$\big(Y(t',m), M(t)\big) \perp\!\!\!\perp T|X$$
$$Y(t',m) \perp\!\!\!\perp M(t)|(T,X),$$

Identifies counterfactual variables as:

$$ADE(t) = \int \begin{pmatrix} E(Y|T=t_1, M=m, X=x) \\ -E(Y|T=t_0, M=m, X=x) \end{pmatrix} dF_{M|T=t,X=x}(m)dF_X(x)$$

$$AIE(t) = \int \begin{pmatrix} E(Y|T=t, M=m, X=x)\cdot \\ \left[dF_{M|T=t_1,X=x}(m) - dF_{M|T=t_0,X=x}(m)\right] \end{pmatrix} dF_X(x).$$

## Interpreting the PO Assumption for the Mediation Model

What does Sequential Ignorability mean?

$\big(Y(t', m), M(t)\big) \perp\!\!\!\perp T | X$

- Assumes that $T$ is exogenous conditioned on $X$.
- No unobserved variable that causes $T$ and $Y$ or $T$ and $M$.

$Y(t', m) \perp\!\!\!\perp M(t) | (T, X)$

- Assumes that $M(t)$ is exogenous conditioned on $X$ and $T$
- Stronger than randomization
- None of those assumptions are testable.

## Can we Obtain the PO Assumption via RCT?

- Plain Randomization on $T$: $Y(t), M(t) \perp\!\!\!\perp T|X$
- Plain Randomization on $M$: $Y(m) \perp\!\!\!\perp M|X$
- Randomization on $T$ and $M$: $Y(t,m) \perp\!\!\!\perp (T,M)|X$
- Which implies that: $Y(t,m) \perp\!\!\!\perp M|(T,X)$
- What does $Y(t',m) \perp\!\!\!\perp M(t)|(T,X)$ mean?

For each participant $\omega$, Randomize $T$, say agent $\omega$ is assigned to $t'$,
Then assign $\omega$ to the mediation value $M_\omega(t)$
that agent $\omega$ would take if $\omega$ were assigned to treatment $t$!

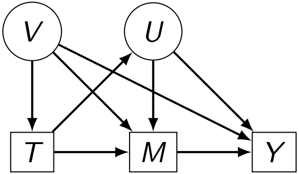
## Reexamining the Mediation Model using SCM

Constructing the Mediation Model using a SCM:

- There are three **observed** variables in the mediation model are: Treatment $T$, mediator $M$ and outcome $Y$.
- Need two more variables to account for **unobserved** confounding effects:
  1. A general *confounder* $V$ is an unobserved exogenous variable that causes $T$, $M$ and $Y$.
  2. The *unobserved mediator* $U$ is caused by $T$ and causes observed mediator $M$.

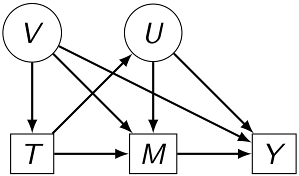
# A General Mediation Model with Confounding Variables

$$\begin{aligned}
\text{Treatment: } & T = f_T(V, \epsilon_T), \\
\text{Unobserved Mediator: } & U = f_U(T, V, \epsilon_U), \\
\text{Observed Mediator: } & M = f_M(T, U, V, \epsilon_M), \\
\text{Outcome: } & Y = f_Y(M, U, V, \epsilon_Y) \\
\text{Independence: } & V, \epsilon_T, \epsilon_U, \epsilon_M, \epsilon_Y.
\end{aligned}$$




- Both variables $T, M$ are endogenous.
- $T \not\perp\!\!\!\perp (M(t), Y(t'))$ and $M \not\perp\!\!\!\perp Y(m)$.

# DAG of a General Mediation Model



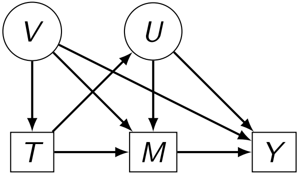

- Both variables $T, M$ are endogenous.
- $T \not\perp\!\!\!\perp (M(t), Y(t'))$ and $M \not\perp\!\!\!\perp Y(m)$.

# Understanding Sequential Ignorability

**Sequential Ignorability** (Imai et al., 2010):

$$\big(Y(t',m), M(t)\big) \perp\!\!\!\perp T|X$$
$$Y(t',m) \perp\!\!\!\perp M(t)|(T,X),$$

What causal assumptions are necessary to render Sequential Ignorability?




- It assumes that $V$ does not exist
- It assumes that $U$ does not cause M (no confounding effect)

## Seeking Identification of the Mediation Model

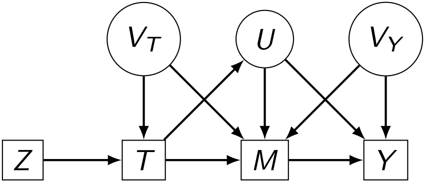
- Mediation model is hopelessly unidentified.
- One alternative: seek for an instrument $Z$ that causes $T$
- and can be used to identify the causal effect of $T$ on $M, Y$
- as well as be used to identify the causal effect of $M$ on $Y$.
- How? By examining the causal relation of unobserved variables!

## The Mediation Model with IV and Partial Confounding

Consider the following model:

$$\begin{aligned}
\text{Treatment: } & T = f_T(Z, V_T, \epsilon_T), \\
\text{Unobserved Mediator: } & U = f_U(T, \epsilon_U), \\
\text{Observed Mediator: } & M = f_M(T, U, V_T, V_Y, \epsilon_M), \\
\text{Outcome: } & Y = f_Y(M, U, V_Y, \epsilon_Y), \\
\text{Independence: } & V_T, V_Y, \epsilon_T, \epsilon_U, \epsilon_M, \epsilon_Y.
\end{aligned}$$

# Mediation Model with IV and Partial Confoundness




- $T$ and $M$ are endogenous
- $T \perp\!\!\!\perp M(t)$ does not hold due to confounder $V_T$,
- $V_Y$ and unobserved mediator $U$ invalidate $M \perp\!\!\!\perp Y(m,t)$
- $T \perp\!\!\!\perp Y(t)$ does not hold due to $V_T, V_Y$.
- Model **still** generates three sets of IV properties!

# A New IV Condition!

The following statistical relations hold in the mediation model:

| Targeted Causal Relation | IV Relevance | Exclusion Restrictions | |
|---|---|---|---|
| Property 1 | for $T \to Y$ | $Z \not\perp\!\!\!\perp T$ | $Z \perp\!\!\!\perp Y(t)$ |
| Property 2 | for $T \to M$ | $Z \not\perp\!\!\!\perp T$ | $Z \perp\!\!\!\perp M(t)$ |
| Property 3 | for $M \to Y$ | $Z \not\perp\!\!\!\perp M \mid T$ | $Z \perp\!\!\!\perp Y(m) \mid T$ |

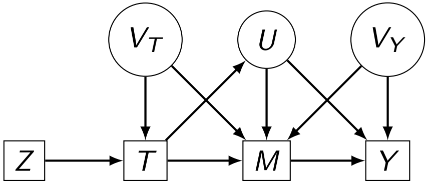
- Prop.1: $Z$ is an IV for $T \to Y$.
- Prop.2: $Z$ is also an IV for $T \to M$.
- Prop.1 and Prop.2 simply state that $Z$ is an IV for $T$
- Prop.3 is the most interesting one:
  $Z$ is an IV for $M \to Y$ when conditional on $T$

## The Third Property $Z \perp\!\!\!\perp Y(m)|T$

- Property 3: $Z \not\perp\!\!\!\perp M|T$ and $Z \perp\!\!\!\perp Y(m)|T$
- $Z$ is an instrument for the causal relation of $M$ on $Y$
- **IF** (and only if) conditioned on $T$.
- $Z \perp\!\!\!\perp Y(m)|T$ holds, but $Z \perp\!\!\!\perp Y(m)$ does not.
- Why?

## Understanding the Property $Z \perp\!\!\!\perp Y(m)|T$

- $Z \perp\!\!\!\perp Y(m)|T$ arises from:
  1. $T$ is caused by both $Z$ and $V_T$ and $V_T \perp\!\!\!\perp Z$
  2. Conditioning on $T$ induces correlation between $Z$ and $V_T$.
  3. Thus, conditioned on $T$, $Z$ affects $M$ (via $V_T$)
  4. $V_T$ becomes a new instrument for $M \to Y$



## Properties of the Mediation Model with Partial Counfoundness

- Assumption on the causal relations among unobserved variables generates identification

One instrument used to evaluate THREE causal effects!

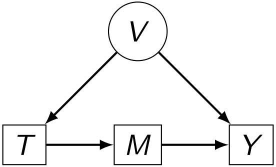
$$E(Y(m,t) - Y(m',t)),\ E(Y(t) - Y(t')),\ E(M(t) - M(t'))$$

## Take Home Message

- Surprisingly SCM and LPO are logically equivalent ( Pearl, 2009, Chapter 7).
- Every assumption/result in SCM can be translated into LPO and vice-versa.
- **Although** equivalent, their tractability differs greatly
- It is difficult to assess independence relations in PO
- The SCM enables you to think outside the box and investigate novel approaches.

## Interpreting a PO Statement for Another Mediation Model

- Would you guess that the relation $M(t) \perp\!\!\!\perp (Y(m), T)$
- is equivalent to assuming the following DAG?



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# Appendix B: Further Remarks on Causality

**Part 3: Causal Calculus**

What can you gain from additional structure?

A General Method to Examine Complex Models

Merging Statical Theory with Causal Analysis

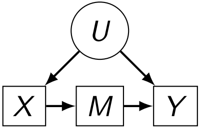
## Part 3 - Causal Calculus

Selected Literature

- Pearl (2009)
  Causal Inference in Statistics: An Overview
- Pearl, J. and Verma, T. (1990).
  A Formal Theory of Inductive Causation.
- Heckman and Pinto (2015)
  Causal Analysis after Haavelmo
- Chalak and White (2011) (You must check this one!)
  An Extended Class of Instrumental Variables for the Estimation of Causal Effects
- White and Chalak (2012)
  Identification and Identification Failure for Treatment Effects Using Structural Systems

## How can we use the SMC to identify the Front-door Model?

$$X = f_X(U, \epsilon_T)$$
$$M = f_M(X, \epsilon_M)$$
$$Y = f_Y(M, U, \epsilon_Y)$$



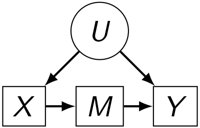

Two Counterfactuals:

$$M(x) = f_M(x, \epsilon_M) \Rightarrow M(x) \perp\!\!\!\perp X$$
$$Y(m) = f_Y(m, U, \epsilon_Y) \text{ but } M \perp\!\!\!\perp U | X \Rightarrow Y(m) \perp\!\!\!\perp M | X$$

Thus the following equalities hold:

- $P(M(x)) = P(M|X = x)$
- $E(Y(m)|X = x) = E(Y|M = m, X = x)$

## Identifying the Counterfactual Mean $E(Y(x))$




Outcome $Y = f_Y(M, U, \epsilon_y)$ generates the following counterfactual:

$$\therefore Y(x) = f_Y(M(x), U, \epsilon_y) \Rightarrow E(Y(x)) = \int E(Y(m))dF_{M(x)}(m)$$

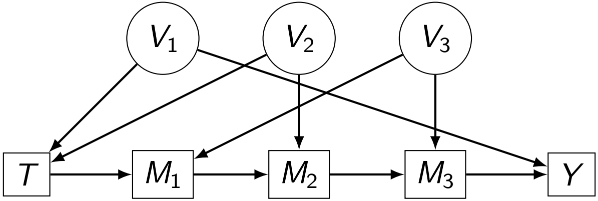
But $P(M(x)) = P(M|X = x)$ and

$$E(Y(m)|X = x') = E(Y|M = m, X = x')$$

$$\Rightarrow E(Y(m)) = \int E(Y|M = m, X = x')dF_X(x')$$

$$\Rightarrow \mathbf{E}(Y(x)) = \int_m \left( \int_{x'} \mathbf{E}(Y|M = m, X = x')dF_X(x') \right) dF_{M|X=x}(m)$$

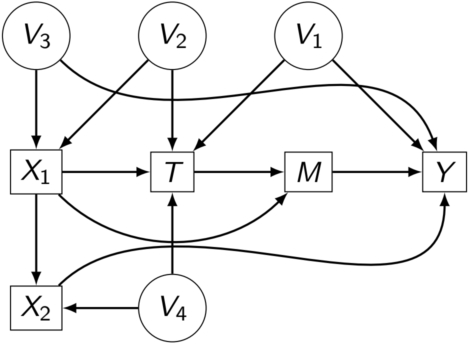
# What about this model?




- $X$ is endogenous, $Y(x) \not\perp\!\!\!\perp X$, indeed, ALL variables are endogenous
- No instruments
- Yet, causal effects are identified:

$$\mathbf{E}(Y(t)) = \int_{t'} \int_{m_1} \int_{m_2} \int_{m_3} \mathbf{E}(Y|m_3, m_2, m_1, \mathbf{T} = \mathbf{t}') \\ dF_{M_3|m_2,m_1,\mathbf{T}=\mathbf{t}}(m_3) \\ dF_{M_2|m_1,\mathbf{T}=\mathbf{t}'}(m_2) \\ dF_{M_1|\mathbf{T}=\mathbf{t}}(m_1) \\ dF_T(\mathbf{t'})$$

# And what about this model?




$$\mathbf{E}(Y(t)) = \int_{t'} \int_{m} \int_{x_1} \mathbf{E}(Y|m, x_1, \mathbf{T} = \mathbf{t}') dF_{M|x_1,\mathbf{T}=\mathbf{t}}(m) dF_{X_1|\mathbf{T}=\mathbf{t}'}(x_1) dF_T(\mathbf{t}')$$

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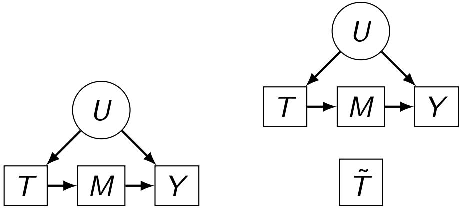
# Appendix C: Some Additional Examples

# Causal Model 1: Revisiting the Front-door Model

| **Empirical Front-door Model** | **Hypothetical Front-door Model** |
|---|---|
| Observed Variables | Observed Variables |
| $T = f_T(V, \epsilon_T)$<br>$M = f_M(T, \epsilon_M)$<br>$Y = f_Y(V, M, \epsilon_Y)$ | $T = f_T(V, \epsilon_T)$<br>$M = f_M(\widetilde{T}, \epsilon_M)$<br>$Y = f_Y(V, M, \epsilon_Y)$<br>$Y = f_Y(V, M, \epsilon_Y)$ |
| Exogenous Variables | Exogenous Variables |
| $V$ | $V, \widetilde{T}$ |
| Unobserved Variables | Unobserved Variables |
| $V = f_V(\epsilon_V)$ | $V = f_V(\epsilon_V)$ |

# Independence Relations
# Hypothetical Front-Door Model




Useful independence relations in the Front-Door hypothetical model:

1. $Y \perp\!\!\!\perp \tilde{T} | (M, T)$
2. $M \perp\!\!\!\perp T | \tilde{T}$
3. $\tilde{T} \perp\!\!\!\perp T$

## General Identification Criteria

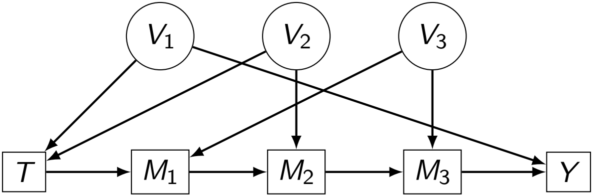
- Given a Causal Model represented by a DAG,
- The counterfactual outcome $Y(t)$ is identified if
- There exists a set of observable variable $K$ that bridges
- The conditional independence $Y \perp\!\!\!\perp \widetilde{T}|(T,K)$ into $T \perp\!\!\!\perp \widetilde{T}$.
- Moreover, the identification formula for $Y(t)$ can be expressed as an alternate pattern.

## Example: Causal Model 2

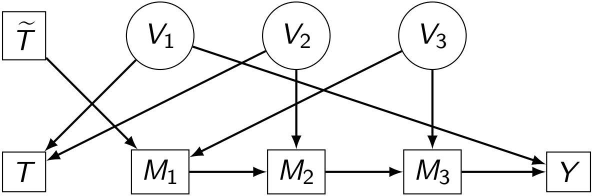
| **Empirical Model** Observed Variables | **Hypothetical Model** Observed Variables |
|---|---|
| $T = f_T(V_1, V_2, \epsilon_T)$ | $T = f_T(V_1, V_2, \epsilon_T)$ |
| $M_1 = f_{M_1}(V_3, T, \epsilon_{M_1})$ | $M_1 = f_{M_1}(V_3, T, \epsilon_{M_1})$ |
| $M_2 = f_{M_2}(V_2, M_1, \epsilon_{M_2})$ | $M_2 = f_{M_2}(V_2, M_1, \epsilon_{M_2})$ |
| $M_3 = f_{M_3}(V_3, M_2, \epsilon_{M_3})$ | $M_3 = f_{M_3}(V_3, M_2, \epsilon_{M_3})$ |
| $Y = f_Y(V_1, M_3, \epsilon_Y)$ | $Y = f_Y(V_1, M_3, \epsilon_Y)$ |
| Exogenous Variables | Exogenous Variables |
| $V_1, V_2, V_3$ | $V_1, V_2, V_3, \widetilde{T}$ |

# DAG of Causal Model 2

## Directed Acyclic Graph of the Empirical Model




## Directed Acyclic Graph of the Hypothetical Model



# Causal Model 2 - Connecting Hypothetical and Empirical

Applying LMC and GA to the hypothetical model generates the following indep. relations:

$$Y \perp\!\!\!\perp \widetilde{T} | (T, M_3, M_2, M_1)$$
$$M_3 \perp\!\!\!\perp T | (\widetilde{T}, M_2, M_1)$$
$$M_2 \perp\!\!\!\perp \widetilde{T} | (T, M_1)$$
$$M_1 \perp\!\!\!\perp T | \widetilde{T}$$
$$\widetilde{T} \perp\!\!\!\perp T \text{ always hold}$$

Observe that:

- The sequence of observed variables $M_1 \to M_2 \to M_3$ forms a **bridge**
- from $Y \perp\!\!\!\perp \widetilde{T} | (T, M_3, M_2, M_1)$ (initial relation)
- to $\widetilde{T} \perp\!\!\!\perp T$ (final relation)

# Causal Model 2 - Connecting Hypothetical and Empirical

Using the two probability rules, we can achieve identification:

**Hypothetical Model**

$$P_H(Y|\widetilde{T} = t) = \sum_{t', m_3, m_2, m_1} P_H(Y|m_3, m_2, m_1, T = t', \widetilde{T} = t)$$
$$P_H(M_3 = m_3|m_2, m_1, T = t', \widetilde{T} = t)$$
$$P_H(M_2 = m_2|m_1, T = t', \widetilde{T} = t)$$
$$P_H(M_1 = m_1|T = t', \widetilde{T} = t)$$
$$P_H(T = t'|\widetilde{T} = t)$$

**Empirical Model**

(alternate pattern)

$$P_E(Y(t)) = \sum_{t', m_3, m_2, m_1} P_E(Y|m_3, m_2, m_1, T = t')$$
$$P_E(M_3 = m_3|m_2, m_1, T = t)$$
$$P_E(M_2 = m_2|m_1, T = t')$$
$$P_E(M_1 = m_1|T = t)$$
$$P_E(T = t')$$

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# Appendix D

## Small Detour: On the Do-Calculus

- Creates a special set of rules that combine:
  1. Graphical conditions
  2. Conditional independence statements
  3. Probability equalities as postulates

In contrast, the hypothetical model framework does not require any tool outside of standard probability theory, provided we endow the space of hypotheticals with a probability measure

**Major Achievement: The do-calculus is Complete!**

## Limitation of the Do-Calculus: IV model is not Identified

- The necessary assumptions the identify the IV model are monotonicity/separability conditions
- These are functional form assumptions
- They refer to properties of the structural functions
- Beyond the DAG information
  (Causal direction among variables remains the same)
- The do-calculus cannot identify the IV model
- The algorithm simply returns that the IV model is not identified

## Causal Model 2 - Comparison Hypothetical vs Do-Calculus Eq.

Equation from do-calculus is different, but equivalent:

**Hypothetical Model** (alternate pattern):

$$P_E(Y(t)) = \sum_{m_3, m_2, m_1, t'} P_E(Y|m_3, m_2, m_1, t')P_E(m_3|m_2, m_1, t)$$
$$P_E(m_2|m_1, t')P_E(m_1|t)P_E(t')$$

**Do-calculus**:

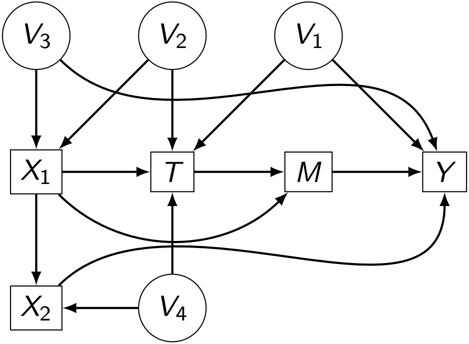
$$P_E(Y(t)) = \sum_{m_1, m_2, m_3} P_E(m_1|t)P_E(m_3|t, m_1, m_2) \cdot$$
$$\left(\sum_{t'} P_E(t')P_E(m_2|t', m_1)\right) \cdot$$
$$\left(\sum_{t', m'_2} P_E(t')P_E(m'_2|t', m_1)P_E(Y|t', m_1, m'_2, m_3)\right)$$

# Causal Model 3

| **Empirical Model**<br>Observed Variables | **Hypothetical Model**<br>Observed Variables |
|---|---|
| $X_1 = f_{X_1}(V_2, V_3, \epsilon_{X_1})$<br>$X_2 = f_{X_2}(V_4, X_1, \epsilon_{X_2})$<br>$T = f_T(V_1, V_2, V_4, X_1, \epsilon_T)$<br>$M = f_M(X_1, T, \epsilon_M)$<br>$Y = f_Y(V_1, V_3, X_2, M, \epsilon_Y)$ | $X_1 = f_{X_1}(V_2, V_3, \epsilon_{X_1})$<br>$X_2 = f_{X_2}(V_4, X_1, \epsilon_{X_2})$<br>$T = f_T(V_1, V_2, V_4, X_1, \epsilon_T)$<br>$M = f_M(X_1, \widetilde{T}, \epsilon_M)$<br>$Y = f_Y(V_1, V_3, X_2, M, \epsilon_Y)$ |
| Exogenous Variables | Exogenous Variables |
| $V_1, V_2, V_3, V_4$ | $V_1, V_2, V_3, V_4, \widetilde{T}$ |
| Unobserved Variables<br>$V_1 = f_{V_1}(\epsilon_{V_1}),\ V_2 = f_{V_2}(\epsilon_{V_2}),$<br>$V_3 = f_{V_3}(\epsilon_{V_3}),\ V_4 = f_{V_4}(\epsilon_{V_4})$ | Unobserved Variables<br>$V_1 = f_{V_1}(\epsilon_{V_1}),\ V_2 = f_{V_2}(\epsilon_{V_2})$<br>$V_3 = f_{V_3}(\epsilon_{V_3}),\ V_4 = f_{V_4}(\epsilon_{V_4})$ |

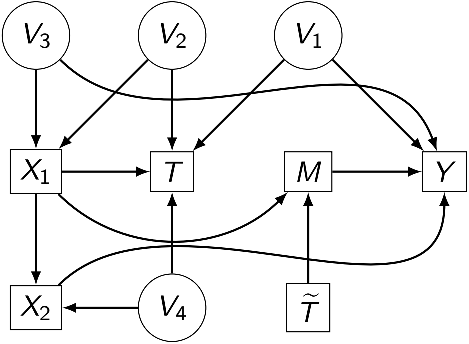
# DAG of Empirical Model 3

## Directed Acyclic Graph of the Empirical Model



# DAG of Hypothetical Model 3

## Directed Acyclic Graph of the Hypothetical Model




$$Y \perp\!\!\!\perp \widetilde{T} | (\widetilde{T}, X_1, M) \quad (3)$$

$$M \perp\!\!\!\perp T | (\widetilde{T}, X_1) \quad (4)$$

## Causal Model 3 - Connecting Hypothetical and Empirical

LMC and GA give you the following conditions:

$$Y \perp\!\!\!\perp \widetilde{T} | (T, M, X_1)$$
$$M \perp\!\!\!\perp T | (\widetilde{T}, X_1)$$
$$X_1 \perp\!\!\!\perp \widetilde{T} | T$$
$$\widetilde{T} \perp\!\!\!\perp T \text{ always hold}$$

- The sequence of observed variables $M \to X_1$ forms a **bridge**
- from $Y \perp\!\!\!\perp \widetilde{T} | (T, X_1, M)$ (initial relation)
- to $\widetilde{T} \perp\!\!\!\perp T$ (final relation)

# Causal Model 3 - Connecting Hypothetical and Empirical

Using the two probability rules, we can achieve identification:

**Hypothetical Model**

$$P_H(Y|\widetilde{T} = t) = \sum_{t',m,x_1} P_H(Y|m, x_1, T = t', \widetilde{T} = t) P_H(M = m|x_1, T = t', \widetilde{T} = t) P_H(X_1 = x_1|T = t', \widetilde{T} = t) P_H(T = t'|\widetilde{T} = t)$$

**Empirical Model**

(alternate pattern)

$$P_E(Y(t)) = \sum_{t',m,x_1} P_E(Y|m, x_1, T = t') P_E(M = m|x_1, T = t) P_E(X_1 = x_1|T = t') P_E(T = t')$$

# Causal Model 3 - Do-calculus Identifying Equation

Equation from do-calculus is different, but equivalent:

**Using Hypothetical Model** (alternate pattern):

$$P_E(Y(t)) = \sum_{m,x_1,t'} P_E(Y|m,x_1,T=t')P_E(m|x_1,T=t)P_E(x_1|T=t')P_E(T=t')$$

**Using Do-calculus:**

$$P_E(Y(t)) = \sum_{x_1,x_2,m} P_E(m|x_1,T=t)P_E(x_2|x_1)P_E(x_1)$$

$$\cdot \frac{\left(\sum_{t'} P_E(Y|x_1,T=t',x_2,m)P_E(x_2|x_1,T=t')P_E(T=t'|x_1)P_E(x_1)\right)}{\left(P_E(x_2|x_1)P_E(x_1)\right)}$$

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