Shadow Prices, Market Wages and Labor Supply

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1. Introduction

Goals:

Derive set of parameters which underlie the functions determining:

1. the probability that a woman works,
2. hours of work,
3. observed wage rate,
4. asking wage or shadow price of time.

2. Shadow Prices and Market Wages

- W^* is the shadow price for the wife's time
- h is the hours of work
- W_m is the spouse's wage
- P is the vector of prices (includes the wage rate)
- A is the asset income
- Z is a vector with household characteristics (number of children, education of family members, etc...)

The shadow price function for the wife's time may be written:

$$W^* = g\left(h, W_m, P, A, Z\right) \tag{1}$$

- W is market wage rate
- E is labor market experience
- S is schooling

The market wage function may be written as:

$$W = B(E, S) \tag{2}$$

with $B_E > 0, B_S > 0$

- If a woman is free to adjust her working hours, a working woman will have the following as equilibrium condition:

$$W = W^*$$

- If she does not work:

$$W \leq W^*$$

3. Estimation

- Specify functional form and stochastic structure for equations (1)and (2)
- Assume we have I observations, indexed $i = 1, 2, \dots, I$

$$l\left(W_i^*\right) = \beta_0 + \beta_1 h_i + \beta_2 W_{mi} + \beta_3 P_i + \beta_4 A_i + \beta_5 Z_i + \varepsilon_i \quad (3)$$

$$l\left(W_i\right) = b_0 + b_1 S_i + b_2 E_i + u_i \quad (4)$$

$$\begin{pmatrix} \varepsilon_i \\ u_i \end{pmatrix} \sim N\left[0_{2\times1}, \Sigma_{2\times2}\right]$$

$$\begin{pmatrix} \varepsilon_i \\ u_i \end{pmatrix} \text{ is independent from } \begin{pmatrix} \varepsilon_j \\ u_j \end{pmatrix} \text{ for } i \neq j, i, j = 1, 2, ..., I.$$

- **Problem:** Observed hours of work depend on the realizations of disturbances (ε_i, u_i)

- To see this, consider a woman i with $l\,(W_i) > l\,(W_i^*)$ at zero hours of work position:

$$l\,(W_i) > l\,(W_i^*) \Rightarrow$$

$$\Rightarrow \varepsilon_i - u_i < b_0 - \beta_0 + b_1 S_i + b_2 E_i - \beta_2 W_{mi} - \beta_3 P_i - \beta_4 A_i - \beta_5 Z_i \tag{5}$$

and hours of work adjust so that $W_i = W_i^*$

- Therefore, hours depend, in part, on the magnitude of the discrepancy $\varepsilon_i - u_i$

- Given that condition (5) holds, the reduced form equations for observed wages and hours become:

$$
\begin{aligned}
h_i &= \frac{b_0}{\beta_1} - \frac{\beta_0}{\beta_1} + \frac{b_1}{\beta_1} S_i + \frac{b_2}{\beta_1} E_i \qquad (6) \\
&\quad - \frac{\beta_2}{\beta_1} W_{mi} - \frac{\beta_3}{\beta_1} P_i - \frac{\beta_4}{\beta_1} A_i - \frac{\beta_5}{\beta_1} Z_i + \frac{u_i - \varepsilon_i}{\beta_1} \\
l(W_i) &= b_0 + b_1 S_i + b_2 E_i + u_i \qquad (7)
\end{aligned}
$$

- We obtain observations on which to estimate (6) and (7) only if condition (5) holds

- For a sample of working women, the distribution of the disturbances of equations (6) and (7) are conditional on inequality (5) and hence are conditional distributions

- Since the same exogenous variables appear in condition (5) and equations (6) and (7), the mean and other moments of these conditional distributions, for a particular observation i, depend on the values of the exogenous variables for the observation

- Consequently, the regressors are correlated with the disturbances and OLS on equations (6) and (7) will not generate unbiased or consistent estimates. The same is true for IV which uses the exogenous variables appearing in condition (5) as instruments

- Let $n\ (h_i, l\ (W_i))$ denote the joint distribution

- Let $\Pr([W_i > W_i^*]_{h=0})$ be the probability that a woman works

- Let $j\ (h_i, l\ (W_i) | [W_i > W_i^*]_{h=0})$ be the conditional distribution (conditional that a woman works)

- Then:

$$j\left(h_i, l\left(W_i\right) \mid [W_i > W_i^*]_{h=0}\right) = \frac{n\left(h_i, l\left(W_i\right)\right)}{\Pr\left([W_i > W_i^*]_{h=0}\right)} \qquad (8)$$

- If a sample of T married women contains K who work and $T - K$ who do not, the likelihood for the entire T observations may be written as:

$$L = \left[\prod_{i=1}^{K} j\left(h_i, l\left(W_i\right) \mid [W_i > W_i^*]_{h=0}\right) \Pr\left([W_i > W_i^*]_{h=0}\right)\right] \cdot \left[\prod_{i=K+1}^{T} \Pr\left([W_i < W_i^*]_{h=0}\right)\right]$$

- Using equation (8), the likelihood collapses to:

$$L = \left[\prod_{i=1}^{K} n\left(h_i, l\left(W_i\right)\right)\right]\left[\prod_{i=K+1}^{T} \Pr\left([W_i < W_i^*]_{h=0}\right)\right] \qquad (9)$$

4. Empirical Results

Appendix 1

- The household utility function:

$$U(X_1, \dots, X_n)$$

where X_1 represents the wife's leisure

- A is asset income
- P_i is the price of the good
- T is the amount of time available to the wife
- h is hours of work, with associated wage rate P_1

- Household solves

$$Max\ U(X_1, \dots, X_n)$$

subject to

$$\begin{aligned} \sum_{i=2}^{n} P_i X_i - A - P_1 h &= 0 \\ T - X_1 - h &= 0 \end{aligned}$$

- So the LaGrangian is

$$U(X_1, ..., X_n) - \lambda \left(\sum_{i=2}^{n} P_i X_i - A - P_1 h \right) - \mu (X_1 + h - T)$$

- FOC:

$$\begin{aligned} U_1 - \mu &= 0 \\ U_i - \lambda P_i &= 0 \quad (i = 1, ..., n) \end{aligned}$$

- The shadow price:

$$\frac{U_1}{\lambda} = \frac{\mu}{\lambda} = W^* = k(h, P_1 h + A, P_2, ..., P_n) \quad (10)$$

where (10) is valid for any arbitrary P_1

- Equilibrium solution with h voluntary chosen requires

$$P_1 = W^*$$

and the relationship between the equilibrium values of W $*$ and h, if one exists, defines the labor supply relationship

- We can always adjoin the value of W^* $at\ h = 0$ $(or\ h < 0)$

- And the continuity of k assures is that adjoined labor supply is continuous and differentiable in equilibrium wages

Appendix 2: Statistical Models

- The joint distribution of ε_i and u_i is assumed to be a bivariate normal distribution with

$$\begin{aligned} E(\varepsilon_i) &= 0 \\ E(u_i) &= 0 \\ E(\varepsilon_i^2) &= \sigma_\varepsilon^2 \\ E(u_i^2) &= \sigma_u^2 \\ E(u_i\varepsilon_i) &= \sigma_{\varepsilon u} = \rho\sigma_u\sigma_\varepsilon \end{aligned}$$

$$\frac{\left(\sigma_\varepsilon^2\sigma_u^2(1-\rho^2)\right)^{-0.5}}{2\pi}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{\varepsilon_i^2}{\sigma_\varepsilon^2}+\frac{u_i^2}{\sigma_u^2}-\frac{2\rho\varepsilon_i u_i}{\sigma_\varepsilon\sigma_u}\right]\right)$$

- Thus

$$\Pr\left([W_i > W_i^*]_{h=0}\right) =$$
$$\Pr\left[\frac{b_0 - \beta_0 + b_1 S_i - \beta_2 (W_m)_i + b_2 E_i - \beta_3 P_i - \beta_4 A_i - \beta_5 Z_i}{(\sigma_\varepsilon^2 + \sigma_u^2 - 2\rho\sigma_\varepsilon\sigma_u)^{0.5}} > \frac{\varepsilon_i - u_i}{(\sigma_\varepsilon^2 + \sigma_u^2 - 2\rho\sigma_\varepsilon\sigma_u)^{0.5}}\right]$$

so that

$$\Pr\left([W_i > W_i^*]_{h=0}\right) = \int_{-\infty}^{J_i} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}r^2\right) dr$$

where

$$J_i = \frac{b_0 - \beta_0 + b_1 S_i - \beta_2 (W_m)_i + b_2 E_i - \beta_3 P_i - \beta_4 A_i - \beta_5 Z_i}{(\sigma_\varepsilon^2 + \sigma_u^2 - 2\rho\sigma_\varepsilon\sigma_u)^{0.5}}$$

- Similarly,

$$\Pr\left([W_i < W_i^*]_{h=0}\right) = \int_{J_i}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}r^2\right) dr$$

- The derivation of the distribution $n\ (h_i, l\ (W_i))$ in (9):

$$\begin{aligned} h_i - D_i &= \frac{u_i - \varepsilon_i}{\beta_1} \\ l[W_i] - F_i &= u_i \end{aligned}$$

- And $\left(\frac{u_i-\varepsilon_i}{\beta_1}, u_i\right)$ are jointly normally distributed with

$$\begin{aligned} E\left(\frac{u_i-\varepsilon_i}{\beta_1}\right) &= 0 \\ E\left(\frac{u_i-\varepsilon_i}{\beta_1}\right)^2 &= \frac{\sigma_\varepsilon^2+\sigma_u^2-2\rho\sigma_\varepsilon\sigma_u}{\beta_1^2} \\ Cov\left(\frac{u_i-\varepsilon_i}{\beta_1}, u_i\right) &= \frac{\sigma_u^2-2\rho\sigma_\varepsilon\sigma_u}{\beta_1^2}. \end{aligned}$$

- Thus,

$$n\left(h_i, l\left(W_i\right)\right) = |\beta_1| \frac{\left(\sigma_\varepsilon^2 \sigma_u^2 (1-\rho^2)\right)^{-0.5}}{2\pi} \exp\left[-\frac{G}{2\left(1-\rho^2\right)}\right]$$

where

$$\begin{aligned} G \;=\; & (h_i - D_i)^2 \left(\frac{\beta_1^2}{\sigma_\varepsilon^2}\right) \\ & -2(h_i - D_i)\left(l[W_i] - F_i\right)\left(\frac{1}{\sigma_\varepsilon^2} - \frac{\rho}{\sigma_\varepsilon \sigma_u}\right) \\ & + \left(l[W_i] - F_i\right)^2 \left(\frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_u^2} - 2\frac{\rho}{\sigma_\varepsilon \sigma_u}\right) \end{aligned}$$

Table 1: Annual Hours Worked

	Intercept	Number of Children Less Than Six	Net Assets	Wage Rate of Husband	Experience	Education	Labor Supply	Standard Deviation
ln Asking Wage	−.623 (.088)	.179 (.019)	$.135 \times 10^{-5}$ $(.055 \times 10^{-5})$	.051 (.007)	—	.0534 (.007)	$.63 \times 10^{-3}$ $(.05 \times 10^{-3})$	.532 (.019)
ln Offered Wage	−.982 (.11)	—	—	—	.048 (.004)	.0761 (.0075)	—	.452 (.0121)
					The estimated correlation of disturbances across equations is			.6541 (.046)

Asymptomatic standard errors in parentheses

Table 2: Annual Weeks Worked

	Intercept	Number of Children Less Than Six	Net Assets	Wage Rate of Husband	Experience	Education	Labor Supply	Standard Deviation
ln Asking Wage	3.1 (1.36)	.149 (.022)	$.50 \times 10^{-6}$ $(.55 \times 10^{-6})$	.046 (.008)	—	.039 (.0124)	.02 (.003)	.671 (.021)
ln Offered Wage	2.75 (1.62)	—	—	—	.0445 (.0062)	.061 (.010)	—	.677 (.018)
The estimated correlation of disturbances across equations is								.83 (.043)

Asymptomatic standard errors in parentheses

Table 3: Estimated Probabilities of Working

Number of Children Less Than Six	Years of Schooling 8	10	12	14	16
0	.30	.38	.47	.56	.66
1	.09	.13	.18	.25	.32
2	.013	.025	.04	.065	.09

Husband's wage rate is $2.50 per hour, net worth is $5,000, and the woman has four years of experience.

Table 4: Annual Hours Worked: Full Information Maximum Likelihood Applied to the Subsample of Working Women

	Intercept	Number of Children Less Than Six	Net Assets	Wage Rate of Husband	Experience	Education	Labor Supply	Standard Deviation
ln Asking Wage	−1.28 (.18)	.0703 (.09)	$.169 \times 10^{-5}$ $(.78 \times 10^{-6})$	.0376 (.01)	—	.0623 (.008)	$.83 \times 10^{-3}$ $(.95 \times 10^{-4})$	.469 (.012)
ln Offered Wage	−.36 (.086)	—	—	—	.0195 (.0025)	.0681 (.007)	—	.507 (.035)
					The estimated correlation of disturbances across equations is			.591 (.09)

Asymptomatic standard errors in parentheses

Table 5: Annual Weeks Worked: Full Information Maximum Likelihood Applied to the Subsample of Working Women

	Intercept	Number of Children Less Than Six	Net Assets	Wage Rate of Husband	Experience	Education	Labor Supply	Standard Deviation
ln Asking Wage	1.55 (.31)	.046 (.03)	$-.36 \times 10^{-7}$ (1.0×10^{-6})	.022 (.012)	—	.0485 (.012)	.043 (.007)	.65 (.015)
ln Offered Wage	3.13 (1.21)	—	—	—	.026 (.0035)	.0561 (.0098)	—	.78 (.05)

The estimated correlation of disturbances across equations is .697 (.07)

Asymptomatic standard errors in parentheses