Causality in Econometrics and Statistics: Structural Models are Causal Models Some Formal Statements Part III on Causality

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Topics to be Covered

- **Contributions**

- What is a causal effect?
  Key concept and discussion on how it is expressed/modeled
- Clarify the benefits of adopting more sophisticated causal analysis.
- Illustrate advantages through selected examples

Topics to be Covered

- **Examine Causal Frameworks**

1. Causal model based on potential outcomes but no choice mechanisms or explanations of outcomes
   The Rubin-Holland causal model.
2. Causal model based on autonomous equations
   Inspired by Haavelmo (1944).
3. Other causal frameworks based on Local Markov Conditions (LMC):
   - Judea Pearls's **Do-calculus** (uses framework of structural equations, but weird calculus).
   - Empirical versus Hypothetical framework of Heckman and Pinto (2015b).

Structure

- **Part 1:** the language of potential outcomes (Holland, 1986).
  - Simplicity: widely used for causal evaluation.
  - Examples: Randomization, Matching, IV and Mediation.
  - Unanswered questions
- **Part 2:** Autonomous Equations (Haavelmo, 1944).
  - Benefits of a proper causal framework
  - Example: The Roy Model, Mediation Model.
  - Statistical tools are ill-suited to examine causality (source of confusion)
- **Part 3:** Hypothetical/Empirical framework (Heckman and Pinto, 2015b) and Do-calculus (Pearl, 2009b)
  - Clarify benefits of enhanced causal framework
  - Examples: based on more complex causal models
  - Compare the approach with previous literature

Selected Literature

- Holland (1986)
  Statistics and Causal Inference (JASA)
- Imbens and Rubin (2015)
- Pearl (2009a)
  Causal Inference in Statistics: An Overview
- Heckman and Pinto (2015b)
  Causal Analysis after Haavelmo
- Freedman (2010)
  Statistical Models and Causal Inference: A Dialogue with the Social Sciences

Frisch: "Causality is in the Mind "

"*. . . we think of a cause as something imperative which exists in the* **exterior world.** *In my opinion this is fundamentally* **wrong**. *If we strip the word cause of its animistic mystery, and leave only the part that science can accept, nothing is left except* a certain way of thinking, [T]he scientific . . . problem of **causality** is essentially a problem regarding our **way of thinking,** not a problem regarding the nature of the exterior world."

—Frisch 1930, p. 36, published 2011

Part 1: The Language of Potential Outcomes

Definition and Applications: RCT, Matching, Meditation, IV

Part 1: The Language of Potential Outcomes
Basic Definitions

- The Rubin-Holland causal framework of potential outcomes.
- Variables in common probability space $(\Omega, \mathcal{F}, P)$
  1. T Treatment choice
  2. Y Outcome
  3. X Baseline Characteristics
- Potential outcome Y of agent ω for fixed $T = t$ is $Y_\omega(t)$.
- Causal effects of t' versus t for ω is $Y_\omega(t) - Y_\omega(t')$.
- The observed outcome is given by Quandt (1958) switching regression:

$$Y = \sum_{t \in \text{supp}(T)} Y(t) \cdot \mathbf{1}[T = t]$$

Part 1: The Language of Potential Outcomes
First Example – RCT

Identification relies on *statistical assumptions*:

Randomized Controlled Trials (RCT): $Y(t) \perp\!\!\!\perp T|X,$

Full Compliance

X are variables used in the randomization protocol.
$Y(t) \perp\!\!\!\perp T|X \Rightarrow$ counterfactual outcomes identified:

$$\mathbf{E}(Y(t)|X) = \left(\sum_{t \in \mathrm{supp}(T)} Y(t) \cdot \mathbf{1}[T = t]|X, \right) \text{ but } Y(t) \perp\!\!\!\perp T|X$$
$$= \left(\sum_{t \in \mathrm{supp}(T)} Y(t) \cdot \mathbf{1}[T = t]|X, T = t \right) = \mathbf{E}(Y|T = t, X),$$

Average causal effects obtained as:

$$E(Y(t_1) - Y(t_0)) = \int \Big(E(Y|T = t_1, X = x) - E(Y|T = t_0, X = x) \Big) dF_X(x)$$

Part 1: The Language of Potential Outcomes
First Example – RCT

- Key ideas of RCT Formalized by R.A. Fisher: Statistical Methods for Research Workers, 1925)
- Average Treatment Effect:

$$
\begin{aligned}
E(Y(t_1) - Y(t_0)) &\equiv \int \Big(Y_\omega(t_1) - Y_\omega(t_0)\Big) dF(\omega) \\
&= \frac{\int_{\omega; T_\omega = t_1} Y_\omega dF(\omega)}{\int_{\omega; T_\omega = t_1} dF(\omega)} - \frac{\int_{\omega; T_\omega = t_0} Y_\omega dF(\omega)}{\int_{\omega; T_\omega = t_0} dF(\omega)} \\
&= \int_{\omega; T_\omega = t_1} Y_\omega \underbrace{\frac{dF(\omega)}{\int_{\omega; T_\omega = t_1} dF(\omega)}} - \int_{\omega; T_\omega = t_0} Y_\omega \underbrace{\frac{dF(\omega)}{\int_{\omega; T_\omega = t_0} dF(\omega)}}
\end{aligned}
$$

- Generally, we assume full support for both $T_\omega = t_1$ and $T - \omega = t_0$.
- Indicated by underbrace: space of ω for which randomization implemented

Part 1: The Language of Potential Outcomes
Second Example – Matching

Statistical assumption that $Y(t) \perp\!\!\!\perp T|X$ is **matching assumption**.

- Agents ω are comparable when conditioned on observed values X,
- Causal effects are weighted average of treated and control participants
- Conditional on their pre-intervention variables X.

1. Matching $\Rightarrow$ exogenous variation of T under X *by assumption*
2. Randomization $\Rightarrow$ exogenous variation of T under X *by design*

Part 1: The Language of Potential Outcomes
Third Example – Mediation Model

- Three observed variables:

1. T is the causal treatment choice
2. M is a mediator caused by T
3. Y is the outcome caused by both T and M

1. $Y_\omega(t)$ is the counterfactual outcome for T fixed at t
2. $Y_\omega(t, m)$ for T and M fixed to (t, m)
3. $M_\omega(t)$ stands for the counterfactual mediator for T fixed at t

Part 1: The Language of Potential Outcomes
Third Example – Mediation Model

- Causal parameters of mediation analysis:

$$\begin{aligned}
\text{Average Total Effect :} \quad ATE(t) &= E(Y(t_1) - Y(t_0)) \\
\text{Average Direct Effect :} \quad ADE(t) &= E(Y(t_1, M(t)) - Y(t_0, M(t))) \\
\text{Average Indirect Effect :} \quad AIE(t) &= E(Y(t, M(t_1)) - Y(t, M(t_0)))
\end{aligned}$$

- The total effect is the sum of direct and indirect effects (Robins and Greenland, 1992)

$$
\begin{aligned}
TE &= E(Y(t_1, M(t_1)) - Y(t_0, M(t_0))) \\
&= \Big(E(Y(t_1, M(t_1))) - E(Y(t_0, M(t_1)))\Big) + \Big(E(Y(t_0, M(t_1)) - Y_i(t_0, M(t_0)))\Big) \\
&\qquad\qquad = DE(t_1) + IE(t_0) \\
&= \Big(E(Y(t_1, M(t_1))) - E(Y(t_1, M(t_0)))\Big) + \Big(E(Y(t_1, M(t_0)) - Y_i(t_0, M(t_0)))\Big) \\
&\qquad\qquad = IE(t_1) + DE(t_0).
\end{aligned}
$$

Part 1: The Language of Potential Outcomes
Third Example – Mediation Model

$$T \rightarrow M \rightarrow Y$$

- Statistical Assumption: **Sequential Ignorability** (Imai et al., 2010): conditional on background variable X:

$$\begin{aligned}\big(Y(t', m), M(t)\big) &\perp\!\!\!\perp T|X \\ Y(t', m) &\perp\!\!\!\perp M(t)|(T, X),\end{aligned}$$

$P(Y(t,m)|X) = P(Y|X, T = t, M = m)$ and $P(M(t)|X) = P(M|X, T = t)$

Part 1: The Language of Potential Outcomes
Third Example – Mediation Model

- Counterfactual variables are identified by:

$$ADE(t) = \int \begin{pmatrix} E(Y|T = t_1, M = m, X = x) \\ -E(Y|T = t_0, M = m, X = x, X = x) \end{pmatrix} dF_{M|T=t,X=x}(m)dF_X(x)$$

$$AIE(t) = \int \begin{pmatrix} E(Y|T = t, M = m, X = x)\cdot \\ \left[dF_{M|T=t_1,X=x}(m) - dF_{M|T=t_0,X=x}(m)\right] \end{pmatrix} dF_X(x).$$

Part 1: The Language of Potential Outcomes
Third Example – Mediation Model

The Sequential Ignorability Assumption

$\big(Y(t', m), M(t)\big) \perp\!\!\!\perp T|X$

- Assumes that T is exogenous conditioned on X.
- No unobserved variable that causes T and Y or T and M.

$$Y(t', m) \perp\!\!\!\perp M(t)|(T, X)$$

- Assumes that M is exogenous conditioned on X and T
- Stronger than randomization
- **None of those assumptions are testable**.

Part 1: The Language of Potential Outcomes
Fourth Example – The Instrumental Variable Model

- Statistical Assumption:

$$\begin{aligned} \text{Exclusion Restriction} &: \; Y(t) \perp\!\!\!\perp Z, \\ \text{IV Relevance} &: \; Z \not\!\perp\!\!\!\perp T \end{aligned}$$

- Differs from the matching (ignorability)
- While matching assumptions suffice to identify causal effects over the common support of X,
- The exclusion restriction does not.
- Imbens and Angrist (1994) Monotonicity $T_\omega(z_0) \leq T_\omega(z_1)$ for all units ω
- Identifies the causal effect of the treatment T for "compliers."

Part 1: The Language of Potential Outcomes
Fourth Example – The Instrumental Variable Model

- The exclusion restrictions are necessary but not sufficient to identify causal effects
- Imbens and Angrist (1994) study a binary T and assume a monotonicity criteria that identifies the Local Average Treatment Effect ($LATE$).
- Vytlacil (2006) studies categorical treatments T and evokes a separability condition that governs the assignment of treatment statuses.
- Heckman and Pinto (2018) present a monotonicity condition that applies to unordered choice models with multiple treatments, they investigate identifying assumptions generated by revealed preference analysis.

- Heckman and Vytlacil (2005) investigate the binary treatment, continuous instruments and assume that the treatment assignment is characterized by a threshold-crossing function.
- Lee and Salanie (2018) assume a generalized set of threshold-crossing rules.
- Altonji and Matzkin (2005); Blundell and Powell (2003, 2004); Imbens and Newey (2007) study control function methods characterised by conditional independence and functional form assumptions.

Part 1: Main Criticisms of the Language of Potential Outcomes

- Not a proper causal framework. Does not assess causal relationships. (What does this mean? See below.)
- Instead, **postulate conditional independence relationships.**
- Causal relationships are *implied*, $Z \to T \to Y$, but never formally articulated.
- Lack of tools to precisely determine causal relationships
- The method defined on the basis of only observed variables.
- **Does not allow for unobserved** variables nor causal relationships
- Rejection of unobservables is a key feature of this approach
- Does not allow for a confounding variable.
- **Does it matter?**

Part 1: Remarks

1. Monotonicity is equivalent to separability in the confounding variables and the instrument Vytlacil (2002).
2. Additional index model structure comes at no cost of generality.
3. Causal analysis using structural equations allows for richer causal analysis.

Part 1: Remarks on the Language of Potential Outcomes for the Mediation Model

1. Sequential Ignorability does not hold under the presence of either unobserved *Confounders* or *Unobserved Mediators* (Heckman and Pinto, 2015a).
2. **Autonomous equations** (Frisch, 1938) allow us to clarify these two sources of confounding
3. Does not allow for the specification of the causal relationships of the unobserved confounding variables.
4. Autonomous equations allow for richer identification and interpretation analysis

Part 2: A Causal Model

Definition, Properties and Core Concepts

Fixing as a Causal Operator

Part 2: A Causal Model – Why bother?

- The benefit of the language of potential outcomes relies on its apparent simplicity.
- But the approach is not sufficiently rich for econometric causal analysis.
- Formal causal framework substantially improves the possibilities of causal analysis.

Part 2: Goals of a Causal Model

- We use insight, linking causality to independent variation of variables in a hypothetical model: *Causality Is In The Mind*
- Build a causal **framework** that solves tasks of causal **identification** and **estimation**:

Task	Description	Requirements
1	Defining Causal Models	A Scientific Theory A Mathematical Framework Required for Formal Causal Models
2	Identifying Causal models from Known Population Distribution Functions of Data	Mathematical Analysis Connect Hypothetical Model with Data Generating Process (Identification in the Population)
3	Estimating models from Real Data	Statistical Analysis Estimation and Testing Theory

Part 2: Components of a Causal Model

- **Causal Model:** defined by a **4** components:

1. **Random Variables** that are observed and/or unobserved by the analyst: $\mathcal{T} = \{Y, U, X, V\}$. [Here: $\mathcal{T}$ is a set of relevant variables.]
2. **Error Terms** that are mutually independent: $\epsilon_Y, \epsilon_U, \epsilon_X, \epsilon_V$.
3. **Structural Equations** that are autonomous : f_Y, f_U, f_X, f_V.

- By **Autonomy** we mean deterministic functions that are "invariant" to changes in their arguments (Frisch, 1938).
- Also known as "Structural" (Hurwicz, 1962).

(3) **Causal Relationships** that map the inputs causing each variable:

$$Y = f_Y(X, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V).$$

- "All causes" model.

The econometric approach explicitly models **unobservables** that drive outcomes and produce selection problems.
Distribution of unobservables is often the object of study.

Part 2: Components of a Causal Model

Given the causal relationships, for instance:

$$\begin{aligned} Y &= f_Y(X, U, \epsilon_Y), & Y \text{ observed} \\ X &= f_X(V, \epsilon_X), & X \text{ observed} \\ U &= f_U(V, \epsilon_U), & U \text{ unobserved} \\ V &= f_V(\epsilon_V), & V \text{ unobserved} \end{aligned}$$

A Few Simple Questions

- Which statistical relationships are generated by this (or any) causal model?
- Is there an equivalence between statistical relationships and causal relationships?

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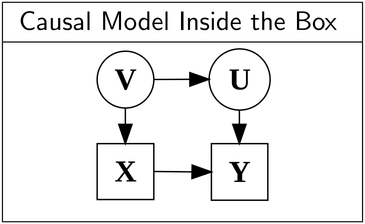
Part 2: Directed Acyclic Graph (DAG) Representation

Model:

$$Y = f_Y(X, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V).$$



Notation of Directed Acyclic Graphs:

- **Children:** Variables directly caused **by** other variables:
  Ex: $Ch(V) = \{U, X\}, \quad Ch(X) = Ch(U) = \{Y\}.$
- **Descendants:** Variables that directly or indirectly cause other variables:
  Ex: $DE(V) = \{U, X, Y\}, \quad D(X) = D(U) = \{Y\}.$
- **Parents:** Variables that directly **cause** other variables:
  Ex: $Pa(Y) = \{X, U\}, \quad Pa(X) = Pa(U) = \{V\}.$

Part 2: Properties of this Causal Framework

- **Recursive Property :** No variable is descendant of itself (acyclic graph).

Why is it useful?

Autonomy + Independent Errors
+ Recursive Property
$\Rightarrow$ Bayesian Network Tools Apply

- **Bayesian Network:** Translates causal links into independence relationships using Statistical/Graphical Tools.
- **Statistical/Graphical Tools:**
  1. Local Markov Condition (**LMC**): a variable is independent of its non-descendants **conditioned on its parents**.
  2. Graphoid Axioms (**GA**): Independence relationshipships, Dawid (1979).
- **Application** of these tools generate relationships such as: $Y \perp\!\!\!\perp V|(U, X), \quad U \perp\!\!\!\perp X|V$

Local Markov Condition (LMC)

(Kiiveri, 1984, Lauritzen, 1996)

- If a model is acyclical, i.e., $Y \notin D(Y) \, \forall \, Y \in \mathcal{T}$ then any variable is independent of its non-descendants, conditional on its parents:

$$\textbf{LMC}: Y \perp\!\!\!\perp \underbrace{\mathcal{T} \setminus (D(Y) \cup Y)}_{\text{set difference}} | Pa(Y) \quad \forall \, Y \in \mathcal{T}.$$

Graphoid Axioms (GA)

(Dawid, 1979)

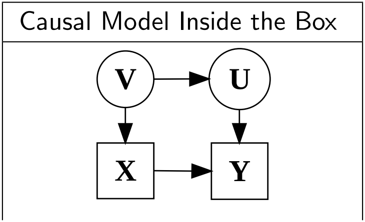
$$
\begin{aligned}
\text{Symmetry: } & X \perp\!\!\!\perp Y|Z \Rightarrow Y \perp\!\!\!\perp X|Z. \\
\text{Decomposition: } & X \perp\!\!\!\perp (W, Y)|Z \Rightarrow X \perp\!\!\!\perp Y|Z. \\
\text{Weak Union: } & X \perp\!\!\!\perp (W, Y)|Z \Rightarrow X \perp\!\!\!\perp Y|(W, Z). \\
\text{Contraction: } & X \perp\!\!\!\perp W|(Y, Z) \text{ and } X \perp\!\!\!\perp Y|Z \Rightarrow X \perp\!\!\!\perp (W, Y)|Z. \\
\text{Intersection: } & X \perp\!\!\!\perp W|(Y, Z) \text{ and } X \perp\!\!\!\perp Y|(W, Z) \Rightarrow X \perp\!\!\!\perp (W, Y) \\
\text{Redundancy: } & X \perp\!\!\!\perp Y|X.
\end{aligned}
$$

Bonus

Exercise: Prove these relationships as a bonus question for the next problem set.

Part 2: Local Markov Condition (LMC)

A variable is independent of its non-descendants conditional on its parents



Causal Model	**LMC Relationships**
$V = f_V(\epsilon_V)$	$V \perp\!\!\!\perp \emptyset \mid \emptyset$
$U = f_U(V, \epsilon_U)$	$U \perp\!\!\!\perp X \mid V$
$X = f_X(V, \epsilon_X)$	$X \perp\!\!\!\perp U \mid V$
$Y = f_Y(X, U, \epsilon_Y)$	$Y \perp\!\!\!\perp V \mid (U, X)$

Equivalence: Assuming a causal Model that defines causal direction is equivalent to assume the set of Local Markov Conditions for each variable of the model.

Causal Model $\Leftrightarrow$ Set of **LMC**s (one for each variable)

Part 2: Analysis of Counterfactuals – the Fixing Operator

- **Fixing:** causal operation sets X-inputs of structural equations to x.

Standard Model	**Model under Fixing**
$V = f_V(\epsilon_V)$	$V = f_V(\epsilon_V)$
$U = f_U(V, \epsilon_U)$	$U = f_U(V, \epsilon_U)$
$X = f_X(V, \epsilon_X)$	$X = x$
$Y = f_Y(X, U, \epsilon_Y)$	$Y = f_Y(x, U, \epsilon_Y)$

- **Importance:** Establishes a framework for counterfactuals.
- **Counterfactual:** $Y(x)$ represents outcome Y when X is fixed at x.
- **Linear Case:** $Y = X\beta + U + \epsilon_Y$ and $Y(x) = x\beta + U + \epsilon_Y$;

Part 2: Joint Distributions

1. **Model Representation under Fixing:**

$$Y = f_Y(x, U, \epsilon_Y); X = x; U = f_U(V, \epsilon_U); V = f_V(\epsilon_V).$$

2. **Standard Joint Distribution Factorization:**

$$\begin{aligned} P(Y, V, U|X = x) &= P(Y|U, V, X = x)P(U|V, X = x)P(V|X = x). \\ &= P(Y|U, V, X = x)P(U|V)\mathbf{P(V|X = x)} \\ &\text{because } U \perp\!\!\!\perp X|V \text{ by LMC.} \end{aligned}$$

3. **Factorization under Fixing X at x:**

$$P(Y, V, U|X \text{ fixed at } x) = P(Y|U, V, X = x)P(U|V)\mathbf{P(V)}.$$

- **Conditioning** X on x affects the distribution of V.
- **Fixing** X on x does **not** affect the distribution of V.

Part 2: Understanding the Fixing Operator (Error Term Representation)

- The definition of causal model permits the following operations:

1. Through **iterated substitution** we can represent all variables as functions of error terms.
2. This representation **clarifies** the concept of fixing.

Part 2: Representing the Model Through Their Error Terms

Standard Model	Model under Fixing
$V = f_V(\epsilon_V)$ $U = f_U(f_V(\epsilon_V), \epsilon_U)$ $X = f_X(f_V(\epsilon_V), \epsilon_X)$	$V = f_V(\epsilon_V)$ $U = f_U(f_V(\epsilon_V), \epsilon_U)$ $X = \mathbf{x}$

Outcome Equation

Standard Model: $Y = f_Y(f_X(f_V(\epsilon_V), \epsilon_X), f_U(f_V(\epsilon_V), \epsilon_U), \epsilon_Y).$

Model under Fixing: $Y = f_Y(\mathbf{x}, f_U(f_V(\epsilon_V), \epsilon_U), \epsilon_Y).$

Part 2: Understanding the Fixing Operator

1. Cumulative error distribution function: $F_{\boldsymbol{\epsilon}}$.

2. **Conditioning**: $(Y = f_Y(f_X(f_U(\epsilon_U), \epsilon_X), f_U(\epsilon_U), \epsilon_Y))$

$$\therefore E(Y|X = \mathbf{x}) = \int_A f_Y(f_X(f_V(\epsilon_V), \epsilon_X), f_U(f_V(\epsilon_V), \epsilon_U), \epsilon_Y) \frac{dF_{\boldsymbol{\epsilon}}(\epsilon)}{\int_A dF_{\boldsymbol{\epsilon}}}$$

Imposes term restriction on values error terms:

$A = \{\epsilon\,;\; f_X(f_V(\epsilon_V), \epsilon_X) = \mathbf{x}\}$

3. **Fixing**: $(Y = f_Y(\mathbf{x}, \epsilon_X), f_U(\epsilon_U), \epsilon_Y))$

$$\therefore E(Y(\mathbf{x})) = \int f_Y(\mathbf{x}, \epsilon_X), f_U(f_V(\epsilon_V), \epsilon_U), \epsilon_Y) dF_{\boldsymbol{\epsilon}}(\epsilon)$$

Imposes no restriction on values assumed by the error terms

Fixing does not belong to nor can it be defined by standard probability theory!!

- Fixing is a **causal operator**, not a statistical operator
- Fixing does not affect the distribution of its ancestors
- Conditioning is a statistical operator
- It affects the distribution of all variables
- Fixing has causal direction
- Conditioning has no direction
- $\therefore$ statisticians have a hard time understanding it

Part 2: Fixing $\neq$ Conditioning

Conditioning: *Statistical* exercise that considers the dependence structure of the data generating process.

$$Y \text{ Conditioned on } X \Rightarrow Y|X = x$$

$$\text{Linear Case: } E(Y|X = x) = x\beta + \boldsymbol{E(U|X = x)}; E(\epsilon_Y|X = x) = 0.$$

Fixing: *causal* exercise that *hypothetically* assigns values to inputs of the autonomous equation we analyze.

$$Y \text{ when } X \text{ is fixed at } x \Rightarrow Y(x) = f_Y(x, U, \epsilon_Y)$$

$$\text{Linear Case: } E(Y(x)) = x\beta + \boldsymbol{E(U)}; E(\epsilon_Y) = 0.$$

Average Causal Effects:

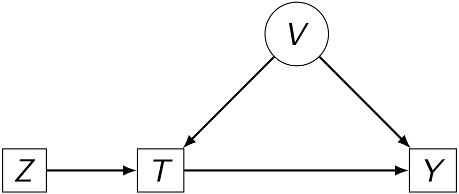
X is *fixed* at x, x' :

$$ATE = \mathrm{E}(Y(x)) - \mathrm{E}(Y(x'))$$

Part 2: A Causal Model – Bayesian Networks

- Bayesian Networks conveniently represents a causal model as a Directed Acyclic Graph (DAG).
- See Lauritzen (1996) for the theory of Bayesian Networks.
- Causal links are directed arrows,
- observed variables displayed as squares and unobserved variables by circles.

Figure 1: DAG for the IV Model



- **LMC** implies: $Y \perp\!\!\!\perp Z | V, T$ and under fixing, $Y(t) \perp\!\!\!\perp T | V$
- Thus, V is a matching variable
- It generates a matching conditional independence relation.
- Note $Y(t)$ can be random variable because of ϵ_Y which is independent of T and V.

Part 2: A Causal Model – Theoretical Benefits

1. Causal directions and counterfactual outcomes are clearly defined,
2. Allows for the investigation of complex causal models.
3. Allows for the definition and examination of unobserved confounding variables.
4. Allows for the precise assumptions regarding the interaction between unobserved confounding variables and observed variables.

Part 2: A Causal Model – Theoretical Benefits

In the language of potential outcomes,
statistical independence relationships among variables are assumed.
In a causal model,
independence relationships come as a consequence of the causal relationships of the model.

Part 2: A Causal Model – Reexamining IV Model

- Generalized Roy Model (Heckman and Vytlacil, 2005) is based on the IV equations
- Under two additional assumptions:

1. the treatment is binary, that is, $\mathrm{supp}(T) = \{0, 1\}$
2. Causal function $T = f_T(Z, V)$
3. Assumption: $T = f_T(Z, V)$ is governed by a separable equation on Z and V, that is $T = \mathbf{1}[\phi(Z) \geq \xi(V)]$.

- The separable equation just stated can be conveniently restated as:

$$T = \mathbf{1}[P \geq U] \tag{1}$$

$$\text{where } P = \mathbf{P}(T = 1|Z) \text{ is the propensity score,}$$

$$\text{and } U = F_{\xi(V)}(\xi(V)) \sim Uniform[0,1]$$

$$U = F_{\xi(V)}(\xi(V)) \sim Uniform[0,1]$$

stands for a transformation of the confounding variable V.

Part 2: A Causal Model – Reexamining IV Model

- Separability is **equivalent to the monotonicity of Imbens and Angrist (1994)** (see Vytlacil (2002)).
- Thus, additional structure imposes no cost of generality
- But allows for a far superior causal and interpretive analysis (Heckman and Vytlacil, 2005).
- The marginal treatment effect:

$$\Delta^{MTE}(p) = E(Y(1) - Y(0)|U = p)$$

- The causal effect of T on Y for the population that is indifferent among treatments at a value $U = p \in [0, 1]$.
- The language of counterfactuals does not allow analysts to state or formalize the separability assumption
- Nor allows for MTE

Part 2: A Causal Model – Benefits of the Roy model

- Powerful analysis.
- Range of causal parameters can be expressed as a weighted average of the $\Delta^{MTE}(p)$:

$$ATE = \int_0^1 \Delta^{MTE}(p) W^{ATE}(p) dp; \qquad W^{ATE}(p) = 1$$

$$TT = \int_0^1 \Delta^{MTE}(p) W^{TT}(p) dp; \qquad W^{TT}(p) = \frac{1 - F_P(p)}{\int_0^1 \left(1 - F_P(t)\right) dt}$$

$$TUT = \int_0^1 \Delta^{MTE}(p) W^{TUT}(p) dp; \qquad W^{TUT}(p) = \frac{F_P(p)}{\int_0^1 \left(1 - F_P(t)\right) dt}$$

$$PRTE = \int_0^1 \Delta^{MTE}(p) W^{PRTE}(p) dp; \qquad W^{PRTE}(p) = \frac{F_{P*}(p) - F_P(p)}{\int_0^1 \left(F_{P*}(p) - F_P(p)\right) dt}$$

$$IV = \int_0^1 \Delta^{MTE}(p) W^{IV}(p) dp; \qquad W^{IV}(p) = \frac{\int_p^1 \left(t - E(P)\right) dF_P(t)}{\int_0^1 \left(t - E(P)\right)^2 dF_P(t)}$$

Part 2: A Causal Model – Reexamining the Mediation Model

- Sequential Ignorability based on strong assumptions
  1. No confounders
  2. No unobserved mediator.
- The model just presented is a general model that allows for these sources of confounding variables.
- The three observed variables are the regular treatment status T, mediator M and outcome Y.
- The additional two variables are **unobserved** variables that account for potential confounding effects:
  1. A general *confounder* V is an unobserved exogenous variable that causes T, M and Y.
  2. The *unobserved mediator* U is caused by T and causes observed mediator M.

Part 2: A Causal Model – Reexamining the Mediation Model

- The three observed variables are the regular treatment status T, mediator M and outcome Y.
- The additional two variables are unobserved variables that account for potential confounding effects:
  1. A general *confounder* V is an unobserved exogenous variable that causes T, M and Y.
  2. The *unobserved mediator* U is caused by T and causes observed mediator M.

$$\text{Treatment: } T = f_T(V, \epsilon_T), \tag{2}$$

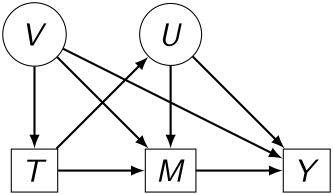
$$\text{Unobserved Mediator: } U = f_U(T, V, \epsilon_U), \tag{3}$$

$$\text{Observed Mediator: } M = f_M(T, U, V, \epsilon_M), \tag{4}$$

$$\text{Outcome: } Y = f_Y(M, U, V, \epsilon_Y) \tag{5}$$

$$\text{Independence: } V, \epsilon_T, \epsilon_U, \epsilon_M, \epsilon_Y. \tag{6}$$

Figure 2: DAG for the Mediation Model with Confounders and Unobserved Mediators



- Sequential Ignorability implies two causal assumptions:
  1. Unobserved confounding V is assumed to be observed (in X);
  2. No Unobserved mediator U causes the mediator M (and outcome Y).
- Very strong faith in quality of available data.

Part 2: A Causal Model – Understanding Sequential Ignorability

- Mediation DAG reveals that Sequential Ignorability assumes that:
  1. the confounding variable V is observed, that is, the pre-treatment variables X; and
  2. that there are no unobserved mediator U.
- Assumption is unappealing
- Solves the identification problem generated by unobserved confounding variables by assuming that they do not exist.
- But additional exogenous variation is needed to solve the general problem.
- What about an IV?

Part 2: A Causal Model – Identification Analysis

- Mediation model is hopelessly unidentified as it stands.
- Both variables T, M are endogenous.
- $T \not\perp\!\!\!\perp (M(t), Y(t'))$ and $M \not\perp\!\!\!\perp Y(m)$.
- One possibility: seek an instrument Z that directly causes T
- Can be used to identify the causal effect of T on M, Y
- Can be used to identify the causal effect of M on Y.
- How? By examining the causal relation of unobserved variables!

Part 2: A Causal Model – Mediation Identification Analysis

Consider the following model:

$$\text{Treatment: } T = f_T(Z, V_T, \epsilon_T), \tag{7}$$

$$\text{Unobserved Mediator: } U = f_U(T, \epsilon_U), \tag{8}$$

$$\text{Observed Mediator: } M = f_M(T, U, V_T, V_Y, \epsilon_M), \tag{9}$$

$$\text{Outcome: } Y = f_Y(M, U, V_Y, \epsilon_Y), \tag{10}$$

$$\text{Independence: } V_T, V_Y, \epsilon_T, \epsilon_U, \epsilon_M, \epsilon_Y. \tag{11}$$

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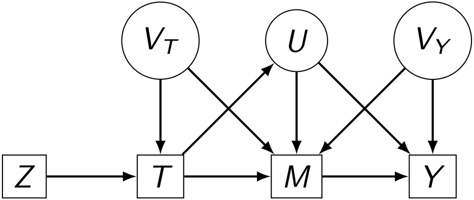
Figure 3: DAG for the Mediation Model with IV and Confounding Variables



- T and M are endogenous.
- $T \perp\!\!\!\perp M(t)$ does not hold due to confounder V_T,
- V_Y and unobserved mediator U invalidate $M \perp\!\!\!\perp Y(m, t)$
- $T \perp\!\!\!\perp Y(t)$ does not hold due to V_T, V_Y.
- Model **still** generates three sets of IV properties! How?

Part 2: A Causal Model – Independence Relations of the Mediation Model

- The following statistical relationships hold in the mediation model (7)–(10):

Targeted Causal Relation	IV Relevance		Exclusion Restrictions
Property 1	for $T \rightarrow Y$	$Z \not\perp\!\!\!\perp T$	$Z \perp\!\!\!\perp Y(t)$
Property 2	for $T \rightarrow M$	$Z \not\perp\!\!\!\perp T$	$Z \perp\!\!\!\perp M(t)$
Property 3	for $M \rightarrow Y$	$Z \not\perp\!\!\!\perp M \| T$	$Z \perp\!\!\!\perp Y(m) \| T$

- **Property 3 is nonstandard.** Prove it!

Part 2: A Causal Model – Properties of the Mediation Model

- Property 1 implies that Z is an instrument for the causal relation of T on Y.
- Property 2 states that Z is also an instrument for T on M.
- Relationships arise from the fact that Z direct causes T
- And does not correlate with the unobserved confounders V_T and V_M.
- Z plays the role of an IV for T
- And observed variables M and Y are outcomes

Part 2: A Causal Model – Properties of the Mediation Model

- Property 3: $Z \not\perp\!\!\!\perp M|T$ and $Z \perp\!\!\!\perp Y(m)|T$
- Z is an instrument for the causal relation of M on Y **IF** (and only if) conditioned on T.
- $Z \perp\!\!\!\perp Y(m)|T$ holds, but $Z \perp\!\!\!\perp Y(m)$ does not.
- Arises from the fact that T is caused by both Z and V_T and because $V_T \perp\!\!\!\perp Z$.
- Conditioning on T induces correlation between Z and V_T.
- But V_T causes M and does not (directly) cause Y.
- Thus, conditioned on T, Z affects M (via V_T)
- And does not affect Y by any channel other than M.

Part 2: A Causal Model – Properties of the Mediation Model

- Assumption on the causal relationships among unobserved variables generates identification

One instrument used to evaluate THREE causal effects!

$$E(Y(m) - Y(m')) \, , \; E(Y(t) - Y(t')) \, , \; E(M(t) - M(t'))$$

Part 2: A Causal Model – A Disagreement
Statistical Tools Versus Causal Analysis

- A causal model allows to clarify a major source of confusion
- Statistical tools are not well-suited to examine causality
- Fixing not defined (it is outside of standard statistics) (Pearl, 2009b; Spirtes et al., 2000)
- Fixing differs from conditioning.
- Conditioning affects the distribution of all variables
- Fixing only affects the distribution of the variables caused by the variable being fixed.
- Fixing has direction while conditioning does not.
- How to solve this problem?

Problem: Causal Concepts are not Well-defined in Statistics

Causal Inference	**Statistical Models**
Directional	Lacks directionality
Counterfactual	Correlational
Fixing	Conditioning
statistical tools do not apply	statistical tools apply

1. **Fixing:** *causal* operation that assigns values to the inputs of structural equations associated to the variable we fix upon.
2. **Conditioning:** *Statistical* exercise that considers the dependence structure of the data generating process.

Problem: Causal Concepts are not Well-defined in Statistics of Potential Outcomes

Some Solutions in the Literature

1. Heckman & Pinto Hypothetical Model.
2. Pearl's do-calculus.

Fixing is a Causal (not statistical) Operation

- **Problem:** Fixing is a Causal Operation defined **Outside** of standard statistics.
- **Comprehension:** Its justification/representation does not follow from standard statistical arguments.
- **Consequence:** Frequent source of **confusion** in statistical discussions.
- **Question:** How can we make statistics converse with causality?

Part 3: The Hypothetical Model – Making Statistics converse with Causality

- Selected Literature
    - Pearl (2009a)
      Causal Inference in Statistics: An Overview
    - Heckman and Pinto (2015b)
      Causal Analysis after Haavelmo
    - Chalak and White (2011)
      An Extended Class of Instrumental Variables for the Estimation of Causal Effects
    - Chalak and White (2012)
      Identification and Identification Failure for Treatment Effects Using Structural Systems

Frisch and Haavemo Contributions to Causality:

1. **Frisch Motto**: "Causality is in the Mind "
2. **Formalized** Yule's credo: *Correlation is not causation.*
3. **Laid** the foundations for *counterfactual* policy analysis.
4. **Distinguished** *fixing* (causal operation) from *conditioning* (statistical operation).
5. **Clarified** *definition* of causal parameters from their *identification* from data.
6. **Developed** Marshall's notion of *ceteris paribus* (1890).

Most Important

Causal effects are determined by the impact of **hypothetical** manipulations of an input on an output.

Key Causal Insights:

1. What are Causal Effects?
   - **Not** empirical descriptions of **actual worlds**,
   - **But** descriptions of **hypothetical worlds**.
2. How are they obtained?
   - **Through** Models – idealized thought experiments.
   - **By** varying–**hypothetically**–the inputs causing outcomes.
3. But what are models?
   - Frameworks defining **causal relationships** among variables.
   - Based on **scientific knowledge**.

Revisiting Ideas on Causality

- **Insight:** express causality through a *hypothetical model* assigning independent variation to inputs determining outcomes.
- **Data:** generated by an empirical model that shares some features with the hypothetical model.
- **Identification:** relies on evaluating causal parameters defined in the *hypothetical model* using data generated by the *empirical model*.
- **Tools:** exploit the language of Directed Acyclic Graphs (DAG).
- **Comparison:** how a causal framework inspired by Haavelmo's ideas relates to other approaches (Pearl, 2009b) .

Introducing the Hypothetical Model: Our Tasks

1. Present **New Causal framework** inspired by the hypothetical variation of inputs.
   - *Hypothetical Model* for Examining Causality
   - Benefits of a *Hypothetical Model*
   - Identification: connecting *Hypothetical* and *Empirical* Models.
2. **Compare** Hypothetical Model approach with **Do-calculus**.
   - Hypothetical Model : relies on standard statistical tools (Allows Statistics to Converse with Causality)
   - Do-calculus: requires *ad hoc* graphical/statistical/probability tools [will leave as an exercise]

How to Connecting Statistics with Causality? Properties the Hypothetical Model

1. **New Model:** Define a Hypothetical Model with desired independent variation of inputs.
2. **Usage:** Hypothetical Model allows us to examine causality.
3. **Characteristic:** usual statistical tools apply.
4. **Benefit:** Fixing translates to statistical conditioning.
5. **Formalizes** the motto "*Causality is in the Mind*".
6. **Clarifies** the notion of identification.

Identification:

Expresses causal parameters defined in the hypothetical model using observed probabilities of the empirical model that governs the data generating process.

Defining The Hypothetical Model

Formalizing Causality Insight

Empirical Model: Governs the data generating process.
Hypothetical Model: Abstract model used to examine causality.

- The hypothetical model stems from the following properties:
  1. **Same** set of structural equations as the empirical model.
  2. **Appends hypothetical variables that we fix**.
  3. **Hypothetical variable** not caused by any other variable.
  4. **Replaces** the input variables we seek to fix by the hypothetical variable, which conceptually can be fixed.

Hypothetical Variables

- **Hypothetical Variable:** $\tilde{X}$ replaces the X-inputs of structural equations.
- **Characteristic:** $\tilde{X}$ is an **external variable**, i.e., no parents.
- **Usage:** hypothetical variable $\tilde{X}$ enables analysts to examine fixing using standard tools of probability.
- **Notation:**
  1. **Empirical Model:** $(\mathcal{T}_{\text{E}}, Pa_{\text{E}}, D_{\text{E}}, Ch_{\text{E}}, \mathbf{P}_{\text{E}}, E_{\text{E}})$ denote– variable set, parents, descendants, Children, Probability and Expectation of the empirical model.
  2. **Hypothetical Model:** $(\mathcal{T}_{\text{H}}, Pa_{\text{H}}, D_{\text{H}}, Ch_{\text{H}}, \mathbf{P}_{\text{H}}, E_{\text{H}})$ denote – variable set,parents, descendants, Children, Probability and Expectation of the hypothetical model.

The Hypothetical Model and the Data Generating Process

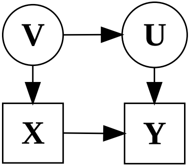
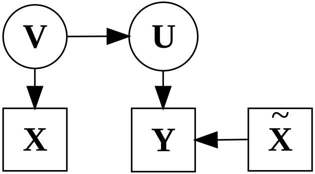
The hypothetical model is not a speculative departure from the empirical data-generating process but an **expanded** version of it.

- Expands the number of random variables in the model.
- Allows for thought experiments.

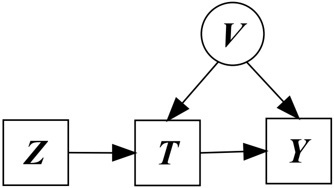
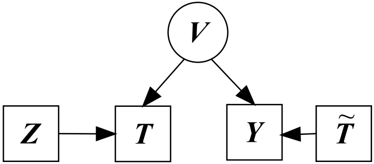
Example of the Hypothetical Model for fixing X

The Associated Hypothetical Model

$$Y = f_Y(\tilde{X}, U, \epsilon_Y); X = f_X(V, \epsilon_X); U = f_U(V, \epsilon_U); V = f_V(\epsilon_V).$$

Empirical Model	Hypothetical Model
	
LMC	LMC
$Y \perp\!\!\!\perp V \| (U, X)$ $U \perp\!\!\!\perp X \| V$	$Y \perp\!\!\!\perp (X, V) \| (U, \tilde{X})$ $U \perp\!\!\!\perp (X, \tilde{X}) \| V$ $\tilde{X} \perp\!\!\!\perp (U, V, X)$ $X \perp\!\!\!\perp (U, Y, \tilde{X}) \| V$

Example of the Standard IV Model : Empirical and Hypothetical Models

Empirical IV Model	Hypothetical IV Model
	

Variable Set	$\mathcal{B}_e = \{V, Z, T, Y\}$	$\mathcal{B}_h = \{V, Z, T, Y, \widetilde{T}\}$
Model Equations	$V = f_V(\epsilon_V)$	$V = f_V(\epsilon_V)$
	$Z = f_Z(\epsilon_Z)$	$Z = f_Z(\epsilon_Z)$
	$T = f_T(Z, V, \epsilon_T)$	$T = f_T(Z, V, \epsilon_T)$
	$Y = f_T(T, V, \epsilon_Y)$	$Y = f_T(\widetilde{T}, V, \epsilon_Y)$

- V is an unobserved vector that generates bias.

Models for Mediation Analysis

1. Empirical Model

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & M \arrow[r] & Y \end{tikzcd}$$

2. Total Effect of X on Y

$$\begin{tikzcd} X & M \arrow[r] & Y \\ & & \tilde{X} \arrow[ul] \arrow[u] \end{tikzcd}$$

3. Indirect Effect of X on Y

$$\begin{tikzcd} X \arrow[rr, bend left] & M \arrow[r] & Y \\ & \tilde{X} \arrow[u] & \end{tikzcd}$$

4. Direct Effect of X on Y for Observed X

$$\begin{tikzcd} X \arrow[r] & M \arrow[r] & Y \\ & & \tilde{X} \arrow[u] \end{tikzcd}$$

Benefits of a Hypothetical Model

- **Formalizes** Haavelmo's insight of Hypothetical variation;
- **Statistical Analysis:** Bayesian Network Tools apply (Local Markov Condition; Graphoid Axioms);
- **Clarifies** the definition of causal parameters;
  1. Causal parameters are defined under the hypothetical model;
  2. Observed data is generated through empirical model;
- **Distinguish** definition from identification;
  1. Identification requires us to **connect** the hypothetical and empirical models.
  2. Allows us to evaluate causal parameters defined in the Hypothetical model using data generated by the Empirical Model.

Benefits of a Hypothetical Model

1. **Versatility:** Targets causal links, not variables.
2. **Simplicity:** Does not require to define any statistical operation outside the realm of standard statistics.
3. **Completeness:** Automatically generates Pearl's do-calculus when it applies (Pinto 2013).

Most Important

Fixing in the empirical model is translated to statistical conditioning in the hypothetical model:

$$\underbrace{E_{\mathrm{E}}(Y(t))}_{\text{Causal Operation Empirical Model}} = \underbrace{E_{\mathrm{H}}(Y|\tilde{T}=t)}_{\text{Statistical Operation Hypothetical Model}}$$

Causality Now Within the Realm of Statistics/Probability!

Some Remarks on Our Causal Framework

- We do not a priori impose statistical relationships among variables, but only causal relationships among variables.
- Statistical relationships come as a **consequence** of applying LMC and GA to models.
- Causal effects are associated with the causal links replaced by hypothetical variables.
- Our framework allows for multiple hypothetical variables associated with **distinct causal effects** (such as **mediation**).
- Easy Manipulation:

$$TT = E_{\mathrm{H}}(Y|\tilde{T} = 1, T = 1) - E_{\mathrm{H}}(Y|\tilde{T} = 0, T = 1)$$
$$TUT = E_{\mathrm{H}}(Y|\tilde{T} = 1, T = 0) - E_{\mathrm{H}}(Y|\tilde{T} = 0, T = 0)$$

Identification

- **Hypothetical Model** allows analysts to define and examine causal parameters.
- **Empirical Model** generates observed/unobserved data;

Clarity: What is Identification?

The capacity to express causal parameters of the hypothetical model through observed probabilities in the empirical model.

Tools: What does Identification requires?

Probability laws that connect *Hypothetical* and *Empirical* Models.

Part 3: The Hypothetical Model versus Empirical Model

- Distribution of variables in hypothetical/empirical models **differs**.
    - $\mathbf{P}_E$ for the probabilities of the empirical model
    - $\mathbf{P}_H$ for the probabilities of the hypothetical model

Counterfactuals obtained by simple conditioning!

$$\mathbf{P}_E(Y(t)) = \mathbf{P}_H(Y|\widetilde{T} = t).$$

Causal parameters are defined as conditional probabilities in the hypothetical model $\mathbf{P}_H$ *and are said to be identified if those can be expressed in terms of the distribution of observed data generated by the empirical model* $\mathbf{P}_E$*.*

Identification

Identification depends on bridging the probabilities of empirical and hypothetical models.

How to connect Empirical and Hypothetical Models?

1. By **sharing** the same error terms and structural equations, conditional probabilities of some variables of the hypothetical model can be written in terms of the probabilities of the empirical model.
2. Conditional **independence properties** of the variables in the hypothetical model also allow for connecting hypothetical and empirical models.
3. Probability Laws are **not** assumed/defined
4. **But** come as a consequence of standard theory of statistic/probability

Three Laws Connecting Hypothetical and Empirical Models (Prove on next homework: 15 bonus points)

1. **L-1:** Let W, Z be any disjoint set of variables in $\mathcal{T}_\mathrm{E} \setminus D_\mathrm{H}(\tilde{X})$ then:

$$\mathbf{P}_\mathrm{H}(W|Z) = \mathbf{P}_\mathrm{H}(W|Z, \tilde{X}) = \mathbf{P}_\mathrm{E}(W|Z) \,\forall\, \{W, Z\} \subset \mathcal{T}_\mathrm{E} \setminus D_\mathrm{H}(\tilde{X}).$$

2. **T-1:** Let W, Z be any disjoint set of variables in $\mathcal{T}_\mathrm{E}$ then:

$$\mathbf{P}_\mathrm{H}(W|Z, X = x, \tilde{X} = x) = \mathbf{P}_\mathrm{E}(W|Z, X = x) \,\forall\, \{W, Z\} \subset \mathcal{T}_\mathrm{E}.$$

3. **Matching:** Let Z, W be any disjoint set of variables in $\mathcal{T}_\mathrm{E}$ such that, in the hypothetical model, $X \perp\!\!\!\perp W|(Z, \tilde{X})$, then

$$\mathbf{P}_\mathrm{H}(W|Z, \tilde{X} = x) = \mathbf{P}_\mathrm{E}(W|Z, X = x),$$

Bonus

C-1: Let $\tilde{X}$ be uniformly distributed in the support of X and let W, Z be any disjoint set of variables in $\mathcal{T}_\mathrm{E}$ then:

$$\mathbf{P}_\mathrm{H}(W|Z, X = \tilde{X}) = \mathbf{P}_\mathrm{E}(W|Z) \,\forall\, \{W, Z\} \subset \mathcal{T}_\mathrm{E}.$$

Some Intuition on Connecting Hypothetical and Empirical Models

Same error terms and structural equations generate:

1. Distribution of **non-children** of $\tilde{X}$ (i.e. $Q \in \mathcal{T}_{\mathrm{E}} \setminus Ch_{\mathrm{H}}(\tilde{X})$) are the same in hypothetical and empirical models.

$$\mathbf{P}_{\mathrm{H}}(Q|Pa_{\mathrm{H}}(Q)) = \mathbf{P}_{\mathrm{E}}(Q|Pa_{\mathrm{E}}(Q)), Q\epsilon(\mathcal{T}_E \setminus Ch_{\mathrm{H}}(\tilde{X}))$$

2. Distribution of **children** of $\tilde{X}$ (i.e. $Q \in Ch_{\mathrm{H}}(\tilde{X})$) are the same in hypothetical and empirical models whenever X and $\tilde{X}$ are conditioned on x.

$$\mathbf{P}_{\mathrm{H}}(Q|Pa_{\mathrm{H}}(Q) \setminus \{\tilde{X}\}, \tilde{X} = x) = \mathbf{P}_{\mathrm{E}}(Q|Pa_{\mathrm{E}}(Q) \setminus \{X\}, X = x).$$

Connecting Empirical and Hypothetical Models

Moreover, we prove that:

1. Distribution of non-descendants of $\tilde{X}$ are the same in hypothetical and empirical models.
2. Distribution of variables conditional on X and $\tilde{X}$ at the same value of x in empirical model and in the hypothetical model is the same as the distribution of variables conditional on $X = x$ in the empirical model.
3. Distribution of an outcome $Y \in \mathcal{T}_{\text{E}}$ when X is *fixed* at x is the same as the distribution of Y conditional on $\tilde{X} = x$ in $Y \in \mathcal{T}_{\text{H}}$.

T–2 : L–1, T–1, and Matching Can Be Rewritten by

- Let (Y, V) be any two disjoint sets of variables in $\mathcal{T}_{\mathrm{E}}$, then:
  1. $\mathbf{P}_{\mathrm{H}}(Y|Pa_{\mathrm{H}}(Y)) = \mathbf{P}_{\mathrm{E}}(Y|Pa_{\mathrm{E}}(Y)) \; \forall \; Y \in \mathcal{T}_{\mathrm{E}} \setminus Ch_{\mathrm{H}}(\widetilde{T})$,
  2. $\mathbf{P}_{\mathrm{H}}(Y|Pa_{\mathrm{H}}(Y), \widetilde{T} = t) = \mathbf{P}_{\mathrm{E}}(Y|Pa_{\mathrm{E}}(Y), T = t) \; \forall \; Y \in Ch_{\mathrm{H}}(\widetilde{T})$.
  3. $\mathbf{P}_{\mathrm{H}}(Y|V, T = t, \widetilde{T} = t) = \mathbf{P}_{\mathrm{E}}(Y|V, T = t)$;
  4. $Y, V \notin D_{\mathrm{H}}(\widetilde{T}) \Rightarrow \mathbf{P}_{\mathrm{H}}(Y|V) = \mathbf{P}_{\mathrm{H}}(Y|V, \widetilde{T}) = \mathbf{P}_{\mathrm{E}}(Y|V)$; .
  5. $T \perp\!\!\!\perp Y|(V, \widetilde{T}) \Rightarrow \mathbf{P}_{\mathrm{H}}(Y|V, \widetilde{T} = t) = \mathbf{P}_{\mathrm{E}}(Y|V, T = t)$.
  6. $\widetilde{T} \sim \mathrm{Unif}(\mathrm{supp}(T)) \Rightarrow \mathbf{P}_{\mathrm{H}}(Y|V, T = \widetilde{T}) = \mathbf{P}_{\mathrm{E}}(Y|V)$;

Bonus

Prove.

Intuition of T–2

- **Item (1):** the distribution of variables not directly caused by the hypothetical variable remains the same in both the hypothetical and the empirical models when conditioned on their parents.
- **Item (2):** Children of $\widetilde{T}$ have the same distribution in both models when conditioned on the same parents.
- **Item (3):** variables in both models share the same conditional distribution when the hypothetical variable $\tilde{T}$ and the variable being fixed T take the same value t.
- **Item (4):** hypothetical variable does not affect the distribution of its non-descendants.
- **Item (5):** refers to the method of matching (Heckman, 2008; Rosenbaum and Rubin, 1983). If T and Y are independent conditioned on V and $\widetilde{T}$, then we can assess the causal effect of T on Y by conditioning on V.

Matching: A Consequence of Connecting Empirical and Hypothetical Models

Matching Property

If there exist a variable V not caused by $\tilde{X}$, such that, $X \perp\!\!\!\perp Y|V, \tilde{X}$, then $E_{\mathrm{H}}(Y|V, \tilde{X} = x)$ under the hypothetical model is equal to $E_{\mathrm{H}}(Y|V, X = x)$ under empirical model.

- **Obs:** LMC for the hypothetical model generates $X \perp\!\!\!\perp Y|V, \tilde{X}$.
- Thus, by matching, treatment effects $E_{\mathrm{E}}(Y(x))$ can be obtained by:

$$E_{\mathrm{E}}(Y(x)) = \underbrace{\int E_{\mathrm{H}}(Y|V = v, \tilde{X} = x) dF_V(v)}_{\text{In Hypothetical Model}}$$

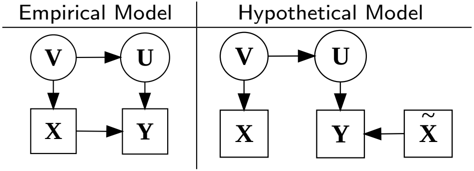
$$= \underbrace{\int E_{\mathrm{E}}(Y|V = v, X = x) dF_V(v)}$$

How to use this Causal Framework?
Rules of Engagement

1. **Define** the Empirical and associated Hypothetical model;
2. **Hypothetical Model:** Generate statistical relationships (LMC,GA);
3. **Express** $\mathrm{P}_{\mathrm{H}}(Y|\widetilde{X})$ in terms of other variables.
4. **Connect** this expression to the Empirical model (T–2).

First Example

1. Defining Hypothetical and Empirical Models



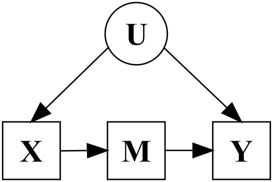
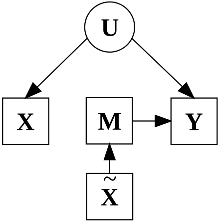
2. Useful Hyp. Model C.I. Relationships: $X \perp\!\!\!\perp Y|(V, \tilde{X})$, $\tilde{X} \perp\!\!\!\perp (U, V, X)$
3. Express $\mathbf{P}_{\mathrm{H}}(Y|\widetilde{X})$ in terms of other variables:

$$\begin{aligned}
\mathbf{P}_{\mathrm{H}}(Y|\widetilde{X}=x) &= \sum_{V} \mathbf{P}_{\mathrm{H}}(Y|\widetilde{X}=x, V)\,\mathbf{P}_{\mathrm{H}}(V|\widetilde{X}=x) \\
&= \sum_{V} \mathbf{P}_{\mathrm{H}}(Y|X=x, \widetilde{X}=x, V)\,\mathbf{P}_{\mathrm{H}}(V) \text{ By C.I.}
\end{aligned}$$

4. Map into the Empirical model:

$$\begin{aligned}
\mathbf{P}_{\mathrm{H}}(Y|\widetilde{X}=x) &= \sum_{V} \mathbf{P}_{\mathrm{H}}(Y|X=x, \widetilde{X}=x, V)\,\mathbf{P}_{\mathrm{H}}(V) \\
&= \sum_{V} \underbrace{\mathbf{P}_{\mathrm{E}}(Y|X=x, V)}_{\text{Item (3) of T-2}} \;\; \underbrace{\mathbf{P}_{\mathrm{E}}(V)}_{\text{Item (1) of T-2}}
\end{aligned}$$

Second Example : The Front-door Model

Empirical Front-door Model	Hypothetical Front-door Model
	
$Pa(U) = \varnothing,$ $Pa(X) = \{U\}$ $Pa(M) = \{X\}$ $Pa(Y) = \{M, U\}$	$Pa(U) = Pa(\tilde{X}) = \varnothing,$ $Pa(X) = \{U\}$ $Pa(M) = \{\tilde{X}\}$ $Pa(Y) = \{M, U\}$

L-2: In the Front-Door hypothetical model:

1. $Y \perp\!\!\!\perp \tilde{X} | M,$
2. $X \perp\!\!\!\perp M$, and
3. $Y \perp\!\!\!\perp \tilde{X} | (M, X)$

Lemma 1

In the Front-Door hypothetical model,

(1) $Y \perp\!\!\!\perp \tilde{X}|M$, *(2)* $X \perp\!\!\!\perp M$, *and (3)* $Y \perp\!\!\!\perp \tilde{X}|(M, X)$

Proof:

1. By LMC for X, we obtain $(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$.
2. By LMC for Y we obtain $Y \perp\!\!\!\perp (X, \tilde{X})|(M, U)$.
3. By Contraction applied to $(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$ and $Y \perp\!\!\!\perp (X, \tilde{X})|(M, U)$ we obtain $(Y, X) \perp\!\!\!\perp \tilde{X}|(M, U)$.
4. By LMC for U we obtain $(M, \tilde{X}) \perp\!\!\!\perp U$.
5. By Contraction applied to $(M, \tilde{X}) \perp\!\!\!\perp U$ and$(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$ we obtain$(X, U) \perp\!\!\!\perp (M, \tilde{X})$.
6. By Contraction on $(Y, X) \perp\!\!\!\perp \tilde{X}|(M, U)$ and $(M, \tilde{X}) \perp\!\!\!\perp U$ we obtain $(Y, X, U) \perp\!\!\!\perp \tilde{X}|M$.
7. Relationships follow from Weak Union and Decomposition.

Using the Hypothetical Model Framework (Front-door)

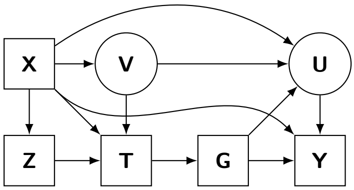
$$
\begin{aligned}
&\mathbf{P}_{\mathrm{H}}(Y|\tilde{X}=x) \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \mathbf{P}_{\mathrm{H}}(Y|M=m,\tilde{X}=x)\,\mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \quad \text{by L.I.E.} \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \mathbf{P}_{\mathrm{H}}(Y|M=m)\,\mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \quad \text{by } Y \perp\!\!\!\perp \tilde{X}|M \text{ of } \textbf{L-2} \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \left(\sum_{x' \in \mathrm{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x'|M=m) \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \left(\sum_{x' \in \mathrm{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x') \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \left(\sum_{x' \in \mathrm{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',\tilde{X}=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x') \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad = \sum_{m \in \mathrm{supp}(M)} \left(\sum_{x' \in \mathrm{supp}(X)} \underbrace{\mathbf{P}_{\mathrm{E}}(Y|M,X=x')}_{\text{by } \textbf{T-1}}\,\underbrace{\mathbf{P}_{\mathrm{E}}(X=x')}_{\text{by } \textbf{L-1}} \right) \underbrace{\mathbf{P}_{\mathrm{E}}(M=m|X=x)}_{\text{by Matching}}.
\end{aligned}
$$

- The second equality from (1) $Y \perp\!\!\!\perp \tilde{X} | M$ of **L-2**.
- The fourth equality from (2) $X \perp\!\!\!\perp M$ of **L-2**.
- The fifth equality from (3) $Y \perp\!\!\!\perp \tilde{X} | (M, X)$ of **L-2**.

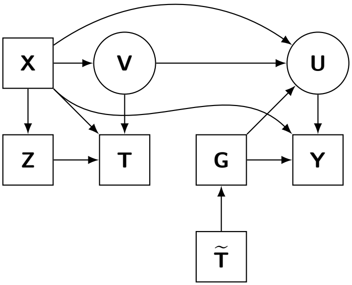
Third Example

1. Defining Hypothetical and Empirical Models

Empirical Causal Model



Hypothetical Causal Model



2. Useful Hypothetical Model Conditional Independence Relationships:

$$Y \perp\!\!\!\perp \widetilde{T} | (G, X), \quad T \perp\!\!\!\perp G | X, \quad Y \perp\!\!\!\perp \widetilde{T} | (G, T), \quad \widetilde{T} \perp\!\!\!\perp X$$

Third Example

3. Express $\mathbf{P}_{\mathrm{H}}(Y|\widetilde{T}=t)$ in terms of other variables:

$$\mathbf{P}_{\mathrm{H}}(Y|\widetilde{T}=t) =$$

$$= \sum_{x\in \mathrm{supp}(X)} \sum_{g\in \mathrm{supp}(G)} \left(\sum_{t'\in \mathrm{supp}(T)} Pr_{\mathrm{H}}(Y|T=t', \tilde{T}=t', G=g, X=x) Pr_{\mathrm{H}}(T=t'|X=x) \right) \times$$

$$\times \left(Pr_{\mathrm{H}}(G=g|\tilde{T}=t) Pr_{\mathrm{H}}(X=x) \right)$$

4. **Identification:** Map into the **Observed** Quantities of the Empirical model:

$$\mathbf{P}_{\mathrm{H}}(Y|\widetilde{T}=t) =$$

$$= \sum_{x\in \mathrm{supp}(X)} \sum_{g\in \mathrm{supp}(G)} \left(\sum_{t'\in \mathrm{supp}(T)} \mathbf{P}_{\mathrm{H}}(Y|T=t', \tilde{T}=t', G=g, X=x)\, \mathbf{P}_{\mathrm{H}}(T=t'|X=x) \right) \times$$

$$\times \left(\mathbf{P}_{\mathrm{H}}(G=g|\tilde{T}=t) Pr_{\mathrm{H}}(X=x) \right)$$

$$= \sum_{x\in \mathrm{supp}(X)} \sum_{g\in \mathrm{supp}(G)} \left(\sum_{t'\in \mathrm{supp}(T)} \underbrace{\mathbf{P}_{\mathrm{E}}(Y|T=t', G=g, X=x)}_{\text{Item (3) of T–2}} \underbrace{\mathbf{P}_{\mathrm{E}}(T=t'|X=x)}_{\text{Item (4) of T–2}} \right) \times$$

$$\times \left(\underbrace{\mathbf{P}_{\mathrm{E}}(G=g|T=t)}_{\text{Item (2) of T–2}} \; \underbrace{\mathbf{P}_{\mathrm{E}}(X=x)}_{\text{Item (1) of T–2}} \right)$$

Part 3: The Hypothetical Model – Two Useful Conditions

Only two conditions **suffice** to investigate the identification of causal parameters!

Theorem 2

For any disjoint set of variables Y, W in $\mathcal{B}_e$, we have that:

$$Y \perp\!\!\!\perp \widetilde{T} | (T, W) \Rightarrow \mathbf{P}_H(Y|\widetilde{T}, T = t', W) = \mathbf{P}_H(Y|T = t', W) = \mathbf{P}_E(Y|T = t', W)$$
$$Y \perp\!\!\!\perp T | (\widetilde{T}, W) \Rightarrow \mathbf{P}_H(Y|\widetilde{T} = t, T, W) = \mathbf{P}_H(Y|\widetilde{T} = t, W) = \mathbf{P}_E(Y|T = t, W)$$

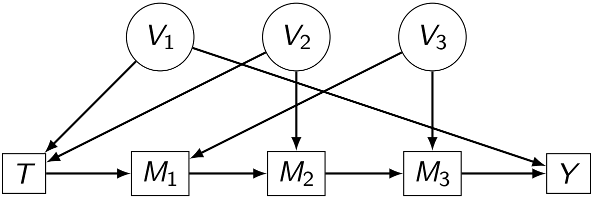
If $Y \perp\!\!\!\perp \widetilde{T} | (T, W)$ or $Y \perp\!\!\!\perp T | (\widetilde{T}, W)$ occurs in the hypothetical model, then we are able to equate variable distributions of the hypothetical and empirical models!

Part 3: Third Example

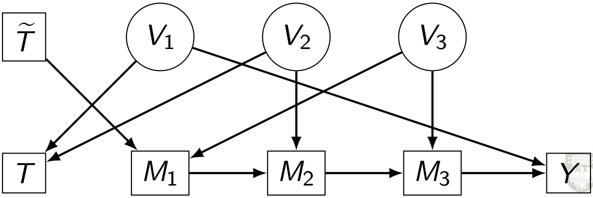
Empirical Model Observed Variables	Hypothetical Model Observed Variables
$T = f_T(V_1, V_2, \epsilon_T)$ $M_1 = f_{M_1}(V_3, T, \epsilon_{M_1})$ $M_2 = f_{M_2}(V_2, M_1, \epsilon_{M_2})$ $M_3 = f_{M_3}(V_3, M_2, \epsilon_{M_3})$ $Y = f_Y(V_1, M_3, \epsilon_Y)$	$T = f_T(V_1, V_2, \epsilon_T)$ $M_1 = f_{M_1}(V_3, \tilde{T}, \epsilon_{M_1})$ $M_2 = f_{M_2}(V_2, M_1, \epsilon_{M_2})$ $M_3 = f_{M_3}(V_3, M_2, \epsilon_{M_3})$ $Y = f_Y(V_1, M_3, \epsilon_Y)$
Exogenous Variables	Exogenous Variables
V_1, V_2, V_3	$V_1, V_2, V_3, \widetilde{T}$

Part 3: The Hypothetical Model – DAG of Example 3

Directed Acyclic Graph of the Empirical Model



Directed Acyclic Graph of the Hypothetical Model



Part 3: The Hypothetical Model – Useful Independence Relationships

In order to identify the causal effect of T on Y, we seek for conditional independence relationships in the hypothetical model that comply with the statements of Theorem 2. Those are the conditional independence relationships (12)–(16) below. For now, we simply state that the following conditional independence relation hold for the hypothetical model:

$$Y \perp\!\!\!\perp \widetilde{T}|(T, M_3, M_2, M_1) \quad (12)$$

$$M_3 \perp\!\!\!\perp T|(M_1, M_2, \widetilde{T}) \quad (13)$$

$$M_2 \perp\!\!\!\perp \widetilde{T}|(T, M_1) \quad (14)$$

$$M_1 \perp\!\!\!\perp T|\widetilde{T} \quad (15)$$

$$T \perp\!\!\!\perp \widetilde{T} \quad (16)$$

Part 3: The Hypothetical Model – Basic Definitions

For sake of notational simplicity, let's consider that all variables are discrete. It is useful to show how Relationships (12)–(16) can be used to factorize the joint distribution of $P(Y, M_3, M_2, M_1, T|\widetilde{T})$:

$$P_h(Y, M_3, M_2, M_1, T, \widetilde{T}) =$$

$$= P_h(Y|M_3, M_2, M_1, T, \widetilde{T})P_h(M_3|M_2, M_1, T, \widetilde{T})P_h(M_2|M_1, T, \widetilde{T})P_h(M_1|T, \widetilde{T})P_h(T|\widetilde{T}), \tag{17}$$

$$= P_h(Y|M_3, M_2, M_1, T)P_h(M_3|M_2, M_1, \widetilde{T})P_h(M_2|M_1, T)P_h(M_1|\widetilde{T})P_h(T). \tag{18}$$

Factorization (17) always hold.
Factorization (18) uses Relationships (12)–(15)
to eliminate variables T or $\widetilde{T}$ of each term of the factorization (17).
Identification formula comes from applying standard statistical tools.

Part 3: The Hypothetical Model – Basic Definitions

We seek to identify $P_e(Y(t))$, expressed by $P_h(Y|\widetilde{T}=t)$.
Can express $P_h(Y|\widetilde{T}=t)$ through the following sum:

$$
\begin{aligned}
& P_h(Y|\widetilde{T}=t) = \\
&= \sum_{t',m_3,m_2,m_1} P_h(Y|m_3,m_2,m_1,T=t')P_h(m_3|m_2,m_1,\widetilde{T}=t)P_h(m_2|m_1,T=t')P_h(m_1|\widetilde{T}=t)P_h(T=t') \\
&= \sum_{t',m_3,m_2,m_1} P_e(Y|m_3,m_2,m_1,T=t')\mathbf{P_e(m_3|m_2,m_1,T=t)}P_e(m_2|m_1,T=t')\mathbf{P_e(m_1|T=t)}P_e(T=t'),
\end{aligned}
$$

Simply uses the Factorization,
Relationships (12)–(15)
And the mapping theorem 2
to equate hypothetical and empirical probabilities.

1. Pearl's (2000) Do-calculus

Link to Pearl Appendix

2. Conclusion

Examined Haavelmo's fundamental contributions

- **Distinction** between causation and correlation (first formal analysis).
- **Distinguished** definition of causal parameters (though process of creating hypothetical models) from their identification from data.
- **Explained** that causal effects of inputs on outputs are defined under abstract models that assign independent variation to inputs.
- **Clarified** concepts that are still muddled in some quarters of statistics.
- **Formalizes** Frisch's notion that causality is in the mind.

Causal Framework Inspired by Haavelmo's Ideas

- **Contribution:** causal framework inspired by Haavelmo,
- **Introduce:** hypothetical models for examining causal effects,
- **Assigns** independent variation to inputs determining outcomes.
- **Enables** us to discuss causal concepts such as Fixing using an intuitive approach.
- **Fixing** is easily translated to statistical conditioning.
- **Eliminates** the need for additional extra-statistical graphical/ statistical rules to achieve identification (in contrast with the do-calculus).
- **Identification** relies on evaluating causal parameters defined in the *hypothetical model* using data generated by the *empirical model*.
- **Achieved** by applying standard statistical tools to fundamentally recursive Bayesian Networks.

Beyond DAG

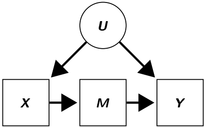
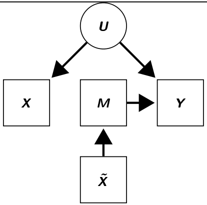
- We discuss the limitations of methods of identification that rely on the fundamentally recursive approach of Directed Acyclic Graphs.
- Haavelmo's framework can be extended to the fundamentally non-recursive framework of the simultaneous equations model without violating autonomy.
- Simultaneous equations are fundamentally non-recursive and falls **outside** of the framework of Bayesian causal nets and DAGs.
- Haavelmo's approach also covers simultaneous causality whereas other frameworks cannot, except through ad hoc rules such as "shutting down" equations;
- Haavelmo's framework allows for a variety of econometric methods can be used to secure identification of this class of models (see, e.g., Matzkin, 2012, 2013.)

Appendix On Do Operators

Comparing Analyses Based on the Do-calculus with those from the Hypothetical Model

- We illustrate the use of the do-calculus and the hypothetical model approaches by identifying the causal effects of a well-known model that Pearl (2009b) calls the "Front-Door model."
- It consists of four variables: (1) an external unobserved variable U; (2) an observed variable X caused by U; (3) an observed variable M caused by X; and (4) an outcome Y caused by U and M.

"Front-Door" Empirical and Hypothetical Models

1. Pearl's "Front-Door" Empirical Model	2. Our Version of the "Front-Door" Hypothetical Model
$\mathcal{T} = \{U, X, M, Y\}$ $\boldsymbol{\epsilon} = \{\epsilon_U, \epsilon_X, \epsilon_M, \epsilon_Y\}$ $Y = f_Y(M, U, \epsilon_Y)$ $X = f_X(U, \epsilon_X)$ $M = f_M(X, \epsilon_M)$ $U = f_U(\epsilon_U)$	$\mathcal{T} = \{U, X, M, Y, \tilde{X}\}$ $\boldsymbol{\epsilon} = \{\epsilon_U, \epsilon_X, \epsilon_M, \epsilon_Y\}$ $Y = f_Y(M, U, \epsilon_Y)$ $X = f_X(U, \epsilon_X)$ $M = f_M(\tilde{X}, \epsilon_M)$ $U = f_U(\epsilon_U)$
 	
$Pa(U) = \varnothing,$ $Pa(X) = \{U\}$ $Pa(M) = \{X\}$ $Pa(Y) = \{M, U\}$	$Pa(U) = Pa(\tilde{X}) = \varnothing,$ $Pa(X) = \{U\}$ $Pa(M) = \{\tilde{X}\}$ $Pa(Y) = \{M, U\}$
$Y \perp\!\!\!\perp X \| (M, U)$ $M \perp\!\!\!\perp U \| X$	$Y \perp\!\!\!\perp (\tilde{X}, X) \| (M, U)$ $M \perp\!\!\!\perp (U, X) \| \tilde{X}$ $X \perp\!\!\!\perp (M, \tilde{X}, Y) \| U$ $U \perp\!\!\!\perp (M, \tilde{X})$ $\tilde{X} \perp\!\!\!\perp (X, U)$
$\mathbf{P}_{\mathrm{E}}(Y, M, X, U) =$ $\mathbf{P}_{\mathrm{E}}(Y \| M, U)\, \mathbf{P}_{\mathrm{E}}(X \| U)\, \mathbf{P}_{\mathrm{E}}(M \| X)\, \mathbf{P}_{\mathrm{E}}(U)$	$\mathbf{P}_{\mathrm{H}}(Y, M, X, U, \tilde{X}) =$ $\mathbf{P}_{\mathrm{H}}(Y \| M, U)\, \mathbf{P}(X \| U)\, \mathbf{P}_{\mathrm{H}}(M \| \tilde{X})\, \mathbf{P}_{\mathrm{H}}(U)\, \mathbf{P}_{\mathrm{H}}(\tilde{X})$
$\mathbf{P}_{\mathrm{E}}(Y, M, U \| do(X) = x) =$ $\mathbf{P}_{\mathrm{E}}(Y \| M, U)\, \mathbf{P}_{\mathrm{E}}(M \| X = x)\, \mathbf{P}_{\mathrm{E}}(U)$	$\mathbf{P}_{\mathrm{H}}(Y, M, U, X \| \tilde{X} = x) =$ $\mathbf{P}_{\mathrm{H}}(Y \| M, U)\, \mathbf{P}(X \| U)\, \mathbf{P}_{\mathrm{H}}(M \| \tilde{X} = x)\, \mathbf{P}_{\mathrm{H}}(U)$

- The do-calculus identifies $\mathbf{P}(Y|do(X))$ through four steps which we now perform.
- Steps 1, 2 and 3 identify $\mathbf{P}(M|do(X))$, $\mathbf{P}(Y|do(M))$ and $\mathbf{P}(Y|M, do(X))$ respectively.

1. Invoking LMC for variable M of DAG $G_{\underline{X}}$, (DAG 1 of Table **??**) generates $X \perp\!\!\!\perp M$. Thus, by Rule 2 of the do-calculus, we obtain $\mathbf{P}(M|do(X)) = \mathbf{P}(M|X)$.

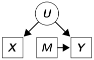
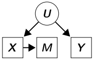
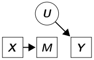
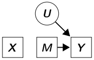
2. Invoking LMC for variable M of DAG $G_{\overline{M}}$, (DAG 1 of Table **??**) generates $X \perp\!\!\!\perp M$. Thus, by Rule 3 of the do-calculus, $\mathbf{P}(X|do(M)) = \mathbf{P}(X)$. In addition, applying LMC for variable M of DAG $G_{\underline{M}}$, (DAG 2 of Table **??**) generates $M \perp\!\!\!\perp Y|X$. Thus, by Rule 2 of do-calculus, $\mathbf{P}(Y|X, do(M)) = \mathbf{P}(Y|X, M)$.

   Therefore
   $$\begin{aligned}\mathbf{P}(Y|do(M)) &= \sum_{x' \in \text{supp}(X)} \mathbf{P}(Y|X = x', do(M))\,\mathbf{P}(X = x'|do(M)) \\ &= \sum_{x' \in \text{supp}(X)} \mathbf{P}(Y|X = x', M)\,\mathbf{P}(X = x'),\end{aligned}$$

   where "supp" means support.

3. Invoking LMC for variable M of DAG $G_{\overline{X},\underline{M}}$, (DAG 3 of Table **??**) generates $Y \perp\!\!\!\perp M|X$.

Do-Calculus and the Front-Door Model

<table>
<tr><td>1. Modified Front-Door Model $G_{\underline{X}} = G_{\overline{M}}$</td><td>2. Modified Front-Door Model $G_{\underline{M}}$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$(Y, M) \perp\!\!\!\perp X \mid U$
$(X, U) \perp\!\!\!\perp M$</td><td>$(X, M) \perp\!\!\!\perp Y \mid U$
$(Y, U) \perp\!\!\!\perp M \mid X$</td></tr>
<tr><td>3. Modified Front-Door Model $G_{\overline{X},\underline{M}}$</td><td>4. Modified Front-Door Model $G_{\overline{X},\overline{M}}$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$(X, M) \perp\!\!\!\perp (Y, U)$</td><td>$(Y, M, U) \perp\!\!\!\perp X$
$U \perp\!\!\!\perp M$</td></tr>
</table>

These rules are intended to supplement standard statistical tools with a new set of “do” operations.

1. Thus, by Rule 2 of the do-calculus, $\mathbf{P}(Y|M, do(X)) = \mathbf{P}(Y|do(M), do(X))$. In addition, applying LMC for variable X of DAG $G_{\overline{X},\overline{M}}$, (DAG 4 of Table **??**) generates $(Y, M, U) \perp\!\!\!\perp X$. By weak union and decomposition, we obtain $Y \perp\!\!\!\perp X|M$. Thus, by Rule 3 of the do-calculus, we obtain that $\mathbf{P}(Y|do(X), do(M)) = \mathbf{P}(Y|do(M))$. Thus $\mathbf{P}(Y|M, do(X)) = \mathbf{P}(Y|do(M), do(X)) = \mathbf{P}(Y|do(M))$.
2. We collect the results from the three previous steps to identify $\mathbf{P}(Y|do(X))$:

$$
\begin{aligned}
&\mathbf{P}(Y|do(X)=x) \\
&= \sum_{m \in \text{supp}(M)} \mathbf{P}(Y|M, do(X)=x)\,\mathbf{P}(M|do(X)=x) \\
&= \sum_{m \in \text{supp}(M)} \underbrace{\mathbf{P}(Y|do(M)=m, do(X)=x)}_{\textbf{Step 3}}\,\mathbf{P}(M=m|do(X)=x) \\
&= \sum_{m \in \text{supp}(M)} \underbrace{\mathbf{P}(Y|do(M)=m)}_{\textbf{Step 3}}\,\mathbf{P}(M=m|do(X)=x) \\
&= \sum_{m \in \text{supp}(M)} \underbrace{\left(\sum_{x' \in \text{supp}(X)} \mathbf{P}(Y|X=x', M)\,\mathbf{P}(X=x')\right)}_{\textbf{Step 2}} \underbrace{\mathbf{P}(M=m|X=x)}_{\textbf{Step 1}}.
\end{aligned}
$$

- We use the do-calculus to identify the desired causal parameter, using the approach inspired by Haavelmo's ideas.
- We replace the relationship of X on M by a hypothetical variable $\tilde{X}$ that causes M.
- We use $\mathbf{P}_{\mathrm{E}}$ to denote the probability of the Front-Door model that generates the data and $\mathbf{P}_{\mathrm{H}}$ for the hypothetical model.

Lemma 3

In the Front-Door hypothetical model,
(1) $Y \perp\!\!\!\perp \tilde{X} | M$,
(2) $X \perp\!\!\!\perp M$, *and*
(3) $Y \perp\!\!\!\perp \tilde{X} | (M, X)$

Proof

By LMC for X, we obtain $(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$. By LMC for Y we obtain $Y \perp\!\!\!\perp (X, \tilde{X})|(M, U)$. By Contraction applied to $(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$ and $Y \perp\!\!\!\perp (X, \tilde{X})|(M, U)$ we obtain $(Y, X) \perp\!\!\!\perp \tilde{X}|(M, U)$. By LMC for U we obtain $(M, \tilde{X}) \perp\!\!\!\perp U$. By Contraction applied to $(M, \tilde{X}) \perp\!\!\!\perp U$ and$(Y, M, \tilde{X}) \perp\!\!\!\perp X|U$ we obtain$(X, U) \perp\!\!\!\perp (M, \tilde{X})$. The second relationship of the Lemma is obtained by Decomposition. In addition, by Contraction on $(Y, X) \perp\!\!\!\perp \tilde{X}|(M, U)$ and $(M, \tilde{X}) \perp\!\!\!\perp U$ we obtain $(Y, X, U) \perp\!\!\!\perp \tilde{X}|M$. The two remaining conditional independence relationships of the Lemma are obtained by Weak Union and Decomposition.

Applying these results,

$$
\begin{aligned}
&\mathbf{P}_{\mathrm{H}}(Y|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \mathbf{P}_{\mathrm{H}}(Y|M=m,\tilde{X}=x)\,\mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \mathbf{P}_{\mathrm{H}}(Y|M=m)\,\mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \left(\sum_{x'\in \operatorname{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x'|M=m) \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \left(\sum_{x'\in \operatorname{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x') \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \left(\sum_{x'\in \operatorname{supp}(X)} \mathbf{P}_{\mathrm{H}}(Y|X=x',\tilde{X}=x',M=m)\,\mathbf{P}_{\mathrm{H}}(X=x') \right) \mathbf{P}_{\mathrm{H}}(M=m|\tilde{X}=x) \\
&\quad= \sum_{m\in \operatorname{supp}(M)} \left(\sum_{x'\in \operatorname{supp}(X)} \underbrace{\mathbf{P}_{\mathrm{E}}(Y|M,X=x')}_{\text{by T1}}\underbrace{\mathbf{P}_{\mathrm{E}}(X=x')}_{\text{by Lemma1}} \right) \underbrace{\mathbf{P}_{\mathrm{E}}(M=m|X=x)}_{\text{by M1}}.
\end{aligned}
$$

- The second equality comes from relationship (1) $Y \perp\!\!\!\perp \tilde{X}|M$ of Lemma 3.
- The fourth equality comes from relationship (2) $X \perp\!\!\!\perp M$ of Lemma 3.
- The fifth equality comes from relationship (3) $Y \perp\!\!\!\perp \tilde{X}|(M, X)$ of Lemma 3.
- The last equality links the distributions of the hypothetical model with the ones of the empirical model.

- The first term uses Theorem 1 to equate $\mathbf{P}_{\mathrm{H}}(Y|X = x', \tilde{X} = x', M = m) = \mathbf{P}_{\mathrm{E}}(Y|M, X = x')$.
- The second term uses the fact that X is not a child of $\tilde{X}$, thus by Lemma, $\mathbf{P}_{\mathrm{H}}(X = x') = \mathbf{P}_{\mathrm{E}}(X = x')$.
- Finally, the last term uses Matching applied to M. Namely, LMC for M generates $M \perp\!\!\!\perp X|\tilde{X}$ in the hypothetical model.
- Then, by Matching, $\mathbf{P}_{\mathrm{H}}(M|\tilde{X} = x) = \mathbf{P}_{\mathrm{E}}(M|X = x)$.

- Both frameworks produce the same final identification formula.
- The methods underlying them differ greatly.
- Concept in the framework inspired by Haavelmo is the notion of a hypothetical model.

The Do-calculus

- **Attempt:** Counterfactual manipulations using the empirical model.
- **Intent:** Expressions obtained from a hypothetical model.
- **Tools:** Uses causal/graphical/statistical rules outside statistics.
- **Fixing:** Uses $do(X) = x$ for fixing X at x in the DAG for all X-inputs (does not allow to target causal links separately).
- **Flexibility:** Does not easily define complex treatments, such as treatment on the treated, i.e., $E_{\mathrm{E}}(Y|X=1,\tilde{X}=1) - E_{\mathrm{E}}(Y|X=1,\tilde{X}=0)$.

In Contrast: Identification using the hypothetical model is transparent and does not require additional causal rules, only standard statistical tools.

Definition the Do-operator (which is Fixing)

The **Do-operator** is based on the **Truncated Factorization** of the probability factor of the fixed variable is deleted:

Let $X \subset V$: Then

$\Pr(V(x) = v) = \Pr(V_1 = v_1, \ldots, V_{m+n} = v_{m+n}, |do(X) = x)$ and:

$$\Pr(V(x) = v) = \begin{cases} \prod_{V_i \in V \setminus X} P(V_i = v_i | pa(V_i)) & \text{if } v \text{ is consistent with } x; \\ 0 & \text{if } v \text{ is inconsistent with } x. \end{cases}$$

Example of the Do-operator

$$\boxed{\mathbf{Z}} \longrightarrow \boxed{\mathbf{X}} \longrightarrow \boxed{\mathbf{Y}}$$

- **Variables:** Y, X, Z
- **Factorization:**

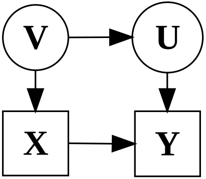
$$\begin{aligned} \Pr(Y, X, Z) &= \Pr(Y|Z, X) \Pr(X|Z) \Pr(Z) \\ &= \Pr(Y|X) \Pr(X|Z) \Pr(Z) \end{aligned}$$

- **Do-operator:** $\Pr(Z, Y|do(X) = x) = \Pr(Y|X = x) \Pr(Z)$
- **Conditional operator:**

$$\begin{aligned} \Pr(Y, Z|X = x) &= \Pr(Y|Z, X = x) \Pr(X|Z, X = x) \Pr(Z|X = x) \\ &= \Pr(Y|X = x) \Pr(Z|X = x) \end{aligned}$$

Do-operator targets variables, not causal links.

Example of the Do-operator



- **Variables:** Y, X, U, V
- **Factorization:** $\Pr(V, U, X, Y) = \Pr(Y|U, X)\Pr(X|V)\Pr(U|V)\Pr(V)$
- **Do-operator:** $\Pr(V, U, Y|do(X) = x) = \Pr(Y|U, X = x)\Pr(U|V)\Pr(V)$
- **Conditional operator:**

$$\begin{aligned}\Pr(V, U, Y|X = x) &= \Pr(Y|U, V, X = x)\Pr(U|V, X = x)\Pr(V|X = x) \\ &= \Pr(Y|U, X = x)\Pr(U|V)\Pr(V|X = x)\end{aligned}$$

Do-operator targets variables, not causal links.

Comparison: Hypothetical Model and Do-Operator

Fixing within Standard Probability Theory

Fixing in the empirical model is translated to statistical conditioning in the hypothetical model:

$$\underbrace{E_{\mathrm{E}}(Y(x))}_{\text{Causal Operation Empirical Model}} \quad = \quad \underbrace{E_{\mathrm{H}}(Y|\tilde{X} = x)}_{\text{Statistical Operation Hypothetical Model}}$$

do-Operator and Statistical Conditioning

Let $\tilde{X}$ be the hypothetical variable in G_{H} associated with variable X in the empirical model G_{E}, such that $Ch_{\mathrm{H}}(\tilde{X}) = Ch_{\mathrm{E}}(X)$, then:

$$\mathbf{P}_{\mathrm{H}}(\mathcal{T}_{\mathrm{E}} \setminus \{X\}|\tilde{X} = x) = \mathbf{P}_{\mathrm{E}}(\mathcal{T}_{\mathrm{E}} \setminus \{X\}|do(X) = x).$$

Defining the Do-calculus

What is the do-calculus?

A set of three graphical/statistical **rules** that **convert** expressions of causal inference into probability equations.

1. **Goal:** Identify causal effects from non-experimental data.
2. **Application:** Bayesian network structure, i.e., Directed Acyclic Graph (DAG) that represents causal relationships.
3. **Identification method:** Iteration of do-calculus rules to generate a function that describes treatment effects statistics as a function of the observed variables only (Tian and Pearl 2002, Tian and Pearl 2003).

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Characteristics of Pearl's Do-Calculus

1. **Information:** DAG only provides information on the causal relation among variables.
2. **Not Suited** for examining assumptions on functional forms.
3. **Identification:** If this information is sufficient to identify causal effects, then:
4. **Completeness**:
   1. There exists a **sequence** of application of the Do-Calculus that
   2. **generates** a formula for causal effects based on observational quantities (Huang and Valtorta 2006, Shpitser and Pearl 2006)
5. **Limitation:** Does not allow for additional information outside the DAG framework.
   1. **Only** applies to the information content of a DAG.
   2. **IV** is not identified through Do-calculus
   3. **Why?** requires assumptions outside DAG: linearity, monotonicity, separability.

Notation for the Do-calculus

More notation is needed to define these rules:

DAG Notation

Let X, Y, Z be arbitrary disjoint sets of variables (nodes) in a causal graph G.

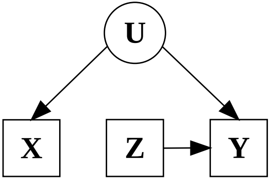
- $G_{\overline{X}}$: DAG that modifies G by deleting the arrows pointing *to* X.
- $G_{\underline{X}}$: DAG that modifies G by deleting arrows emerging *from* X.
- $G_{\overline{X},\underline{Z}}$: DAG that modifies G by deleting arrows pointing *to* X and *emerging* from Z.

Examples of DAG Notation

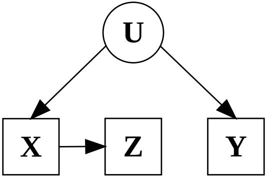
G	$G_{\underline{X}}$	$G_{\overline{X}}$	$G_{\overline{X},\underline{U}}$
$\begin{tikzcd} \boldsymbol{V} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\ \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \end{tikzcd}$	$\begin{tikzcd} \boldsymbol{V} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\ \boldsymbol{X} & \boldsymbol{Y} \end{tikzcd}$	$\begin{tikzcd} \boldsymbol{V} \arrow[r] & \boldsymbol{U} \arrow[d] \\ \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \end{tikzcd}$	$\begin{tikzcd} \boldsymbol{V} \arrow[r] & \boldsymbol{U} \\ \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \end{tikzcd}$

Example of DAG Notation

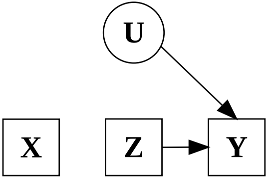
$G_{\underline{X}} = G_{\overline{Z}}$



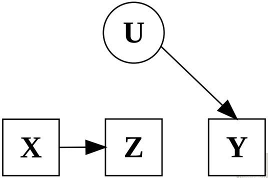
$G_{\underline{Z}}$



$G_{\overline{X},\overline{Z}}$



$G_{\overline{X},\underline{Z}}$



G

$$\begin{tikzcd}
 & \boldsymbol{V} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\
\boldsymbol{Z} \arrow[r] & \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \\
 & \boldsymbol{W} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$G_{\overline{X}}$

$$\begin{tikzcd}
 & \boldsymbol{V} \arrow[r] & \boldsymbol{U} \arrow[d] \\
\boldsymbol{Z} & \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \\
 & \boldsymbol{W} \arrow[ur] &
\end{tikzcd}$$

$G_{\overline{X},\overline{Y}}$

$$\begin{tikzcd}
 & \boldsymbol{V} \arrow[r] & \boldsymbol{U} \arrow[d] \\
\boldsymbol{Z} & \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \\
 & \boldsymbol{W} \arrow[ur] &
\end{tikzcd}$$

$G_{\overline{X},\overline{Z(W)}}$

$$\begin{tikzcd}
 & \boldsymbol{V} \arrow[r] & \boldsymbol{U} \arrow[d] \\
\boldsymbol{Z} & \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \\
 & \boldsymbol{W} \arrow[ur] &
\end{tikzcd}$$

Do-calculus Rules

- Assumes the Local Markov Condition and independence of ϵ.

Let G be a DAG and let X, Y, Z, W be any disjoint sets of variables. The do-calculus rules are:

- **Rule 1:** *Insertion/deletion of observations:*
  $Y \perp\!\!\!\perp Z|(X, W)$ under $G_{\overline{X}}$
  $$\Rightarrow \mathbf{P}(Y|do(X), Z, W) = \mathbf{P}(Y|do(X), W).$$
- **Rule 2:** *Action/observation exchange:*
  $Y \perp\!\!\!\perp Z|(X, W)$ under $G_{\overline{X}, \underline{Z}}$
  $$\Rightarrow \mathbf{P}(Y|do(X), do(Z), W) = \mathbf{P}(Y|do(X), Z, W).$$
- **Rule 3:** *Insertion/deletion of actions:*
  $Y \perp\!\!\!\perp Z|(X, W)$ under $G_{\overline{X}, \overline{Z(W)}}$
  $$\Rightarrow \mathbf{P}(Y|do(X), do(Z), W) = \mathbf{P}(Y|do(X), W),$$
  where $Z(W)$ is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$.

Understanding the Rules of Do-Calculus

Let G be a DAG then for any disjoint sets of variables X, Y, Z, W :
Rule 1: Insertion/deletion of observations

$$\text{If } \underbrace{Y \perp\!\!\!\perp Z|(X, W)}_{\text{Statistical Relation}} \text{ under } \underbrace{G_{\overline{X}}}_{\text{Graphic Criterion}} \text{ then}$$

$$\underbrace{\Pr(Y|do(X), Z, W) = \Pr(Y|do(X), W)}_{\text{Equivalent Probability Expression}}$$

Do-Calculus Exercise

$$G$$

$$\begin{tikzcd} \boldsymbol{V} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\ \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \end{tikzcd}$$

$$G_{\underline{X}}$$

$$\begin{tikzcd} \boldsymbol{V} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\ \boldsymbol{X} & \boldsymbol{Y} \end{tikzcd}$$

1. LMC to X under $G_{\underline{X}}$ generates $X \perp\!\!\!\perp (U, Y)|V \Rightarrow X \perp\!\!\!\perp (U, Y)|V$.
2. Now if $X \perp\!\!\!\perp (U, Y)|V$ holds under $G_{\underline{X}}$, then, by **Rule 2**,

$$\mathbf{P}(Y|do(X), V) = \mathbf{P}(Y|X, V). \tag{19}$$

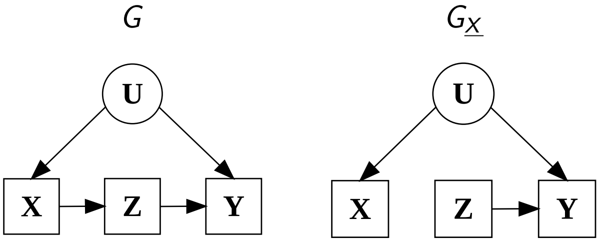
$$\therefore E(Y|do(X) = x) = \underbrace{\int E(Y|V = v, do(X) = x) dF_V(v)}_{\text{Using do(X),i.e. Fixing } X}$$

$$= \underbrace{\int E(Y|V = v, X = x) dF_V(v)}_{\text{Replace ``do'' with Standard Statistical Conditioning}} \text{ by Equation(19)}$$

Do-Calculus Exercise : The Front-door Model

Using the Do-Calculus : Task 1 – Compute $\Pr(Z|do(X))$

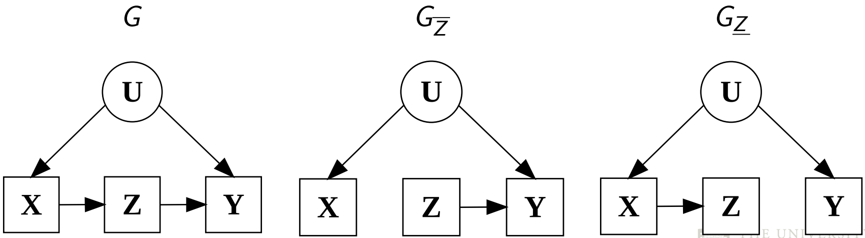
$X \perp\!\!\!\perp Z$ in $G_{\underline{X}}$, by **Rule 2**, $\Pr(Z|do(X)) = \Pr(Z|X)$.



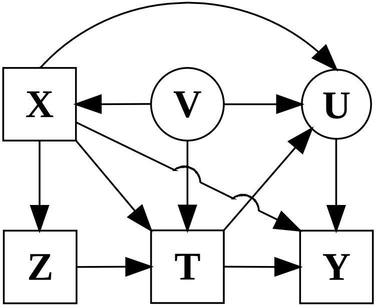
Using the Do-Calculus : Task 2 – Compute $\Pr(Y|do(Z))$

$Z \perp\!\!\!\perp X$ in $G_{\overline{Z}}$, by **Rule 3**, $\Pr(X|do(Z)) = \Pr(X)$
$Z \perp\!\!\!\perp Y|X$ in $G_{\underline{Z}}$, by **Rule 2**, $\Pr(Y|X, do(Z)) = \Pr(Y|X, Z)$

$$\therefore \Pr(Y|do(Z)) = \sum_X \Pr(Y|X, do(Z)) \Pr(X|do(Z))$$
$$= \sum_X \Pr(Y|X, Z) \Pr(X)$$



Generalized Roy Model



This figure represents causal relationships of the Generalized Roy Model. Arrows represent direct causal relationships. Circles represent unobserved variables. Squares represent observed variables

Key Aspects of the Generalized Roy Model

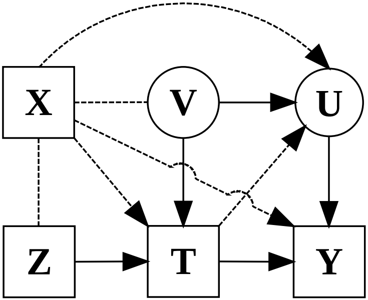
1. T is caused by Z, V;
2. U mediates the effects of V on Y (that is V causes U);
3. T and U cause Y and
4. Z (instrument) not caused by V, U and does not directly cause Y, U.

We are left to examine the cases whether:

1. V causes X (or vice-versa),
2. X causes Z (or vice-versa),
3. X causes T,
4. X causes U,
5. T causes U, and
6. X causes Y.

The combinations of all these causal relationships generate 144 possible models (Pinto, 2013).

Key Aspects of the Generalized Roy Model (Pinto, 2013)



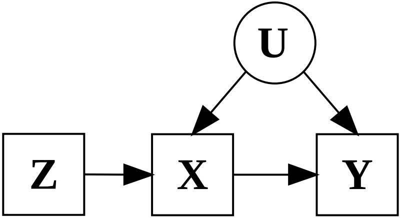
Dashed lines denote causal relationships that may not exist or, if they exist, the causal direction can go either way. Dashed arrows denote causal relationships that may not exist, but, if they exist, the causal direction must comply the arrow direction.

Marginalizing the Generalized Roy Model

- We examine the identification of causal effects of the Generalized Roy Model using a simplified model w.l.o.g.
- Suppress variables X and U.
- This simplification is usually called marginalization in the DAG literature (Koster (2002), Lauritzen (1996), Wermuth (2011)).

Marginalizing the Generalized Roy Model

$$G = G_{\overline{Z}}$$



This figure represents causal relationships of the Marginalized Roy Model. Arrows represent direct causal relationships. Circles represent unobserved variables. Squares represent observed variables
Note: Z is exogenous, thus conditioning on Z is equivalent to fixing Z.

Examining the Marginalized Roy Model – 1/4

- $Y \perp\!\!\!\perp Z$ in $G_{\overline{X}}$, by **Rule 1**

$$\Pr(Y|do(X), Z) = \Pr(Y|do(X))$$

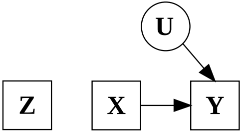
- $Y \perp\!\!\!\perp Z$, in $G_{\overline{X},\overline{Z}}$, by **Rule 3**

$$\Pr(Y|do(X), Z) = \Pr(Y|do(X))$$

- $Y \perp\!\!\!\perp Z|X$ in $G_{\overline{X},\underline{Z}}$, by **Rule 2**

$$\Pr(Y|do(X), do(Z)) = \Pr(Y|do(X), Z)$$

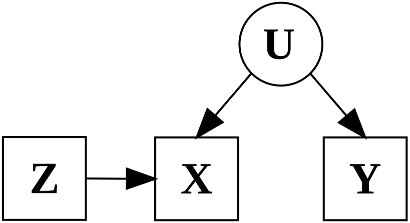
$$G_{\overline{X}} = G_{\overline{X},\overline{Z}} = G_{\overline{X},\underline{Z}}$$



Examining the Marginalized Roy Model – 2/4

- Under $G_{\overline{X}}$, $Y \not\!\perp\!\!\!\perp X$, thus **Rule 2** does not apply.
- Under $G_{\underline{X},\overline{Z}}$, $Y \not\!\perp\!\!\!\perp X|Z$, thus **Rule 2** does not apply.

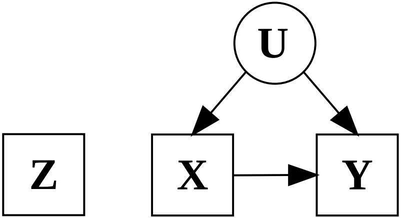
$$G_{\underline{X}} = G_{\underline{X},\overline{Z}}$$



Examining the Marginalized Roy Model – 3/4

- $G_{\underline{Z}} \Rightarrow Y \perp\!\!\!\perp Z$, thus by **Rule 2** $\Pr(Y|do(Z)) = \Pr(Y|Z)$.

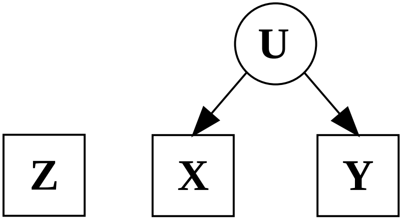
$G_{\underline{Z}}$



Examining the Marginalized Roy Model – 4 of 4 Modifications

- Under $G_{\underline{X},\underline{Z}}$, $Y \not\!\perp\!\!\!\perp (X, Z)$, thus **Rule 2** does not apply.

$G_{\underline{X},\underline{Z}}$



Conclusion of Do-calculus and the Roy Model

The Do-Calculus applied to the Marginalized Roy Model generates:

1. $\Pr(Y|do(X), do(Z)) = \Pr(Y|do(X), Z) = \Pr(Y|do(X))$,
2. $\Pr(Y|do(Z)) = \Pr(Y|Z)$

These relationships only corroborate the exogeneity of the instrumental variable Z and are not sufficient to identify $\Pr(Y|do(X))$.

Identification of the Roy Model

To identify the Roy Model, we make assumption on how Z impacts X, i.e. monotonicity/separability.

These assumptions **cannot** be represented in a DAG.

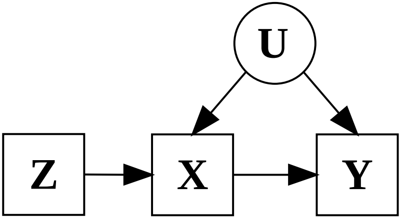
These assumptions are associated with properties of **how** Z causes X and not only **if** Z causes X.

3. Limitations of Do-calculus for Econometric Identification

Failure of Do-Calculus
Does not Generates Standard IV Results

The simplest instrumental variable model consists of four variables:

1. A confounding variable U that is external and unobserved.
2. An external instrumental variable Z.
3. An observed variable X caused by U and Z.
4. An outcome Y caused by U and X.



4.1 Do-Calculus Non-identification of the IV Model

- **Limitation:** IV model is not identified by literature that relies exclusively on DAGs.
- **Why?:** IV identification relies on assumptions outside the scope of DAG literature.
- **LMC:** generates the conditional independence relationships: $Y \perp\!\!\!\perp Z|(U, X)$ and $U \perp\!\!\!\perp Z$.
- **TSLS:** $X \not\perp\!\!\!\perp Z$ holds, thus, the IV model satisfy the necessary criteria to apply the method of Two Stage Least Squares (TSLS).
- **Assumption Outside of DAGs:** TSLS identifies the IV model under linearity.

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Do-Calculus and IV

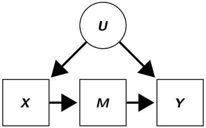
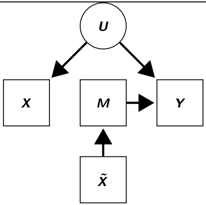
The Do-Calculus applied to the IV Model generates:

1. $\Pr(Y|do(X), do(Z)) = \Pr(Y|do(X), Z) = \Pr(Y|do(X))$,
2. $\Pr(Y|do(Z)) = \Pr(Y|Z)$

Only establishes the exogeneity of the instrumental variable Z.
Insufficient to identify $\Pr(Y|do(X))$.

- The instrumental variable model is not identified applying the rules of the do-calculus.
- Indeed, in this framework it is impossible to identify the causal effect of X on Y without additional information.
- The instrumental variable model is identified under further assumptions such as linearity, separability, monotonicity.
- However, these assumptions are outside the scope of the do-calculus.

"Front-Door" Empirical and Hypothetical Models

1. Pearl's "Front-Door" Empirical Model	2. Our Version of the "Front-Door" Hypothetical Model
$\mathcal{T} = \{U, X, M, Y\}$ $\boldsymbol{\epsilon} = \{\epsilon_U, \epsilon_X, \epsilon_M, \epsilon_Y\}$ $Y = f_Y(M, U, \epsilon_Y)$ $X = f_X(U, \epsilon_X)$ $M = f_M(X, \epsilon_M)$ $U = f_U(\epsilon_U)$	$\mathcal{T} = \{U, X, M, Y, \tilde{X}\}$ $\boldsymbol{\epsilon} = \{\epsilon_U, \epsilon_X, \epsilon_M, \epsilon_Y\}$ $Y = f_Y(M, U, \epsilon_Y)$ $X = f_X(U, \epsilon_X)$ $M = f_M(\tilde{X}, \epsilon_M)$ $U = f_U(\epsilon_U)$
	
$Pa(U) = \varnothing$, $Pa(X) = \{U\}$ $Pa(M) = \{X\}$ $Pa(Y) = \{M, U\}$	$Pa(U) = Pa(\tilde{X}) = \varnothing$, $Pa(X) = \{U\}$ $Pa(M) = \{\tilde{X}\}$ $Pa(Y) = \{M, U\}$
$Y \perp\!\!\!\perp X \| (M, U)$ $M \perp\!\!\!\perp U \| X$	$Y \perp\!\!\!\perp (\tilde{X}, X) \| (M, U)$ $M \perp\!\!\!\perp (U, X) \| \tilde{X}$ $X \perp\!\!\!\perp (M, \tilde{X}, Y) \| U$ $U \perp\!\!\!\perp (M, \tilde{X})$ $\tilde{X} \perp\!\!\!\perp (X, U)$
$\mathbf{P}_{\mathrm{E}}(Y, M, X, U) =$ $\mathbf{P}_{\mathrm{E}}(Y \| M, U)\, \mathbf{P}_{\mathrm{E}}(X \| U)\, \mathbf{P}_{\mathrm{E}}(M \| X)\, \mathbf{P}_{\mathrm{E}}(U)$	$\mathbf{P}_{\mathrm{H}}(Y, M, X, U, \tilde{X}) =$ $\mathbf{P}_{\mathrm{H}}(Y \| M, U)\, \mathbf{P}(X \| U)\, \mathbf{P}_{\mathrm{H}}(M \| \tilde{X})\, \mathbf{P}_{\mathrm{H}}(U)\, \mathbf{P}_{\mathrm{H}}(\tilde{X})$
$\mathbf{P}_{\mathrm{E}}(Y, M, U \| do(X) = x) =$ $\mathbf{P}_{\mathrm{E}}(Y \| M, U)\, \mathbf{P}_{\mathrm{E}}(M \| X = x)\, \mathbf{P}_{\mathrm{E}}(U)$	$\mathbf{P}_{\mathrm{H}}(Y, M, U, X \| \tilde{X} = x) =$ $\mathbf{P}_{\mathrm{H}}(Y \| M, U)\, \mathbf{P}(X \| U)\, \mathbf{P}_{\mathrm{H}}(M \| \tilde{X} = x)\, \mathbf{P}_{\mathrm{H}}(U)$

4. Summary of Do-calculus and Haavelmo

Summarizing Do-calculus of Pearl (2009b) and Haavelmo's Inspired Framework

- **Common Features** of Haavelmo and Do Calculus:
  1. **Autonomy** (Frisch, 1938)
  2. **Errors Terms:** ϵ mutually independent
  3. **Statistical Tools:** LMC and GA apply
  4. **Counterfactuals:** Fixing or Do-operator is a Causal, not statistical, Operation.
- **Distinct Features** of Haavelmo and Do Calculus:

	Haavelmo	**Do-calculus**
Approach:	Thinks Outside the Box	Applies Complex Tools
Introduces:	Hypothetical Model	Graphical Rules
Identification:	Connects P_H and P_E	Iteration of Rules
Versatility:	Basic Statistics Apply	Extra Notation/Tools

Return to main text