

Notes on Frisch Demands*

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Simple Two Good Model to Interpret Frisch Demands

This is a model for each period of a general life cycle model.
Abstract from interest rates and time preference.

$$\begin{aligned} \max U(C, L) \\ \text{s.t. } PC + WL = E \end{aligned}$$

E is the resources available to be spent within the period. Assume fixed for the moment.

$$\begin{aligned} U_C &= \lambda P \\ U_L &= \lambda W \end{aligned}$$

C = Consumption

L = Leisure

λ = LaGrange Multiplier associated with the budget constraint

Treat leisure as an ordinary good for the moment.

Comparative Statics of Model

Totally Differentiate

$$\begin{bmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{bmatrix} \begin{bmatrix} dC \\ dL \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dP \\ \lambda dW \\ -dE + CdP + LdW \end{bmatrix}$$

Let bars denote determinants

$$\frac{\partial L}{\partial E} = \frac{\begin{vmatrix} U_{CC} & 0 & -P \\ U_{CL} & 0 & -W \\ -P & -1 & 0 \end{vmatrix}}{|\cdot|}$$

$$\text{Notation } |\cdot| = \begin{vmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{vmatrix} > 0 \text{ (second order conditions)}$$

$$\frac{dL}{dE} = \left\{ \frac{\overbrace{(-U_{CC}W + PU_{CL})}^{>0}}{\underbrace{|\cdot|}_{>0}} \right\} > 0 \text{ (From Assumed Normality)}$$

Now this can be related to displacements for λ :

$$\begin{aligned} \left. \frac{\partial \lambda}{\partial W} \right|_{E=\bar{E}} &= \frac{\begin{vmatrix} U_{CC} & U_{CL} & 0 \\ U_{CL} & U_{LL} & \lambda \\ -P & -W & 0 \end{vmatrix}}{|\cdot|} \\ &= \frac{-\lambda}{|\cdot|} \begin{vmatrix} U_{CC} & U_{CL} \\ -P & -W \end{vmatrix} \\ &= (-\lambda) \frac{(-U_{CC}W + U_{CL}P)}{|\cdot|} \\ &= \lambda \frac{(U_{CC}W - U_{CL}P)}{|\cdot|} \\ &< 0 \text{ from normality of leisure} \end{aligned}$$

Thus

$$\left. \frac{\partial \lambda}{\partial W} \right|_{E=\bar{E}} = -\frac{\partial L}{\partial E}$$

(Intuition: The cost of a component of utility rises, the marginal utility of income declines.)

Observe

$$\frac{\partial \lambda}{\partial E} = - \frac{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}}{|\cdot|}$$

$$\frac{d\lambda}{dE} < 0 \text{ under strict concavity}$$

(Diminishing Marginal Utility of Income)

Take total differential:

$$d\lambda = \frac{\partial \lambda}{\partial P} \Big|_{E=\bar{E}} dP + \frac{\partial \lambda}{\partial W} \Big|_{E=\bar{E}} dW + \frac{\partial \lambda}{\partial E} (dE - CdP - LdW)$$

(This sign pattern assumes that all goods are normal.)

Consider

$$\left. \frac{\partial L}{\partial W} \right|_{\lambda}$$

Compensate the agent to keep λ fixed. Let dK be the compensation.

$$d\lambda = 0 = \left(\left. \frac{\partial \lambda}{\partial W} \right|_{\bar{U}} \right)^{(-)} dW + \left(\frac{\partial \lambda}{\partial E} \right)^{(-)} (dK - LdW)$$

$$0 = \frac{\partial \lambda}{\partial W} dW - LdW + dK$$

$$dK = \left(L - \frac{\left(\left. \frac{\partial \lambda}{\partial W} \right|_{\bar{U}} \right)^{(+)}}{\frac{\partial \lambda}{\partial E}} \right) dW$$

\therefore The compensation required to keep λ constant is smaller than what is required to keep utility constant for the same change in the wage.

So

Hicks Slutsky compensated
substitution effect

$$\frac{\partial L}{\partial W} \Big|_{\lambda} = \left(\frac{\partial L}{\partial W} \Big|_{U=\bar{U}} - L \frac{\partial L}{\partial E} \right) + \frac{\partial L}{\partial E} \left(\frac{\partial K}{\partial W} \right)$$

compensation to
keep λ fixed

$$\begin{aligned} \frac{\partial L}{\partial W} \Big|_{\lambda} &= \frac{\partial L}{\partial W} \Big|_{U=\bar{U}} + \frac{\partial L}{\partial E} \left(\frac{\partial K}{\partial W} - L \right) \\ &= \frac{\partial L}{\partial W} \Big|_{U=\bar{U}} + \frac{\partial L}{\partial E} \left(-\frac{\frac{\partial \lambda}{\partial W} \Big|_{\bar{U}}}{\frac{\partial \lambda}{\partial E}} \right) \end{aligned}$$

Clearly $\frac{\partial L}{\partial W} \Big|_{\lambda} \leq \frac{\partial L}{\partial W} \Big|_{U=\bar{U}} \leq 0$

Digression

Direct derivation (assume λ fixed).

$$U_C = \lambda P$$

$$U_L = \lambda W$$

$$U_{CC}dC + U_{CL}dL = \lambda dP$$

$$U_{CL}dC + U_{LL}dL = \lambda dW$$

Take Cramer's Rule:

$$\begin{aligned}\frac{\partial L}{\partial W} \Big|_{\lambda} &= \frac{\begin{vmatrix} U_{CC} & 0 \\ U_{CL} & \lambda \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}} \\ &= \frac{(U_{CC}\lambda)}{U_{CC}U_{LL} - U_{CL}^2} < 0\end{aligned}$$

Claimed result follows because

$$\begin{aligned}U_{CC} &< 0 \\ U_{CC}U_{LL} - U_{CL}^2 &> 0 \\ &\text{(concavity)}\end{aligned}$$

$$\begin{aligned} \left. \frac{\partial C}{\partial W} \right|_{\lambda} &= \frac{\begin{vmatrix} 0 & U_{CL} \\ \lambda & U_{LL} \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}} \\ &= \frac{-\lambda U_{CL}}{U_{CC}U_{LL} - U_{CL}^2} \end{aligned}$$

$$U_{CL} > 0 \implies \left. \frac{\partial C}{\partial W} \right|_{\lambda} < 0$$

$$U_{CL} < 0 \implies \left. \frac{\partial C}{\partial W} \right|_{\lambda} > 0$$

(See e.g. Heckman, 1974, AER)

Back to the main thread:

$$0 \geq \frac{\partial L}{\partial W} \Big|_{U=\bar{U}} \geq \frac{\partial L}{\partial W} \Big|_{\lambda}$$

Intuition: we have to pay less compensation to keep λ fixed than U fixed and leisure is a normal good, so we get less leisure with λ fixed and hence the λ -constant wage response is more negative.

$$\frac{\partial L}{\partial W} \Big|_{U=\bar{U}} \geq \frac{\partial L}{\partial W} \quad (\text{From normality of leisure})$$

Thus we do not know how to order

$$\underbrace{\frac{\partial L}{\partial W}}_{\text{uncompensated}} \text{ and } \frac{\partial L}{\partial W} \Big|_{\lambda}$$

We have thus far abstracted from the fact that people have a time endowment, T . (Implicitly, we set $T = 0$). The Hicks-Slutsky uncompensated wage effect for leisure is (for $T \geq 0$)

$$\frac{\partial L}{\partial W} = \frac{\partial L}{\partial W} \Big|_{U=\bar{U}} + (T - L) \frac{\partial L}{\partial E}$$

This makes the wage effect on labor supply ambiguous.

Define $h = T - L$

$$\frac{\partial h}{\partial L} = -1$$

$$\frac{\partial h}{\partial W} = \frac{\partial h}{\partial W} \Big|_{U=\bar{U}}^{(+)} + h \left(\frac{\partial h}{\partial E} \right)^{(-)}$$

$$\frac{\partial h}{\partial W} \Big|_{\lambda} = \frac{\partial h}{\partial W} \Big|_{U=\bar{U}}^{(+)} + \frac{\partial h}{\partial E}^{(-)} h + \frac{\partial h}{\partial E}^{(-)} \left(\frac{\partial K}{\partial W} \right)$$

What is dK ?

$$d\lambda = 0 = \left(\frac{\partial \lambda}{\partial W} \right) \Big|_{\bar{U}}^{(-)} dW + \left(\frac{\partial \lambda}{\partial E} \right)^{(-)} (dK + h dW)$$

$$0 = \frac{\frac{\partial \lambda}{\partial W} \Big|_{\bar{U}}}{\frac{\partial \lambda}{\partial E}} dW + h dW + dK$$

$$\frac{\partial K}{\partial W} = - \underbrace{\left(\frac{\frac{\partial \lambda}{\partial W} \Big|_{\bar{U}}}{\frac{\partial \lambda}{\partial E}} + h \right)}_{+} < 0$$

So

$$\frac{\partial h}{\partial W} \Big|_{\lambda} = \frac{\partial h}{\partial W} \Big|_{U=\bar{U}} - \underbrace{\frac{\partial h}{\partial E} \left(\frac{\frac{\partial \lambda}{\partial W} \Big|_{\bar{U}}}{\frac{\partial \lambda}{\partial E}} \right)}_{(+)}$$
$$\therefore 0 \leq \frac{\partial h}{\partial W} \Big|_{U=\bar{U}} \leq \frac{\partial h}{\partial W} \Big|_{\lambda}$$

and with respect to the uncompensated Marshallian elasticity:

$$\frac{\partial h}{\partial W} < \frac{\partial h}{\partial W} \Big|_{U=\bar{U}} \leq \frac{\partial h}{\partial W} \Big|_{\lambda}$$

In the case of labor supply more income is transferred out periods where wages increase. dK is larger (more negative) in this case than the previous one considered.

(There is a more negative wage effect on borrowing, *i.e.* a force toward saving. This is intuitively so because wage increase generates income in the period which is likely transferred to other periods.)

- This is for a single period.
- We connect back to the multiperiod model by using the intertemporal arbitrage condition

$$\lambda_{t+1} = \frac{1 + \beta}{1 + r} \lambda_t$$

(see BHH).

- This determines the allocations of the E_t across the periods in a two stage budgeting procedure.
- In those periods where λ_t is high ship resources in (borrow).
- Where it is low, ship out (save).
- As we showed, holding $U = \bar{U}$, $W \uparrow \lambda \downarrow$.
- Tends to create a force for saving in periods of wage increases.
- Possibly offsetting it, is substitution toward consumption when leisure goes down.