Income Dynamics and Life-cycle Inequality: Mechanisms And Controversies

by Richard Blundell
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James J. Heckman

THE UNIVERSITY OF CHICAGO

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Consumption and Income Inequality

- The income variance increases with permanent income shocks.
- The variance of consumption also increases with permanent income shocks.
- How does this align with the Mincer model?
- The degree to which these move in line will depend on the degree of precautionary savings and access to credit.
- Recent evidence on the growth in consumption inequality over the life cycle for different birth cohorts in the UK and the US shows a strong increase in inequality across cohorts.

- Younger birth cohorts face higher overall consumption inequality during their working life than similarly aged older cohorts.

- Figures 1 and 2 show the evidence from the UK1 and from the US2 respectively.

- Income inequality growth displays some similarities, but a clearly different pattern.

- See Figure 3 for the UK, for example.

Fig. 1. Variance of Log Non-durable Consumption by Age, UK

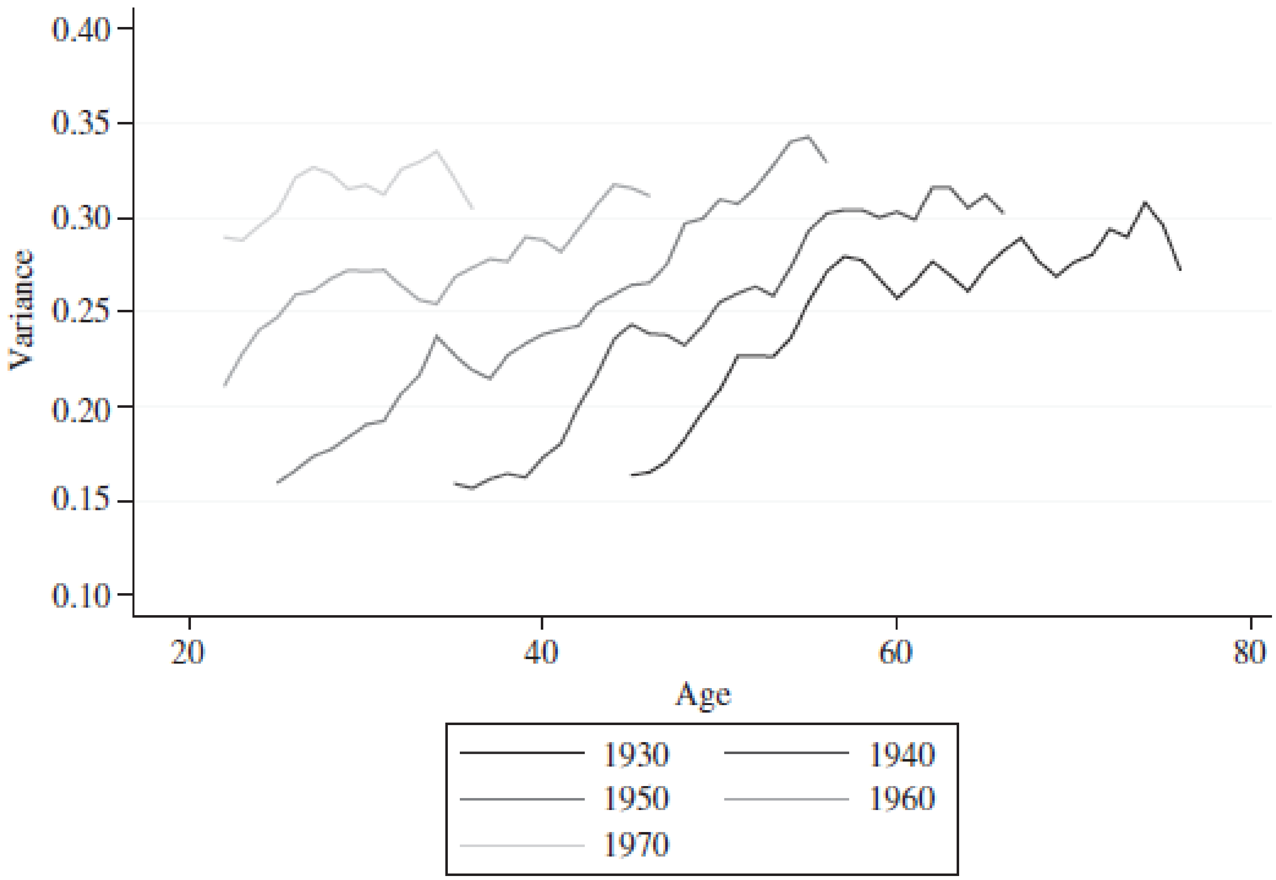


Fig. 2. Variance of Log Non-durable Consumption by Age, US

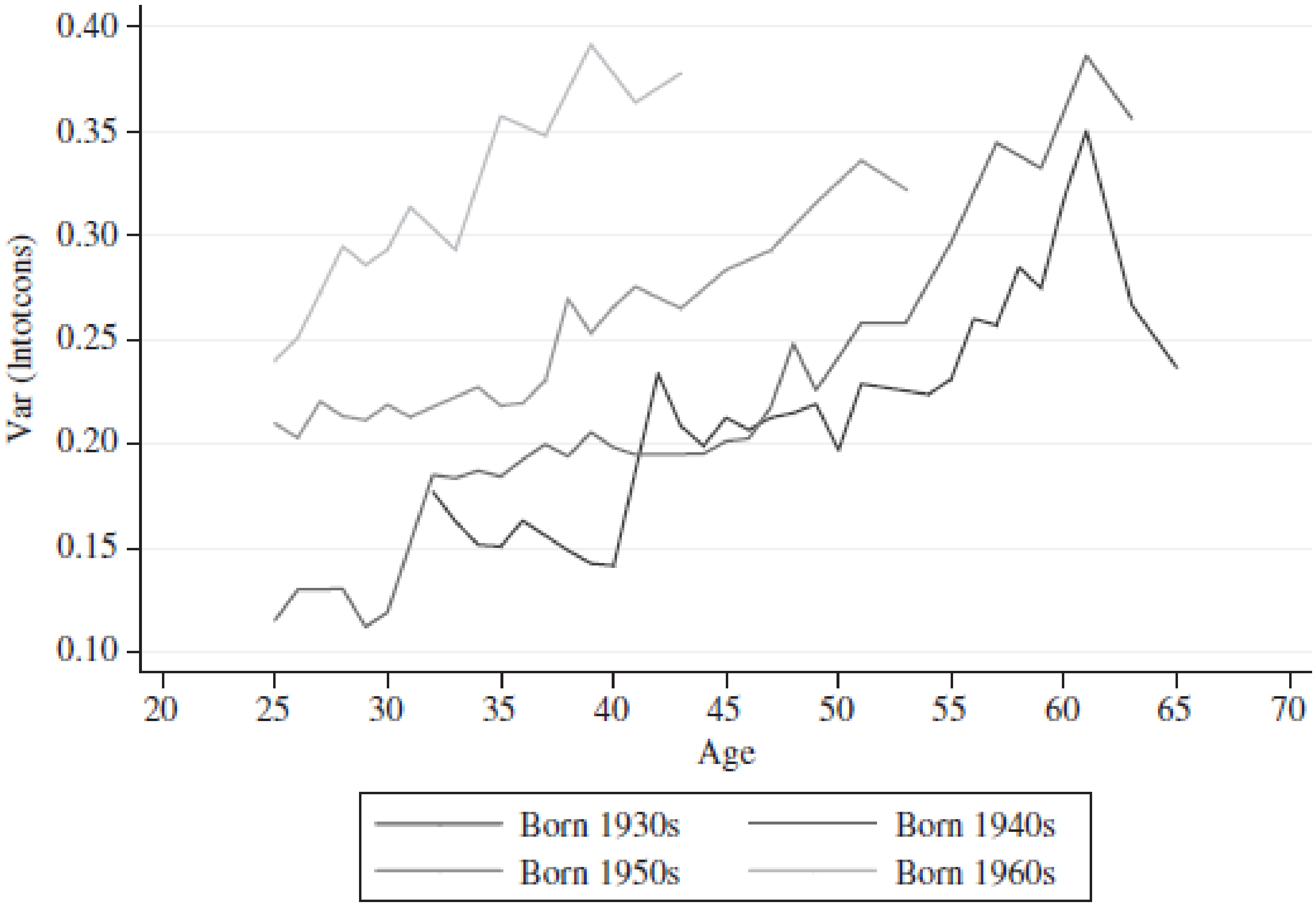
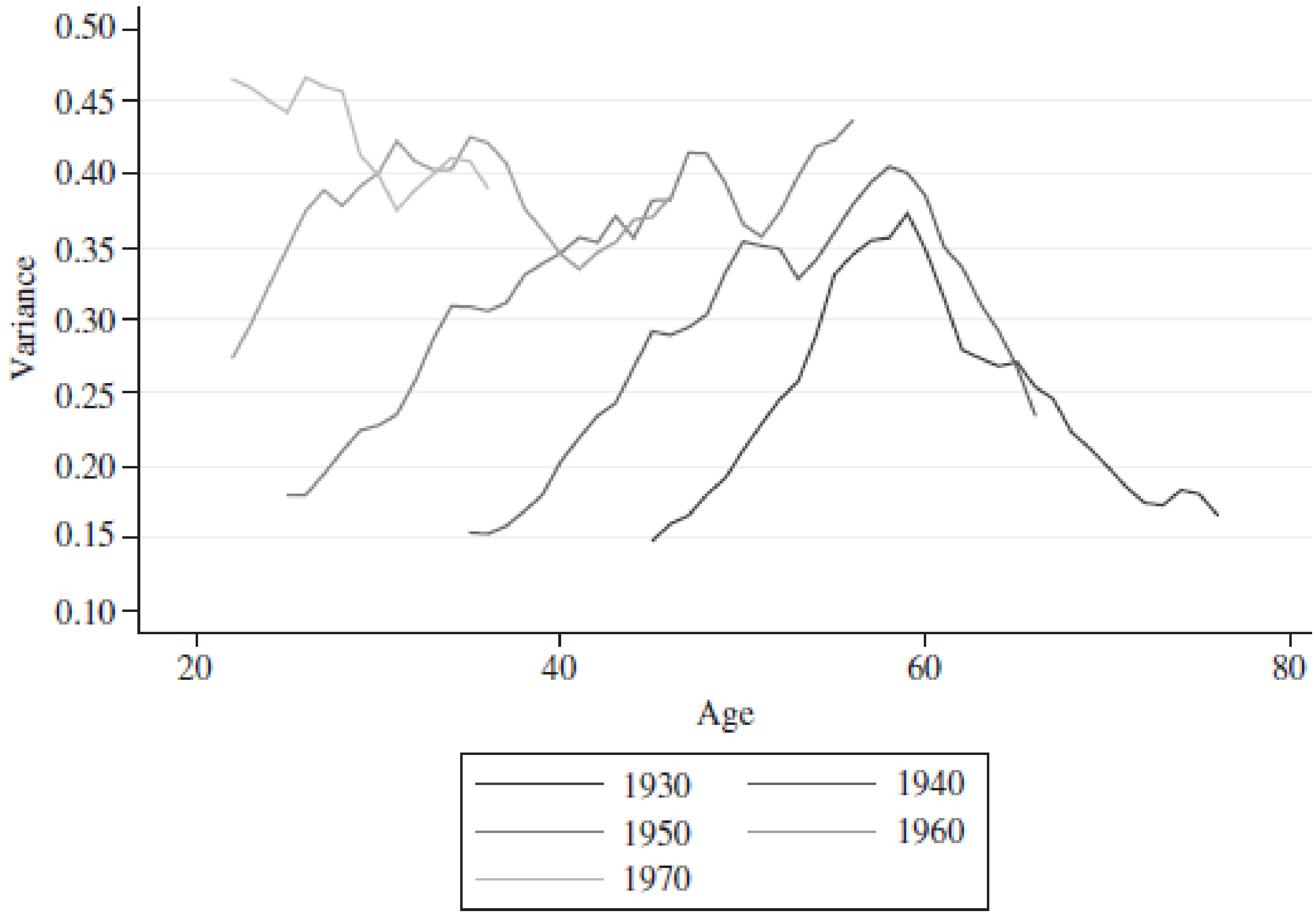


Fig. 3. Variance of Log Disposable Income by Age, UK



- Similar pattern: variance ↑ over cohorts

Panel Data Income Dynamics

- Non-stationarity and the persistence of shocks.

- Assume log income $y_{i,a,t}(\equiv \ln Y_{i,a,t})$ for consumer i, at age a, time period t, observable characteristics $Z_{i,a,t}$:

$$y_{i,a,t} = B'_{i,a,t} f_i + Z'_{i,a,t} \varphi + y^P_{i,a,t} + y^T_{i,a,t}, \qquad (1)$$

- For any cohort, a "reasonably general" specification for the idiosyncratic effects $B'_{i,t} f_i$:

Ability ↓

$$B'_{i,t} f_i = f_{0i} + p_t f_{1i}, \qquad (2)$$

↑ Factor loading (price)

- f_{0i}: individual effect.
- p_t price at t.
- $p_t f_t$ can also represent an idiosyncratic trend at time t.
- $y^T_{i,t}$: represented by a low-order MA(q):

$$v_{it} = \sum_{j=0}^{q} \theta_j \varepsilon_{i,t-j} \text{ with } \theta_0 \equiv 1, \qquad (3)$$

- y^P_{it}:

$$y^P_{it} = \rho y^P_{it-1} + \zeta_{it}. \qquad (4)$$

- Remove deterministic term $Z'_{i,t}\varphi$ from $y_{i,t}$:

$$y_{i,t} = p_i f_{1i} + f_{0i} + y^P_{i,t} + \sum_{j=0}^{q} \theta_j \varepsilon_{i,t-j}, \qquad (5)$$

- If $q=1$, then this implies a key quasi-difference moment restriction

$$\text{cov}(\Delta^\rho y_t, \Delta^\rho y_{t-2}) = \text{var}(f_0)(1-\rho)^2 + \text{var}(f_1)\Delta^\rho p_t \Delta^\rho p_{t-2} - \rho\theta_1 \text{var}(\varepsilon_{t-2}), \quad (6)$$

where $\Delta^\rho = (1 - \rho L)$ is the quasi-difference operator.

- For large $q=1$ and small θ_1, (6) implies

$$\text{cov}(\Delta y_t, \Delta y_{t-2}) \simeq \text{var}(f_1)\Delta p_t \Delta p_{t-2}. \quad (7)$$

- For near unit root permanent shocks and innovation transitory shocks, if $(var\ (f_1) = 0)$
- No autocovariances of order two or above remaining in the growth rates of the income variable *y*.
- Allowing for a higher MA process relaxes this; but at some point, the autocovariance structure for income growth drops to zero.

1.1. Idiosyncratic Trends

- The trend term $p_t f_{1i}$ in (5) could take a number of forms. Two alternatives worth highlighting are as follows:

  a) deterministic idiosyncratic trend:

$$p_t f_{1i} = r(t) f_{1i},$$

  where r is a known function of t, e.g. $r(t) = t$, (e.g., Gorman model)

  b) stochastic trend in 'ability prices':

$$p_t = p_{t-1} + \xi_t.$$

  with $E_{t-1}\xi_t$.

1.2. The Permanent–transitory Model of Income Dynamics

- Permanent–transitory decomposition provides a useful baseline.
- Rewrite (5) as

$$y_{it} = y_{it}^P + y_{it}^T \tag{8}$$

$$y_{it}^P = y_{it-1}^P + \zeta_{it} \tag{9}$$

- Transitory or mean reverting component $y_{it}^T = v_{i,t}$,

$$v_{it} = \sum_{j=0}^{q} \theta_j \varepsilon_{i,t-j} \text{ with } \theta_0 \equiv 1. \tag{10}$$

- Autocovariances of $\Delta y_{it} (= \xi_{it} + \Delta v_{it})$

$$\operatorname{cov}(\Delta y_t, \Delta y_{t+s}) = \begin{cases} \operatorname{var}(\zeta_t) + \operatorname{var}(\Delta v_t) & \text{for } s = 0 \\ \operatorname{cov}(\Delta v_t, \Delta v_{t+s}) & \text{for } s \neq 0. \end{cases} \tag{11}$$

- Figures 4 and 5: Income dynamics in Norwegian Population Register Panel.

- Figures plot the variances of the permanent shocks to labour market income and disposable income for men during their working life.

Fig. 4. Variance of Permanent Shocks by Age, Norway

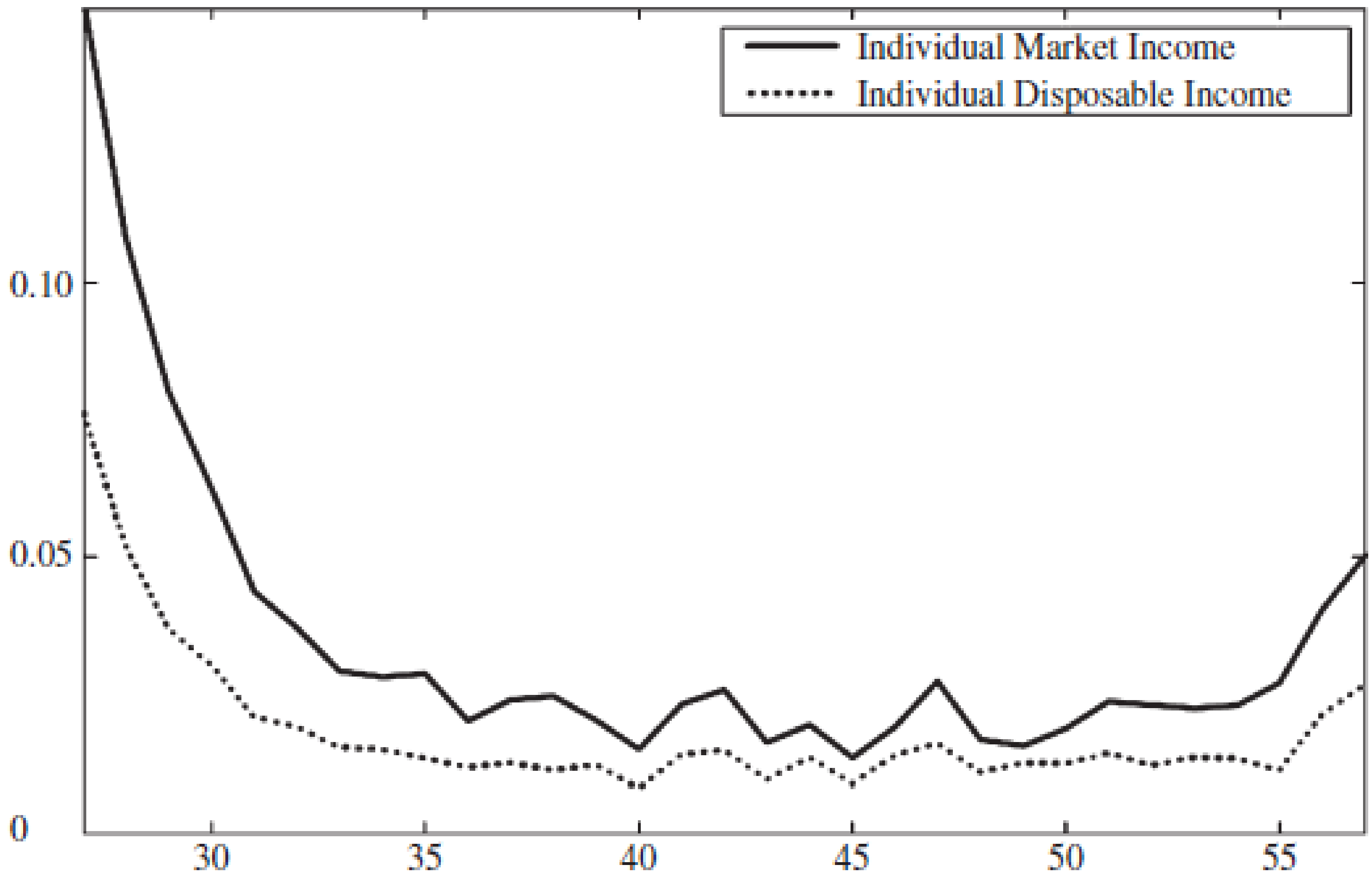
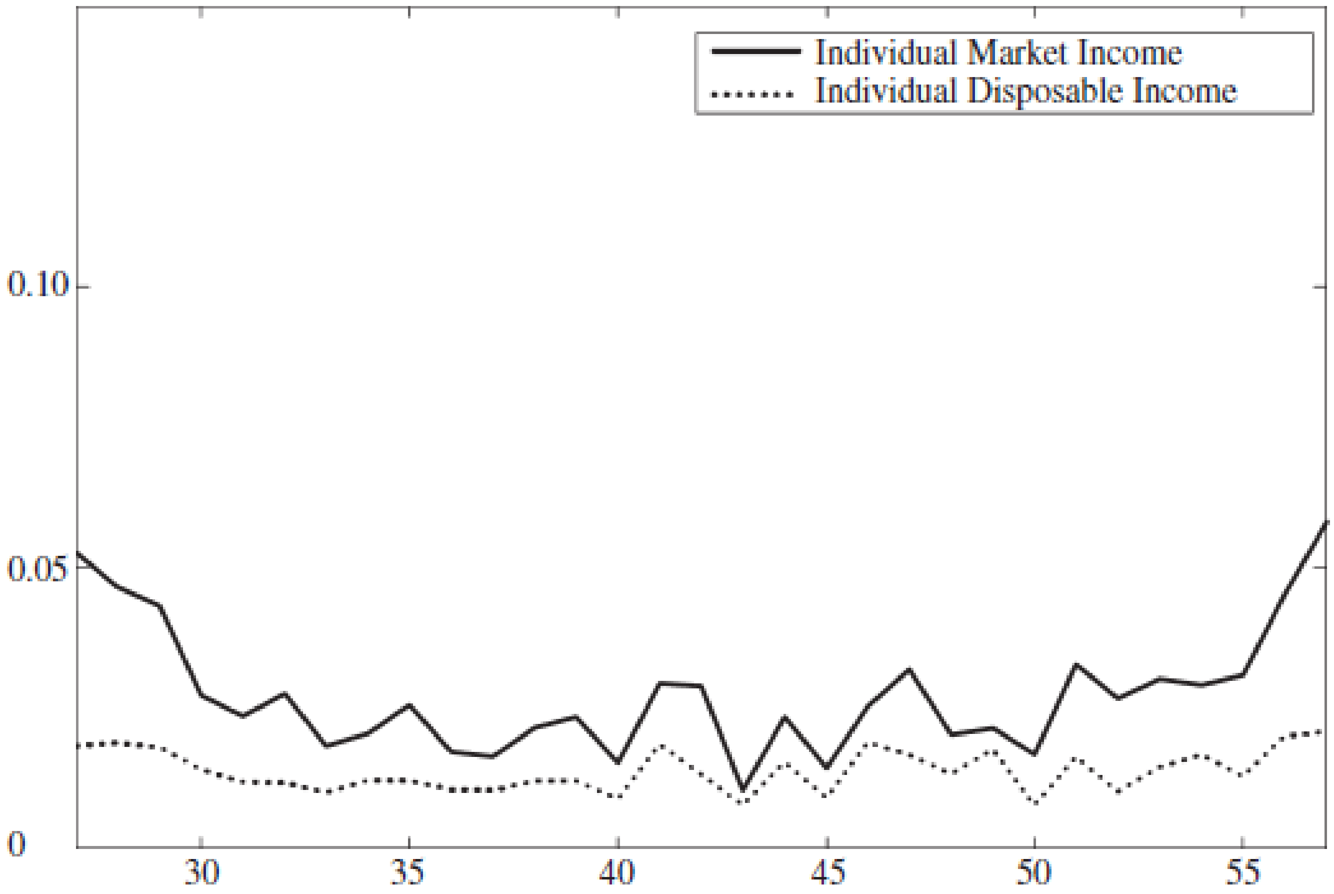


Fig. 5. Variance of Permanent Shocks by Age (Low Educated), Norway



- Second Figure: separates out the less-educated group and shows the strong increase in the variance of permanent shocks at older working ages for this group.

- Overall U shape for variances over the life cycle may reflect an aggregation over high-educated workers, whose shocks are largest earlier their lifetime, and low-educated workers, who face larger variances to persistent income shocks later in their working life.

1.3. Some (Simple) Empirics of Income Dynamics

- As noted above, forecastable components and differential trends are most important early in the life cycle.
- Tables 1 and 2: the head is male, lives in a couple and prime aged (aged between 30 and 60 years).
- Selection removes the early career trends and the later career health effects.
- Moreover, the baseline specifications (8)–(10) allow for general fixed effects and initial conditions.

Table 1: The Autocovariance Matrix of Income Growth PSID

Year	$var(\Delta y_t)$	$cov(\Delta y_{t+1}, \Delta y_t)$	$cov(\Delta y_{t+2}, \Delta y_t)$
1980	0.0832	−0.0196	−0.0018
	(0.0089)	(0.0035)	(0.0032)
1981	0.0717	−0.0220	−0.0074
	(0.0075)	(0.0034)	(0.0037)
1982	0.0718	−0.0226	−0.0081
	(0.0051)	(0.0035)	(0.0026)
1983	0.0783	−0.0209	−0.0094
	(0.0066)	(0.0034)	(0.0042)
1984	0.0805	−0.0288	−0.0034
	(0.0055)	(0.0036)	(0.0032)
1985	0.1090	−0.0379	−0.0019
	(0.0180)	(0.0074)	(0.0038)
1986	0.1023	−0.0354	−0.0115
	(0.0077)	(0.0054)	(0.0038)
1987	0.1116	−0.0375	0.0016
	(0.0097)	(0.0051)	(0.0046)
1988	0.0925	−0.0313	−0.0021
	(0.0080)	(0.0042)	(0.0032)
1989	0.0883	−0.0280	−0.0035
	(0.0067)	(0.0059)	(0.0034)
1990	0.0924	−0.0296	−0.0067
	(0.0095)	(0.0049)	(0.0050)
1991	0.0818	−0.0299	NA
	(0.0059)	(0.0040)	
1992	0.1177	NA	NA
	(0.0079)		

Source. Blundell *et al.* (2008).

Table 2: The Autocovariance Matrix of Income Growth BHPS

Year	$var(\Delta y_t)$	$cov(\Delta y_{t+1}, \Delta y_t)$	$cov(\Delta y_{t+2}, \Delta y_t)$
1980	0.1429	−0.0504	−0.0044
	(0.0071)	(0.0042)	(0.0039)
1981	0.0717	−0.0220	−0.0074
	(0.0075)	(0.0034)	(0.0037)
1982	0.0718	−0.0226	−0.0081
	(0.0051)	(0.0035)	(0.0026)
1983	0.0783	−0.0209	−0.0094
	(0.0066)	(0.0034)	(0.0042)
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	(0.0059)	(0.0040)	
1992	0.1177	NA	NA
	(0.0079)		

Source. Blundell and Etheridge (2008).

2. Intertemporal Choice and the Evolution of the Consumption Distribution

2.1. Self-insurance?

- Individuals can self-insure using a credit market with access to a risk-free bond with real return r_{t+j}.

- Consumption and income are linked through the intertemporal budget constraint:

$$A_{i,t+j+1} = (1 + r_{t+j})(A_{i,t+j} + Y_{i,t+j} - C_{i,t+j}) \text{ with } A_{i,T} = 0. \tag{12}$$

- Constant relative risk averse (CRRA) preferences

$$u_t(C_{i,t+j}, Z_{i,t+j}) \equiv \frac{1}{(1+\delta)^j} \frac{C_{i,t+j}^{\beta} - 1}{\beta} \mathrm{e}^{Z'_{i,t+j}\vartheta}, \tag{13}$$

- FOC:

$$C_{i,t-1}^{\beta-1} = \frac{1 + r_{t-1}}{1+\delta} \mathrm{e}^{\Delta Z'_{i,t}\vartheta_t} \mathrm{E}_{t-1} C_{i,t}^{\beta-1}.$$

$$\Delta c_{i,t} \simeq \Delta Z'_{i,t}\vartheta'_t + \eta_{i,t} + \Gamma_{i,t}, \qquad (14)$$

- $c_{i,t} \equiv \Delta \log C_{i,t}$,
- $\vartheta'_t = (1-\beta)^{-1}\vartheta_{t,}$
- $\eta_{i,t}$ is a consumption growth shock with $E_{t-1}\eta_{i,t} = 0, \Gamma_{i,t}$ captures any slope in the consumption path due to interest rates, impatience or precautionary savings
- Error in the approximation is $\mathcal{O}\left(E_{t-1}\eta^2_{i,t}\right)$.
- Conveniently, with CRRA preferences, $\Gamma_{i,t}$ is independent of $C_{i,t}$.

2.2. Linking the Evolution of the Consumption and Income Distributions

- For log income growth in the permanent–transitory model (9, 10):

$$\Delta y_{i,t+k} = \zeta_{i,t+k} + \sum_{j=0}^{q} \theta_j \varepsilon_{i,t+k-j}. \tag{15}$$

- The intertemporal budget constraint (12):

$$\sum_{k=0}^{T-t} Q_{t+k} C_{i,t+k} = \sum_{k=0}^{L-t} Q_{t+k} Y_{i,t+k} + A_{i,t},$$

- Y is the level of income,
- T is death,
- L is retirement, and
- Q_{t+k} is discount factor: $1/\prod_{i=1}^{k}(1+r_{t+i}), k = 1, \dots, T-t$ (and $Q_t = 1$).

- Define

$$\pi_{i,t} = \sum_{k=0}^{L-t} Q_{t+k} Y_{i,t-k} \Big/ \sum_{k=0}^{L-t} Q_{t+k} Y_{i,t-k} + A_{i,t}, \tag{16}$$

- Share of future labour income in current human and financial wealth

$$\gamma_t \simeq \frac{r}{1+r}\left[1 + \sum_{j=1}^{q} \theta_j \Big/ (1+r)^j\right] \tag{17}$$

- Annuity factor (for $r_t = r$).

- Blundell et al. (2013): stochastic individual element $\eta_{i,t}$ in consumption growth (14) is approximated by

$$\eta_{i,t} \simeq \pi_{i,t}\left(\zeta_{i,t} + \gamma_{Lt}\varepsilon_{i,t}\right),$$

where

$$\pi_{it} \approx \frac{\text{Human wealth}_{it}}{\text{Assets}_{it} + \text{Human wealth}_{it}}$$

Link to Further Discussion

2.3. When Does Consumption Inequality Measure Welfare Inequality?

3. Partial Insurance

- Extends (encompasses) the previous model

3.1. Consumption Dynamics with Partial Insurance

- Blundell et al. (2008) introduce transmission parameters ϕ_t and ψ_t.
- For any birth cohort, the consumption growth relationship (18) is

$$\Delta \ln C_{it} \cong \Gamma_{it} + \Delta Z_{it}' \varphi^c + \xi_{it} + \phi_t \zeta_{it} + \psi_t \varepsilon_{it}, \tag{19}$$

- Partial insurance w.r.t. permanent shocks implies $0 \leq 1 - \phi_t \leq 1$
- Full insurance: ϕ_t=0
- Partial insurance w.r.t. transitory shocks implies $0 \leq 1 - \psi_t \leq 1$.
- Full insurance: ψ_t=0

- The expressions $1 - \phi_t$ and $1 - \psi_t$ then measure the fractions insured and subsume π_t and γ_t from the self-insurance model.

- A (latent) factor structure that provides the panel data moments linking the evolution of distribution of consumption to the evolution of labour income distribution.

- Describes how consumption updates in response to income shocks.

3.2. The Key Panel Data Moments

- Taking the models for income dynamics (15) and consumption dynamics (19) together.

- Derive the second-order panel data variances and autocovariances that serve to identify the unknown transmission parameters, ϕ_t and ψ_t, of the partial-insurance specification.

- The autocovariance structure for log-adjusted income growth ($\Delta y_t \equiv \Delta \ln Y_t - \Delta Z_t' \phi^y$) is given in (11).

- For log consumption $\left(\Delta_{c_t} \equiv \Delta \ln C_{it} \cong \Gamma_{it} + \Delta Z_{it}' \varphi^c\right)$:

$$\operatorname{cov}(\Delta c_t, \Delta c_{t+s}) = \phi_t^2 \operatorname{var}(\zeta_t) + \psi_t^2 \operatorname{var}(\varepsilon_t) + \operatorname{var}(\xi_t) \qquad (20)$$

for *s=0* and zero otherwise.

- Cross-moments between income and consumption growth:

$$\operatorname{cov}(\Delta c_t, \Delta y_{t+s}) = \begin{cases} \phi_t \operatorname{var}(\zeta_t) + \psi_t \operatorname{var}(\varepsilon_t) \\ \psi_t \operatorname{cov}(\varepsilon_t, \Delta v_{t+s}) \end{cases} \tag{21}$$

- Summary of the key panel data moments is given by

$$\begin{aligned} \operatorname{var}(\Delta y_t) &= \operatorname{var}(\zeta_t) + \operatorname{var}(\Delta\varepsilon_t) \\ \operatorname{cov}(\Delta y_{t+1}, \Delta y_t) &= -\operatorname{var}(\varepsilon_t) \\ \operatorname{var}(\Delta c_t) &= \phi_t^2 \operatorname{var}(\zeta_t) + \psi_t^2 \operatorname{var}(\varepsilon_t) + \operatorname{var}(\xi_{it}) + \operatorname{var}(u_{it}^c) \\ \operatorname{var}(\Delta c_t, \Delta c_{it+1}) &= -\operatorname{var}(u_{it}^c) \\ \operatorname{cov}(\Delta c_t, \Delta y_t) &= \phi_t \operatorname{var}(\zeta_t) + \psi_t \operatorname{var}(\varepsilon_t) \\ \operatorname{cov}(\Delta c_t, \Delta y_{t+1}) &= -\psi_t \operatorname{var}(\varepsilon_t) \end{aligned} \tag{22}$$

3.3. Identification

(follows from using sample moments)

3.6. Partial-insurance Parameters for the US

- The PSID contains a measure of total food expenditures (food!)

- To line up the measures as best as is possible, five-quarter respondents only (annual expenditure measures) from the CEX are utilised.

- Otherwise, the sample selection is similar to that for the PSID. Further detail of this approach and a comparison of both data sources are in Blundell et al. (2004), which builds on the earlier work of Skinner (1987).

- Table 3 presents the implied autocovariance structure between consumption and income growth.

- Table 4 provides the estimates of the partial-insurance parameters / and w from (19) for the baseline specification. It also shows results for specifications which allow the transmission parameters to differ by birth cohort and by education level.

Table 3: The Autocovariance Matrix of Consumption Growth in the US

Year	$var(\Delta c_t)$	$cov(\Delta c_{t+1}, \Delta c_t)$	$cov(\Delta c_{t+2}, \Delta c_t)$
1980	0.1275	−0.0526	0.0022
	(0.0097)	(0.0076)	(0.0056)
1981	0.1197	−0.0573	0.0025
	(0.0116)	(0.0084)	(0.0043)
1982	0.1322	−0.0641	0.0006
	(0.0110)	(0.0087)	(0.0060)
1983	0.1532	−0.0691	−0.0056
	(0.0159)	(0.0100)	(0.0067)
1984	0.1869	−0.1003	−0.0131
	(0.0173)	(0.0163)	(0.0089)
1985	0.2019	−0.0872	NA
	(0.0244)	(0.0194)	
1986	0.1628	NA	NA
	(0.0184)		
1987	NA	NA	NA
1988	NA	NA	NA
1989	NA	NA	NA
1990	0.1751	−0.0602	−0.0057
	(0.0221)	(0.0062)	(0.0067)
1991	0.1646	−0.0696	NA
	(0.0142)	(0.0100)	
1992	0.1467	NA	NA
	(0.0130)		

Source. Blundell *et al.* (2008).

Table 4: Partial-insurance Parameter Estimates

Transmission parameters	Whole sample	No college	College	Born 1940s	Born 1930s
ϕ (Partial-insurance permanent shock)	0.6423	0.9439	0.4194	0.7928	0.6889
	(0.0945)	(0.1783)	(0.0924)	(0.1848)	(0.2393)
ψ (Partial-insurance transitory shock)	0.0533	0.0768	0.0675	0.0273	−0.0381
	(0.0435)	(0.0602)	(0.0550)	(0.0705)	(0.0737)

Notes. Standard errors in parenthesis. This Table reports DWMD results of the parameters of interest. See Blundell *et al.* (2008) for results allowing for time-varying variances of measurement error in consumption.
Source. Blundell *et al.* (2008).

- 65% of permanent income shocks not insured.
- All of transitory shocks insured.

3.7. The Importance of Measuring Assets

Table 5: Wealth and Durables

Consumption: Income: Sample:	Non-durable Net income Baseline	Non-durable Net income Low wealth	Non-durable Net income High wealth	Total Net income Low wealth
ϕ (Partial-insurance permanent shock)	0.6423	0.8489	0.6248	1.0342
	(0.0945)	(0.2848)	(0.0999)	(0.3517)
ψ (Partial-insurance transitory shock)	0.0533	0.2877	0.0106	0.3683
	(0.0435)	(0.1143)	(0.0414)	(0.1465)

Note. See Table 4.

- To assess the importance of this mechanism for low-wealth families, we can examine the same selection of low-wealth households but now include durable expenditures in our consumption measure.

- In the final column in Table 5, the transmission parameter for transitory shocks is now even larger than column 2 and the permanent shock parameter has a point estimate of unity.

- Once durable expenditures are included, consumption growth is even more sensitive to transitory shocks for low-wealth families.

- Transmission parameters subsume self-insurance and do not allow us to separate the various insurance mechanisms.

3.8. Excess Insurance?

- 35% of permanent shocks not reflected in changes of consumption: why?

4. Additional ‘Insurance’ Mechanisms

4.1. Taxes and Transfers

Table 6: Taxation and Other Earnings

Consumption: Income:	Non-durable Net income	Non-durable Earnings only	Non-durable Male earnings
ϕ (Partial-insurance permanent shock)	0.6423	0.3700	0.2245
	(0.0945)	(0.0574)	(0.0493)
ψ (Partial-insurance transitory shock)	0.0533	0.0633	0.0502
	(0.0435)	(0.0309)	(0.0294)

Note. See Table 4.

- A simple way to assess the ‘insurance’ value of the tax and transfer system in the context of the partial-insurance approach is to examine the impact on the insurance parameters of changing the income definition to be gross of taxes and transfers.

- A reduction in the transmission parameters would indicate the degree of additional insurance.

- The second column of Table 6 shows the results of such an experiment using the partial-insurance modelling framework and PSID–CEX data source analysis above.

- The reduction in the estimated transmission parameter for permanent shocks / from 0.64 to 0.37 indicates the important role of taxes and transfers in insuring family incomes.

- The final column points to the importance of family labour supply to which we now turn.

4.2. Family Labour Supply

- To investigate the family labour supply, Blundell et al. (2012) use the enhanced data from the post-1996 PSID to estimate a model of consumption inequality and family labour supply for couples.

- The new asset data allow a direct measure of pit and the more comprehensive consumption data avoid the need for imputation.

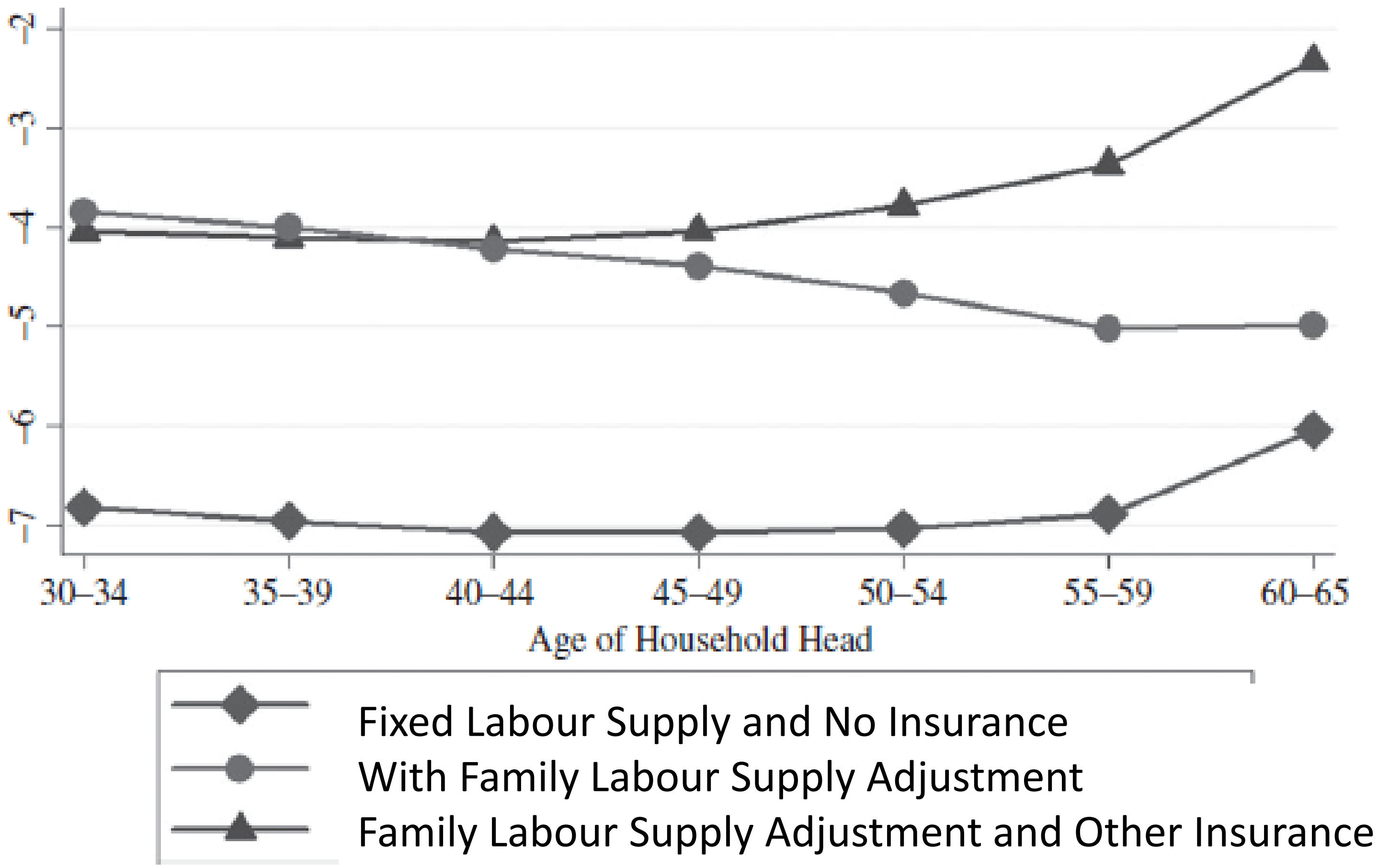
- Their analysis extends previous work and expresses the distributional dynamics of consumption and earnings growth as functions of Frisch elasticities, ‘insurance parameters’ and wage shocks.

- The impact of a permanent shock to male wages $w_m$is shown to generalize the transmission parameter $\pi_{i,t}$ in expression (18) to take the form:

$$\pi_{i,t} s_{i,m,t} \frac{\eta_{c,p}\left(1+\eta_{h_m,w_m}\right)}{\eta_{c,p}+\left(1-\pi_{i,t}\right)\bar{\eta}_{h,w}}, \tag{24}$$

- $s_{i,m,t}$ is the share of the male earnings in future human capital wealth, and the $\eta_{c,p}$ and $\eta_{h,w}$ parameters are the Frisch consumption and hours of work elasticities respectively.

Fig. 6. The Average Impact on Consumption of a Permanent Shock to Male Wage, US 10% ↓



Appendix

Approximation of the Euler equation

- Blundell, Pistaferri and Preston (2008) consider the consumption problem faced by household i of age a in period t.
- Assuming that preferences are of the CRRA form, the objective is to choose a path for consumption C so as to:

$$\max_{C} E_a \sum_{j=0}^{A-a} \beta^j \frac{C_{i,a+j,t+j}^{1-\gamma} - 1}{1-\gamma} e^{Z'_{i,a+j,t+j}\vartheta_{a+j}}. \qquad (6)$$

where $Z_{i,a+j,t+j}$ incorporates taste shifters (such as age, household composition, etc.), and we denote with $E_a(.) = E(.|\Omega_{i,a,t})$.

- Maximization of (6) is subject to the budget constraint which in the self-insurance model assumes individuals have access to a risk free bond with real return r

$$A_{ia+j+1} = (1+r)\left(A_{i,a+j,t+j} + Y_{i,a+j,t+j} - C_{i,a+j,t+j}\right) \tag{7}$$
$$A_{i,A} = 0 \tag{8}$$

with $A_{i,a,t}$ given.

- Blundell, Pistaferri and Preston (2008) set the retirement age after which labor income falls to zero at L, assumed known and certain, and the end of the life-cycle at age A.

- They assume that there is no uncertainty about the date of death.
- With budget constraint (7), optimal consumption choices can be described by the Euler equation (assuming for simplicity that there is no preference heterogeneity, or $\vartheta_a = 0$):

$$C_{i,a-1,t-1}^{-\gamma} = \beta \left(1 + r\right) E_{a-1} C_{i,a,t}^{-\gamma}. \tag{9}$$

- As it is, equation (9) is not useful for empirical purposes.

- Blundell, Pistaferri and Preston (2008) show that the Euler equation can be approximated as follows:

$$\Delta \log C_{i,a,t} \simeq \eta_{i,a,t} + f^{C}_{i,a,t}$$

where $\eta_{i,a,t}$ is a consumption shock with $E_{a-1}(\eta_{i,a,t}) = 0$, $f^{c}_{i,a,t}$ captures any slope in the consumption path due to interest rates, impatience or precautionary savings and the error in the approximation is $O(E_a \eta^2_{i,a,t})$.

- Suppose that any idiosyncratic component to this gradient to the consumption path can be adequately picked up by a vector of deterministic characteristics $\Gamma^{c}_{i,a,t}$ and a stochastic individual element $\xi_{i,a,t}$

$$\Delta \log C_{i,a,t} - \Gamma^{c}_{i,a,t} = \Delta c_{i,a,t} \simeq \eta_{i,a,t} + \xi_{i,a,t}.$$

- Assume log income is

$$\log Y_{i,a,t} = p_{i,a,t} + \varepsilon_{i,a,t} \tag{10}$$
$$p_{i,a,t} = \Gamma^{y}_{i,a,t} + p_{i,a-1,t-1} + \zeta_{i,a,t} \tag{11}$$

where $\Gamma^{y}_{i,a,t}$ represent observable characteristic influencing the growth of income.

- Income growth can be written as:

$$\Delta \log Y_{i,a,t} - \Gamma^{y}_{i,a,t} = \Delta y_{i,a,t} = \zeta_{i,a,t} + \Delta \varepsilon_{i,a,t}.$$

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- The (ex-post) intertemporal budget constraint is

$$\sum_{j=0}^{A-a} \frac{C_{i,a+j,t+j}}{(1+r)^j} = \sum_{j=0}^{L-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j} + A_{i,a,t}$$

where A is the age of death and L is the retirement age.

- Applying the approximation above and taking differences in expectations gives

$$\eta_{i,a,t} \simeq \Xi_{i,a,t}\left[\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t}\right]$$

where π_a is an annuitization factor defined below in technical notes, $\Xi_{i,a,t} = \frac{\sum_{j=0}^{A-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j}}{\sum_{j=0}^{A-a} \frac{Y_{i,a+j,t+j}}{(1+r)^j} + A_{i,a,t}}$ is the share of future labor income in current human and financial wealth, and the error of the approximation is $O(\left[\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t}\right]^2 + E_{a-1}\left[\zeta_{i,a,t} + \pi_a \varepsilon_{i,a,t}\right]^2)$.

- Then

$$\Delta \log C_{i,a,t} \simeq \xi_{i,a,t} + \Xi_{i,a,t}\zeta_{i,a,t} + \pi_a \Xi_{i,a,t}\varepsilon_{i,a,t} \qquad (12)$$

with a similar order of approximation error.

- The random term $\xi_{i,a,t}$ can be interpreted as the innovation to higher moments of the income process.

- The interpretation of the impact of income shocks on consumption growth in the PIH model with CRRA preferences is straightforward.
- For individuals a long time from the end of their life with the value of current financial assets small relative to remaining future labor income, $\Xi_{i,a,t} \simeq 1$, and permanent shocks pass through more or less completely into consumption whereas transitory shocks are (almost) completely insured against through saving.
- Precautionary saving can provide effective self-insurance against permanent shocks only if the stock of assets built up is large relative to future labor income, which is to say $\Xi_{i,a,t}$ is appreciably smaller than unity, in which case there will also be some smoothing of permanent shocks through self insurance.

- The most important feature of the approximation approach is to show that the effect of an income shock on consumption depends not only on the persistence of the shock and the planning horizon (as in the LC-PIH case with quadratic preferences), but also on preference parameters.
- *Ceteris paribus*, the consumption of more prudent households will respond less to income shocks.
- The reason is that they can use their accumulated stock of precautionary wealth to smooth the impact of the shocks (for which they were saving precautiously against in the first place).

Technical Discussion Drawn from Blundell, Low, and Preston (2008)
Approximating the Euler Equation

- The household plan at age t is to maximize the expected remaining lifetime utility:

$$E_t \sum_{\tau=0}^{T-t} \frac{U(c_{i,t+\tau})}{(1+\delta)^{\tau}}$$

- We begin by calculating the error in approximating the Euler equation.

$$E_t U'\left(c_{it+1}\right) = U'\left(c_{it}\right)\left(\frac{1+\delta}{1+r}\right) = U'\left(c_{it} e^{\kappa_{it+1}}\right) \qquad (13)$$

for some κ_{it+1}

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- By exact Taylor expansion of period $t+1$ marginal utility in $\ln c_{it+1}$ and around $\ln c_{it} + \kappa_{it+1}$, there exists a $\tilde{c}$ between $c_{it}e^{\kappa_{it+1}}$ and c_{it+1} such that

$$U'(c_{it+1}) = U'\left(c_{it}e^{\kappa_{it+1}}\right)\left[1 + \frac{1}{\kappa\left(c_{it}e^{\kappa_{it+1}}\right)}\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2}\beta\left(\tilde{c}, c_{it}e^{\kappa_{it+1}}\right)\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\right] \quad (14)$$

where $\kappa(c) \equiv U'(c)/cU''(c) < 0$ and $\beta(\tilde{c}, c) \equiv [\tilde{c}^2 U'''(\tilde{c}) + \tilde{c}U''(\tilde{c})]/U'(c)$.

- Taking expectations

$$E_t U'(c_{it+1}) = U'\left(c_{it} e^{\kappa_{it+1}}\right)\left[1 + \frac{1}{\kappa\left(c_{it} e^{\kappa_{it+1}}\right)} E_t\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2} E_t\left\{\beta\left(\tilde{c}, c_{it} e^{\kappa_{it+1}}\right)\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\right\}\right] \tag{15}$$

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- Substituting for $E_t U'(c_{it+1})$ from (13)

$$\frac{1}{\kappa(c_{it}e^{\kappa_{it+1}})} E_t\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right] + \frac{1}{2} E_t\left\{\beta(\tilde{c}, c_{it}e^{\kappa_{it+1}})\left[\Delta \ln c_{it+1} - \kappa_{it+1}\right]^2\right\} = 0 \quad (16)$$

and thus

$$\Delta \ln C_{it+1} = \kappa_{it+1} - \frac{\kappa(c_{it}e^{\kappa_{it+1}})}{2} E_t\left\{\beta\left(\tilde{c}, c_i t e^{\kappa_{it+1}}\right)\left[\Delta \ln c_{it} + 1 e^{\kappa_{it+1}}\right]^2\right\} + \varepsilon_{it+1} \quad (17)$$

where the consumption innovation ε_{it+1} satisfies $E_t \varepsilon_{it+1} = 0$. As $E_t \varepsilon_{it+1}^2 \to 0$, $\beta(\tilde{c}, c_{it}e^{\kappa_{it+1}})$ tends to a constant and therefore by Slutsky's theorem

$$\Delta \ln c_{it+1} = \varepsilon_{it+1} + \kappa_{it+1} + \mathcal{O}\left(E_t\left|\varepsilon_{it+1}\right|^2\right). \quad (18)$$

- If preferences are CRRA then κ_{it+1} does not depend on $c_i t$ and is common to all households, say κ_{t+1}.
- The log of consumption therefore follows a martingale process with common drift

$$\Delta \ln c_{it+1} = \varepsilon_{it+1} + \kappa_{t+1} + \mathcal{O}\left(E_t \left|\varepsilon_{it+1}\right|^2\right). \tag{19}$$

Approximating the Lifetime Budget Constraint

- The second step in the approximation is relating income risk to consumption variability.
- In order to make this link between the consumption innovation ε_{it+1} and the permanent and transitory shocks to the income process, we loglinearise the intertemporal budget constraint using a general Taylor series approximation (extending the idea in Campbell 1993).

- Define a function $F\colon \mathbb{R}^{N+1} \to \mathbb{R}$ by $F(\xi) = \ln \sum_{j=0}^{N} \exp \xi_j$.
- By exact Taylor expansion around an arbitrary point $\xi^0 \in \mathbb{R}^{N+1}$

$$F(\xi) = \ln \sum_{j=0}^{N} \exp \xi_j^0 + \sum_{j=0}^{N} \frac{\exp \xi_j^0}{\sum_{k=0}^{0} \exp \xi_k^0} (\xi_j - \xi_j^0) + \frac{1}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} \frac{\partial^2 F(\tilde{\xi})}{\partial \xi_j \partial \xi_k} \left(\xi_j - \xi_j^0\right) \left(\xi_k - \xi_k^0\right) \tag{20}$$

where $\tilde{\xi}$ lies between ξ and ξ^0 and is used to make the expansion exact.

- The coefficients in the remainder term are given by

$$\frac{\partial^2 F(\tilde{\xi})}{\partial \xi_j \partial \xi_k} = \frac{\exp \tilde{\xi}_j}{\sum_k \exp \tilde{\xi}_k} \left(\delta_{jk} - \frac{\exp \tilde{\xi}_j}{\sum_k \exp \tilde{\xi}_k} \right), \tag{21}$$

where δ_{jk} denotes the Kronecker delta.

- These coefficients are bounded because $0 < \exp \tilde{\xi}_j / \sum_k \exp \tilde{\xi}_k < 1$.

- Hence, taking expectations of (21) subject to information set $\mathcal{I}$

$$E_{\mathcal{I}}\left[F(\xi)\right] = \ln \sum_{j=0}^{N} \exp \xi_j^0 + \sum_{j=0}^{N} \frac{\exp \xi_j^0}{\sum_{k=0}^{N} \exp \xi_k^0} (E_{\mathcal{I}} \xi_j - \xi_j^0)$$
$$+ \frac{1}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} E_{\mathcal{I}} \left(\frac{\partial^2 F(\tilde{\xi})}{\partial \xi_j \partial \xi_k} (\xi_j = \xi_j^0)(\xi_k - \xi_k^0) \right). \tag{22}$$

- We apply this expansion firstly to the expected present value of consumption, $\sum_{j=0}^{T-t} c_{it+j}(1+r)^{-j}$.
- Let $N = T - t$ and let

$$\begin{aligned} \xi_j &= \ln c_{it+j} - j \ln(1+r) \\ \xi_j^0 &= E_{t-1} \ln c_{it+j} - j \ln(1+r), \quad i = 0, \dots, T-t. \end{aligned} \tag{23}$$

- Then, substituting equation (23) into equation (22) and noting only the order of magnitude for the remainder term,

$$E_{\mathcal{I}}\left[\ln\sum_{j+0}^{T-t}\frac{c_{it+j}}{(1+r)^j}\right]=\ln\sum_{j+0}^{T-t}\exp\left[E_{t-1}\ln c_{it+j}-j\ln(1+r)\right]$$
$$+\sum_{j=0}^{T-t}\theta_{it+j}\left[E_{\mathcal{I}}\ln c_{it+j}-E_{t-1}\ln c_{it+j}\right]$$
$$+\mathcal{O}(E_{\mathcal{I}}\|\varepsilon_{it}^{T}\|^2) \qquad (24)$$

where

$$\theta_{it+j}=\frac{\exp\xi_j^0}{\sum_{k=0}^{N}\exp\xi_k^0}=\frac{\exp\left[E_{t-1}\ln c_{it+j}-j\ln(1+r)\right]}{\sum_{k=0}^{T-t}\exp\left[E_{t-1}\ln c_{it+k}-kl(1-r)\right]}$$

and ε_{it}^{T} denotes the vector of future consumption innovations $(\varepsilon_{it},\varepsilon_{it+1},\ldots,\varepsilon_{iT})'$.

- The term θ_{it+j} can be seen as an annuitisation factor for consumption.
- We now apply the expansion (22) to the expected present vale of resources, $\sum_{j=0}^{R-t-1}(1+r)^{-j}y_{it+j} + A_{iT+1}(1+r)^{-(T-t)}$.
- Let $N = R - T$ and let

$$\begin{aligned}
\xi_j &= \ln y_{it+j} - j\ln(1-r) \\
\xi_j^0 &= E_{t-1}\ln y_{it+j} - j\ln(1+r) \quad j = 0, \ldots, R-t-1 \\
\xi_N &= \ln\left[A_{it} - A_{iT+1}(1+r)^{-(T-t)}\right] \\
\xi_N^0 &= E_{t-1}\ln[A_{it} - A_{iT-1}(a+r)^{-(T-t)}].
\end{aligned} \tag{25}$$

- Then, substituting equation (25) into equation (22), and again noting only the order of magnitude for the remainder term,

$$
\begin{aligned}
& E_{\mathcal{I}} \ln \left(\sum_{j=0}^{R-r-1} \frac{y_{it+j}}{(1+r)^j} + A_{it} - \frac{A_{iT+1}}{(1+r)^{T-t}} \right) = \\
& \quad \ln \left[\sum_{j=0}^{R-t-1} \exp \left[E_{t-1} \ln y_{it+j} - j \ln(1+r) \right] + \exp E_{t-1} \ln \left[A_{it} - \frac{A_{iT+1}}{(1+r)^{T-t}} \right] \right] \\
& \quad + \pi_{it} \sum_{j=0}^{R-t-1} \alpha_{t+j} \left[E_{\mathcal{I}} \ln y_{it+j} - E_{t-1} \ln y_{it+j} \right] \\
& \quad + (1-\pi_{it}) \left[E_{\mathcal{I}} \ln \left[A_{it} - \frac{A_{iT+1}}{(1+r)^{T-1}} \right] - E_{t-1} \ln \left[A_{it} - \frac{A_{iT+1}}{(a+r)^{T-t}} \right] \right] \\
& \quad + \mathcal{O} \left(E_{t-1} \| (\nu_{it}^{R-1}) \|^2 \right) \qquad (26)
\end{aligned}
$$

where

$$\pi_{t+j} =$$

$$= \frac{\exp[E_{t-1} \ln y_{it+j} - j \ln(1+r)]}{\sum_{k=0}^{R-t-1} \exp[E_{t-1} \ln y_{it+k} - k \ln(1+r)]}$$

$$= \frac{\exp\left[\sum_{k=0}^{j}(\eta_{t+k} + E_{t-1}\bar{u}_{t+k}) + E_{t-1}\bar{u}_{t+j} - j \ln(1+r)\right]}{\sum_{k=0}^{R-t-1} \exp\left[\sum_{t=0}^{k}(\eta_{t+l} + E_{t-1}\bar{u}_{t+l}) + E_{t-1}\bar{u}_{t+k} - k \ln(1+r)\right]}$$

(This is the same as $\pi_{i,a,t}$ in Blundell et al., 2008.)

...can be seen as an annuitisation factor for income (common within a cohort because of the assumption of common income trends) and

$$\begin{aligned}\Xi_{i,a,t} =& 1 - \frac{\exp \xi_N^0}{\sum_{k=0}^{N} \exp \xi_k^0} \\ =& \frac{\sum_{j=0}^{R-t-1} \exp[E_{t-1} \ln y_{t+j} - j \ln(1+r)]}{\sum_{j=0}^{R-t-1} \exp[E_{t-1} \ln y_{it+j} - j \ln(1+r)] + \exp E_{t-1} \ln[A_{it} - A_{iT+1}/(1+R)^{T-t}]}\end{aligned}$$

is (roughly) the share of expected future labor income in current human and financial wealth (net of terminal assets) and ν_{it}^{R-1} denotes the vector of future income shocks $(\nu_{it}', \nu_{it+1}', \dots, \nu_{iR-1}')'$.

This corresponds to the $\Xi_{i,a,t}$.

- We are able to equate the subjects of equations (24) and (26) because the realised budget must balance and $\sum_{j=0}^{R-t} \frac{c_{it+j}}{(1+r)^j}$ and $\sum_{j=0}^{R-t-1} \frac{y_{it+j}}{(1+r)^j} + A_{it} - \frac{A_i T+1}{(1+r)^{T-t}}$ therefore have the same distribution.

- We use (24) and (26), taking differences between expectations at the start of the period, before the shocks are realised, and at the end of the period, after the shocks are realised.
- This gives

$$\begin{aligned}\varepsilon_{it} &+ \mathcal{O}(E_t \parallel \epsilon_{it}^T \parallel^2 + E_{t-1} \parallel \epsilon_{it}^T \parallel^2) \\ &= \pi_{it}(v_{it} + \alpha_t u_{it}) + \pi_{it}\Omega_t \\ &\quad + \mathcal{O}(E_t \parallel \nu_{it}^{R-1} \parallel^2 + E_{t-1} \parallel \nu_{it}^{R-1} \parallel^2),\end{aligned}$$

where the left hand side is the innovation to the expected present value of consumption and the right hand side is the innovation to the expected present value of income and

$$\Omega_t = \sum_{j=0}^{R-t-1} \pi_{t+j} \sum_{k=0}^{j} (E_t - E_{t-1})\omega_{t+k}$$

captures the revision to expectations of current and future common shocks.

- Squaring the two sides, taking expectations and inspecting terms reveals that the terms which are $\mathcal{O}(E_t \parallel \varepsilon_{it}^T \parallel^2 + E_{t-1} \parallel \varepsilon_{it}^T \parallel^2)$ are $\mathcal{O}(E_t \parallel \nu_{it}^{R-1} \parallel^2 + E_{t-1} \parallel \nu_{it}^{R-1} \parallel^2)$.
- Furthermore, since, for all $j \geq 0$, $\parallel \nu_{it+j} \parallel^2 = \mathcal{O}_p(E_t \parallel \nu_{it+j} \parallel^2)$ by Chebyshev's inequality, $E_t \parallel \nu_{it}^{R-1} \parallel^2 = \mathcal{O}_p(E_{t-1} \parallel \nu_{it}^{R-1} \parallel^2)$.
- Thus

$$\varepsilon_{it} = \Xi_{i,a,t}(v_{it} + \pi_{i,t} u_{it}) + \Xi_{i,a,t}\Omega_t + \mathcal{O}_p(E_{t-1} \parallel \nu_{it}^{R-1} \parallel^2)$$

and therefore

$$\Delta \ln c_{it} = \kappa_t + \Xi_{i,a,t}(v_{it} + \alpha_t u_{it}) + \Xi_{i,a,t}\Omega_t + \mathcal{O}_p(E_{t-1} \parallel \nu_{it}^{R-1} \parallel^2).$$

End of Digression.

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2.3. When Does Consumption Inequality Measure Welfare Inequality?

- Does consumption better reflects household welfare or some measure of current income?

- Define $\bar{Y}_i$ as that *certain* present discounted value of lifetime income which would allow the individual to achieve the same expected utility.

- The consumption stream $\bar{C}_i = \bar{C}(EU_i)$ would be chosen if $\bar{Y}_i$ satisfies

$$\sum_t u_t(\bar{C}_{it}) \equiv \mathrm{E}\left[\sum_t u_t(C_{it})\right] = EU_i.$$

- $\bar{C}$ is certainly equivalent.

- Comparisons across individuals facing different income risk.

- Constant Absolute Risk Aversion (CARA) preferences

$$u_t(C_{it}) = -\alpha_t \exp(-\gamma_t C_{it}) \quad \alpha_t, \gamma_t > 0, t > 0.$$

- Blundell and Preston (1998): $C_{it} \geq C_{jt}$ implies $EU_i \geq EU_j$ whenever individuals *i* and *j* share the same year of birth if and only if $C_i = \bar{C}(EU_i)$, whatever the distribution of future income.

- CRRA case (13): imply $C_{i0} < \tilde{C}_{i0}$, i.e. that there is 'excess' precautionary saving if higher incomes decrease risk aversion.

- Consumption overestimates the welfare cost of income risk.

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