

The Normal Generalized Roy Model

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and Program Evaluation Approaches to Evaluating Policy
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Example: The Generalized Roy Model for the Normal Case

$$Y_1 = \mu_1(X) + U_1$$

$$Y_0 = \mu_0(X) + U_0$$

$$C = \mu_C(Z) + U_C$$

Net Benefit: $I = Y_1 - Y_0 - C$

$$I = \underbrace{\mu_1(X) - \mu_0(X) - \mu_C(Z)}_{\mu_D(Z)} + \underbrace{U_1 - U_0 - U_C}_{-V}$$

$$(U_0, U_1, U_C) \perp\!\!\!\perp (X, Z)$$

$$E(U_0, U_1, U_C) = (0, 0, 0)$$

$$V \perp\!\!\!\perp (X, Z)$$

- It is heuristically useful to think of V as an unobserved cost, e.g., an unobserved cost for example?

- Assume normally distributed errors.
- Assume Z contains X but may contain other variables (exclusions)

$$\begin{aligned}\text{Observed } Y : \quad Y &= DY_1 + (1 - D)Y_0 \\ D &= 1(I \geq 0) = 1(\mu_D(Z) \geq V)\end{aligned}$$

- Assume $V \sim N(0, \sigma_V^2)$

- Propensity Score:

$$\Pr(D = 1 \mid Z = z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$E(Y \mid D = 1, X = x, Z = z) = \mu_1(X) + \underbrace{E(U_1 \mid \mu_D(z) \geq V)}_{K_1(P(z))}$$

because $(X, Z) \perp\!\!\!\perp (U_1, V)$.

- Under normality we obtain

$$E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right) = \frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \tilde{\lambda}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

- Why?

$$U_1 = \text{Cov} \left(U_1, \frac{V}{\sigma_V} \right) \frac{V}{\sigma_V} + \varepsilon_1$$

$$\varepsilon_1 \perp\!\!\!\perp V$$

$$\begin{aligned}
 E \left(\frac{V}{\sigma_V} \mid \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right) &= \frac{\int_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} t \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}{\int_{-\infty}^{\frac{\mu_D(z)}{\sigma_V}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} = \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right) \\
 &= \frac{\frac{-1}{\sqrt{2\pi}} e^{(-\frac{1}{2}) \left(\frac{\mu_D(z)}{\sigma_V} \right)^2}}{\Phi \left(\frac{\mu_D(z)}{\sigma_V} \right)} = \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right) = \frac{-\phi \left(\frac{\mu_D(z)}{\sigma_V} \right)}{\Phi \left(\frac{\mu_D(z)}{\sigma_V} \right)}
 \end{aligned}$$

- Notice

$$\lim_{\mu_D(z) \rightarrow \infty} \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right) = 0$$
$$\lim_{\mu_D(z) \rightarrow -\infty} \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right) = -\infty$$

- Propensity score:

$$P(z) = \Pr(D = 1 \mid Z = z) = \Phi \left(\frac{\mu_D(z)}{\sigma_V} \right)$$
$$\therefore \left(\frac{\mu_D(z)}{\sigma_V} \right) = \Phi^{-1} (\Pr(D = 1 \mid Z = z))$$

- Thus we can replace $\frac{\mu_D(z)}{\sigma_V}$ with a known function of $P(z)$

- Notice that because $(X, Z) \perp\!\!\!\perp (U, V)$, Z enters the model (conditional on X) only through $P(Z)$.
- This is called *index sufficiency*.

- Put all of these results together to obtain

$$E(Y | D = 1, X = x, Z = z) = \mu_1(x) + \left(\frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right)$$

$$= E(Y_1 | D = 1, X = x, Z = z) = \mu_1(x) + \left(\frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right)$$

$$\tilde{\lambda}(z) = E \left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} < \frac{\mu_D(z)}{\sigma_V} \right) < 0$$

$$\lambda(z) = E \left(\frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} \geq \frac{\mu_D(z)}{\sigma_V} \right) > 0$$

$$E(Y | D = 0, X = x, Z = z) = \mu_0(x) + \left(\frac{\text{Cov}(U_0, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \lambda \left(\frac{\mu_D(z)}{\sigma_V} \right)$$

$$\text{Var} \left(\frac{V}{\sigma_V} \right) = 1$$

$$\frac{V}{\sigma_V} = -\frac{(U_1 - U_0 - U_C)}{\sigma_V}$$

$$\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) = -\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) + \text{Cov}\left(U_0, \frac{V}{\sigma_V}\right) + \text{Cov}\left(U_C, \frac{V}{\sigma_V}\right)$$

In Roy model case ($U_C = 0$),

$$\begin{aligned}\text{Cov}\left(U_1, \frac{V}{\sigma_V}\right) &= -\text{Cov}\left(U_1, \frac{U_1 - U_0}{\sigma_V}\right) \\ &= -\frac{\text{Cov}(U_1 - U_0, U_1)}{\sqrt{\text{Var}(U_1 - U_0)}}\end{aligned}$$

- We can identify $\mu_1(x), \mu_0(x)$
- From Discrete Choice model we can identify

$$\frac{\mu_D(z)}{\sigma_V} = \frac{\mu_1(x) - \mu_0(x) - \mu_C(z)}{\sigma_V}$$

- If we have a regressor in X that does not affect $\mu_C(z)$ (say regressor x_j , so $\frac{\partial \mu_C(z)}{\partial x_j} = 0$), we can identify σ_V and $\mu_C(z)$.
- \therefore We can identify the net benefit function and the cost function up to scale.
- \therefore We can compute *ex-ante* subjective net gains up to scale.

- Method generalizes: Don't need normality
- Need "Large Support" assumption to identify ATE and TT
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$$\begin{aligned}
 E(Y \mid D = 1, X = x, Z = z) &= \mu_1(x) + \overbrace{K_1(P(z))}^{\text{control function}} \\
 E(Y \mid D = 0, X = x, Z = z) &= \mu_0(x) + \underbrace{K_0(P(z))}_{\text{control function}}
 \end{aligned}$$

$$\lim_{P(z) \rightarrow 1} E(Y \mid D = 1, X = x, Z = z) = \mu_1(x)$$

$$\lim_{P(z) \rightarrow 0} E(Y \mid D = 0, X = x, Z = z) = \mu_0(x)$$

- If we have this condition satisfied, we can identify ATE

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- ATE is defined in a limit set. This is true for any model with selection on unobservables (IV; selection models)

- What about treatment on the treated?

$$E(Y_1 - Y_0 \mid D = 1, X = x, Z = z)$$

- a From the data, we observe

$$E(Y_1 | D = 1, X = x, Z = z)$$

- b Can also create it from the model
- c $E(Y_0 | D = 1, X = x, Z = z)$ is a counterfactual

We know

$$E(Y_0 | D = 0, X = x, Z = z) = \mu_0(x) + \text{Cov} \left(U_0, \frac{V}{\sigma_V} \right) \lambda \left(\frac{\mu_D(Z)}{\sigma_V} \right)$$

(this is data)

- d We seek

$$E(Y_0 | D = 1, X = x, Z = z) = \mu_0(x) + \text{Cov} \left(U_0, \frac{V}{\sigma_V} \right) \tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right)$$

- But under normality, we know $\text{Cov} \left(U_0, \frac{V}{\sigma_V} \right)$
- We know $\frac{\mu_D(z)}{\sigma_V}$
- $\tilde{\lambda}(\cdot)$ is a known function.
- Can form $\tilde{\lambda} \left(\frac{\mu_D(z)}{\sigma_V} \right)$ and can construct counterfactual.

- More generally, without normality (but with $(X, Z) \perp\!\!\!\perp (U, V)$)

$$E(Y_1 | D = 1, X, Z) = E(Y | D = 1, X = x, Z = z) = \mu_1(x) + K_1(P(z))$$

$$E(Y_0 | D = 0, X, Z) = E(Y | D = 0, X = x, Z = z) = \mu_0(x) + \tilde{K}_0(P(z))$$

$$\text{where } K_1(P(z)) = E(U_1 | D = 1, X = x, Z = z) = E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V}\right)$$

$$\tilde{K}_1(P(z)) = E\left(U_1 \mid \frac{\mu_D(z)}{\sigma_V} \leq \frac{V}{\sigma_V}\right)$$

$$\tilde{K}_0(P(z)) = E\left(U_0 \mid \frac{\mu_D(z)}{\sigma_V} \leq \frac{V}{\sigma_V}\right)$$

- Use the transformation

$$\frac{F_V}{\sigma_V} \left(\frac{\mu_D(z)}{\sigma_V} \right) = P(z)$$

$$\frac{F_V}{\sigma_V} \left(\frac{V}{\sigma_V} \right) = U_D \quad (\text{a uniform random variable})$$

$$D = 1 \left(\frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right) = 1 (P(z) \geq U_D)$$

$$K_1(P(z)) = E(U_1 | P(z) > U_D)$$

$$K_1(P(z))P(z) + \tilde{K}_1(P(z))(1 - P(z)) = 0$$

\therefore we can construct $\tilde{K}_1(P(z))$

- Symmetrically

$$\tilde{K}_0(P(z)) = E(U_0 \mid P(z) \leq U_D)$$

$$K_0(P(z)) = E(U_0 \mid P(z) > U_D)$$

$$(1 - P(z))\tilde{K}_0(P(z)) + P(z)K_0(P(z)) = 0$$

- ∴ If we have “identification at infinity,” we can construct

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

- We can construct TT

$$E(Y_1 - Y_0 \mid D = 1, X = x, Z = z) =$$

$$= \underbrace{[\mu_1(x) + K_1(P(z))]}_{\text{factual}} - \underbrace{[\mu_0(x) + K_0(P(z))]}_{\text{counterfactual}}$$

- We can form $\mu_1(x) + K_1(P(z))$ from data
- We get $\mu_0(x)$ from limit set $P(z) \rightarrow 0$ identifies $\mu_0(x)$
- We can form $K_0(P(z)) = -\tilde{K}_0(P(z)) \frac{P(z)}{1-P(z)}$
- ∴ Can construct the desired counterfactual mean.

- Notice how we can get Effect of Treatment for People at the Margin:

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z)$$

- Under normality we have (as a result of independence)

$$\begin{aligned} E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) \\ &= \mu_1(x) - \mu_0(x) + E\left(U_1 - U_0 \mid \frac{\mu_D(z)}{\sigma_V} = \frac{V}{\sigma_V}, X = x, Z = z\right) \\ &= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \frac{\mu_D(z)}{\sigma_V} \end{aligned}$$

In the Roy model case where $U_C = 0$ but $\mu_C(z) \neq 0$

$$\begin{aligned} &= \mu_1(x) - \mu_0(x) - \sigma_V \left(\frac{\mu_D(z)}{\sigma_V}\right) \\ &= \mu_1(x) - \mu_0(x) - \mu_D(z) \\ &= \mu_C(z) \end{aligned}$$

(marginal gain = marginal cost)

- MTE for Normal Model:

$$\begin{aligned} E(Y_1 - Y_0 \mid V = v, X = x, Z = z) &= \\ &= \mu_1(x) - \mu_0(x) + \text{Cov} \left(U_1 - U_0, \frac{V}{\sigma_V} \right) v \end{aligned}$$

- Effect of Treatment for People at the Margin picks $v = \frac{\mu_D(z)}{\sigma_V}$

- Remember

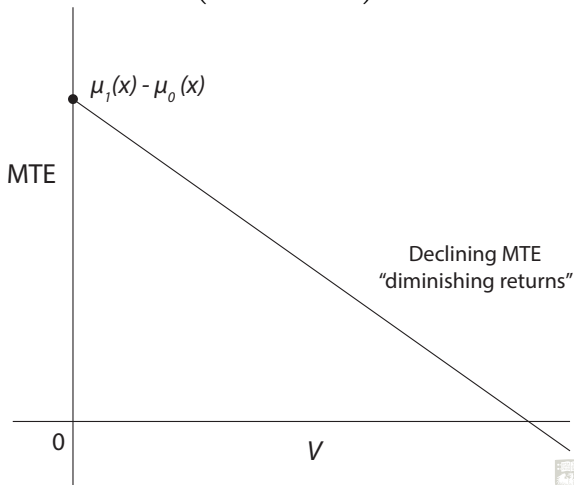
$$\begin{aligned}\frac{V}{\sigma_V} &= -\frac{\{U_1 - U_0 - U_C\}}{\sigma_V} \\ &= -\frac{\text{Cov}(U_1 - U_0, V)}{(\text{Var}(U_1 - U_0)) + \text{Cov}(U_1 - U_0, U_C)}\end{aligned}$$

- Roy Model: $U_C \equiv 0$

Meritocratic Society (Becker, 1967)

- Remember: Think of V as a cost component, i.e., psychic cost.

$$\text{Cov} \left(U_1 - U_0, \frac{V}{\sigma_V} \right) \leq 0$$

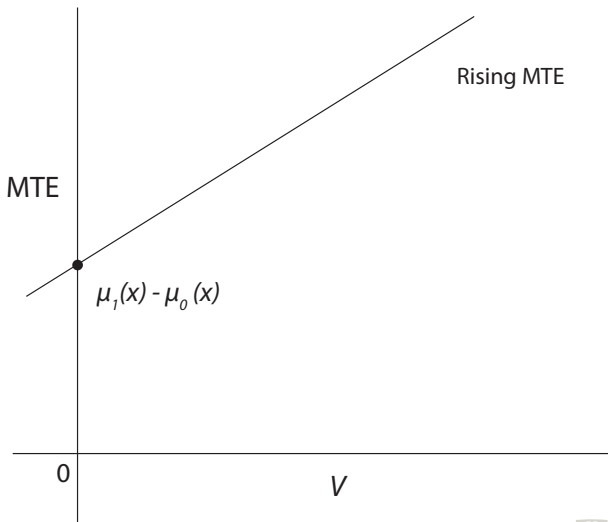


- Suppose

$$\text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \geq 0$$
$$\therefore \frac{\text{Cov}(U_1 - U_0, U_C)}{\sigma_V} > 0$$

- Unobserved components of costs rise with gross gains

Anti-Meritocratic Society



- Notice we can use the result that

$$\frac{\mu_D(z)}{\sigma_V} = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(P(z))$$
$$V = F^{-1}_{\left(\frac{V}{\sigma_V}\right)}(U_D)$$

- Effect of Treatment for People at Margin of Indifference Between Taking Treatment and Not:

$$\begin{aligned}
 E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) &= \\
 &= \mu_1(x) - \mu_0(x) + \text{Cov} \left(U_1 - U_0, \frac{V}{\sigma_V} \right) \underbrace{F_{\left(\frac{V}{\sigma_V}\right)}^{-1} (P(z))}_{\substack{\text{Propensity} \\ \text{score} \\ \downarrow \\ \text{can estimate this}}}
 \end{aligned}$$

- MTE:

$$\begin{aligned}
 E(Y_1 - Y_0 \mid U_D = u_D, X = x, Z = z) &= \\
 &= \mu_1(x) - \mu_0(x) + \text{Cov} \left(U_1 - U_0, \frac{V}{\sigma_V} \right) F_{\left(\frac{V}{\sigma_V}\right)}^{-1} (u_D)
 \end{aligned}$$

- Notice from definition of TT

$$\begin{aligned}
 E(Y_1 - Y_0|D = 1, X = x, Z = z)P(z) \\
 &= [\mu_1(x) - \mu_0(x)]P(z) \\
 &+ E(U_1 - U_0|D = 1, X = x, Z = z)P(z)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial[E(Y_1 - Y_0|D = 1, X = x, Z = z)P(z)]}{\partial P(z)} \\
 &= \mu_1(x) - \mu_0(x) + E(U_1 - U_0|X = x, P(z) = U_D) \\
 &= \text{MTE}
 \end{aligned}$$

- Marginal change in TT
- Also $\text{MTE} = \frac{\partial E(Y|Z=z)}{\partial P(z)}$

Problem: Prove this claim

- Hint: Read “Building Bridges”

- Recent Advances in Econometrics:
 - a Relax normality
 - b Do not assume linearity of $\mu_1(X)$ and $\mu_0(X)$ in terms of X
 - c Do not require identification at infinity but only because they abandon pursuit of ATE, TT, TUT or else assume that $(Y_1, Y_0) \perp\!\!\!\perp D \mid X$ (matching assumption)
 - d Identification at infinity in some version or the other is required for ATE, TT, TUT as long as there is selection on unobservables (i.e., $(Y_1, Y_0) \not\perp\!\!\!\perp D \mid X$)