Human Capital Accumulation and Earnings Dynamics over the Life Cycle: the Ben-Porath Model and Beyond

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1. Introduction

2. Basic Ben-Porath Model

- At each point in time, the agent's current stock of human capital, $H(t)$, and the rental rate of human capital, R, determine the amount of her potential earnings: $Y(t) = RH(t)$.
- The agent chooses two type of inputs in order to produce human capital:
  1. a fraction of her current stock of human capital, $I(t)$, with $I(t) \in [0,1]$;
  2. market goods, $D(t)$.
- Therefore, the cost of human capital investments includes both foregone earnings, $RI(t)H(t)$, and cost of the purchased market goods, $P_D D(t)$, where P_D is the price of the market goods.

- Then, the agent's disposable earnings in period t, $E(t)$, are equivalent to her potential earnings in period t, $Y(t)$, less the total costs:

$$E(t) = RH(t) - RI(t)H(t) - P_D D(t). \tag{1}$$

Assumption 1

(Strict Concavity of the Production Function) $\forall\ t \in [0, T]\ F(\cdot, \cdot)$ *is strictly concave in both of its arguments.*

Definition 1

(Law of Motion for Human Capital Stock in the Basic Ben-Porath Specification)

$$\dot{H}(t) = F(\underbrace{I(t)H(t)}_{\text{Neutrality}}, D(t)) - \sigma H(t). \tag{2}$$

- Embeds a neutrality assumption.
- This assumption simplifies our calculations by neutralizing the effect of $H(t)$ on the optimal decision of time investment.

Problem 2

(Life-cycle Individual's Problem in the Basic Ben-Porath Model)

$$\max_{I_t, D_t} \int_0^T \exp(-rt)\left[RH(t)(1 - I(t)) - P_D D(t)\right]$$

s.t.

$$H(0) = H_0$$
$$\dot{H}(t) = F\left(I(t)H(t), D(t)\right) - \sigma H(t)$$

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- The present value Hamiltonian associated to the agent's maximization problem is

$$\mathcal{H}(\cdot) = \exp(-rt)\left[RH(t) - RI(t)H(t) - P_D D(t)\right] + \mu(t)\dot{H}(t) \tag{3}$$

- $\mu(t)$: shadow price of the human capital stock.
- Thus, the following conditions must be satisfied for the interior solution.

Condition 1

(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Ben-Porath Model)

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \Leftrightarrow \exp(-rt)R = \mu(t)F_1\left(I(t)H(t), D(t)\right) \tag{4}$$

$$\textbf{(}\boldsymbol{H(t)}\textbf{ cancels on both sides)} \tag{5}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial D(t)} = 0 \Leftrightarrow \exp(-rt)P_D = \mu(t)F_2\left(I(t)H(t), D(t)\right) \tag{6}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu}(t) \Leftrightarrow \exp(-rt)R\left(1 - I(t)\right) + \mu(t)\left(F_1\left(I(t)H(t), D(t)\right)I(t) - \sigma\right) = -\dot{\mu}(t) \tag{7}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H}(t) \Leftrightarrow \dot{H}(t) = F\left(I(t)H(t), D(t)\right) - \sigma H(t) \tag{8}$$

$$\textit{Transversality}: \lim_{t \to T} \mu(t)H(t) = 0 \tag{9}$$

where F_j is the first order derivative of the production function F with respect to argument j.

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Link to

Appendix 1: Optimal Control

- To simplify notation, combine the two terms with intertemporal meaning in the life-cycle decision problem into one term through $g(t) \equiv \exp(rt)\mu(t)$.
- Then, combine (4) and (7) to get

$$\dot{\mu}(t) = -\exp(-rt)R + \mu(t)\sigma \tag{10}$$

- Note that $\dot{g}(t) = \dot{\mu}(t)\exp(rt) + r\mu(t)e^{rt}$.
- Use (10) to obtain:

$$\dot{g}(t) = (\sigma + r)g(t) - R. \tag{11}$$

- Equation (9) implies that $\mu(T) = 0$, and therefore $g(T) = 0$ provided that $H(T) = 0$ has no economic sense.
- Solve (11):

$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp((\sigma + r)(t - T))\right], \tag{12}$$

- $\therefore \dot{g}(t) < 0$.

- To wrap up the discussion, note that the optimality conditions for the interior solution are:

$$\begin{aligned} g(t)F_1(I(t)H(t), D(t)) &= R \\ g(t)F_2(I(t)H(t), D(t)) &= P_D. \end{aligned} \tag{13}$$

- The system in (13) consists of two equations and two unknowns that solve for the Marshallian demand for $I(t)H(t)$ and $D(t)$.
- **Question:** What happens when $\frac{\partial^2 F}{\partial IH \partial D} > 0$?

2.1 Earnings Dynamics

- Consider both the slope and curvature of the earnings dynamics in the case with no $D(t)$
- $F_{D(t)} = 0$
- Production function takes the single argument $I(t)H(t)$
- Without loss of generality, assume $R \equiv 1$

2.1.1 The Slope of Earnings Dynamics

Claim 1

(Earnings over Time with no Depreciation) *Let* $\sigma = 0$. *Then, when the optimal solution for* $I(t)$ *is interior,* $\dot{E}(t) > 0$.

Proof.

Differentiate (1) and use (2) to write

$$\begin{aligned}\dot{E}(t) &= \dot{H}(t) - \dot{\overbrace{I(t)H(t)}} \\ &= F\left(I(t)H(t)\right) - I(t)\dot{H}(t) \\ &> 0 \end{aligned} \tag{14}$$

where the inequality follows because the Marshallian demands for $I(t)H(t)$ is decreasing over time. □

Claim 2

(Earnings over Time with Depreciation) Let $\sigma > 0$. Then, $\dot{E}(t) \lesseqgtr 0$.

Proof.

Follow the same steps as in the proof of Claim 1 and note that the term $\sigma H(t)$ appears in the expression for $\dot{E}(t)$. This term could be

$$\lesseqgtr F\left(I(t)H(t)\right) - \overbrace{I(t)H(t)}^{\cdot}.$$

$\square$

2.1.2 The Curvature of Earnings Dynamics

Claim 3

(Concavity of the Earnings Function with no Depreciation) Assume that $\eta \equiv \left(1 - \frac{F'F'''}{(F'')^2}\right) < 0$. *Then, the earnings function is strictly concave.*

Proof.

First note that $\dot{E}(t) > 0$ by Claim 1. Since $F_{D(t)} = 0$ we can write the first order condition for investment as

$$g(t)F'(I(t)H(t)) = 1 \tag{15}$$

and differentiate it with respect to t to get

$$\dot{g}(t)F'(I(t)H(t)) + g(t)F''(I(t)H(t))\,\dot{\overbrace{I(t)H(t)}} = 0$$
$$\Leftrightarrow$$
$$\dot{\overbrace{I(t)H(t)}} = -\left(\frac{\dot{g}(t)}{g(t)}\right)\left[\frac{F'}{F''}\right]. \tag{16}$$

Moreover, drop the argument t to simplify notation, and note that

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{F''^2}\right]\left[\frac{F'}{F''}\right] \tag{17}$$

where we substitute in (16). Further, note that

$$\begin{aligned}
\dot{E} &= F(IH) - \overset{\cdot}{\overbrace{IH}} - \sigma H \\
\ddot{E} &= F'(IH)\overset{\cdot}{\overbrace{IH}} - I\ddot{H} - \sigma\dot{H} \\
&= \frac{1}{g}\overset{\cdot}{\overbrace{IH}} - \overset{\cdot\cdot}{\overbrace{IH}} - \sigma\dot{H}.
\end{aligned} \tag{18}$$

and from (11) obtain $\frac{\ddot{g}}{g} = r\frac{\dot{g}}{g}$.

Thus,

$$
\begin{aligned}
\ddot{E} &= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g}+\frac{\dot{g}}{g}\left(1-\frac{F'F'''}{F''^2}\right)\right]+\left[r\frac{\dot{g}}{g}-\left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g}+\frac{\dot{g}}{g}\left(1-\frac{F'F'''}{F''^2}\right)-\frac{gr-\dot{g}}{g}\right] \\
&= -\frac{\dot{g}}{g}\frac{F'}{F''}\left[\frac{1}{g}+\frac{\dot{g}}{g}\left(1-\frac{F'F'''}{F''^2}\right)-\frac{1}{g}\right] \\
&= -\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}\left(1-\frac{F'F'''}{F''^2}\right) \qquad (19)
\end{aligned}
$$

Third equality uses (11), i.e., $gr-\dot{g}=1$. F is strictly concave and therefore $-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}>0$. Since $\left(1-\frac{F'F'''}{F''^2}\right)<0$ the claim follows. □

- Therefore, $E(t)$ is concave if and only if $\eta<0$, which implies a necessary condition for concavity: $F^{'''}>0$.

Example 3

(Human Capital Production Functions and Earnings Concavity)

- Power Production Function 1 : consider the case of $F(x) = \frac{Ax^{\alpha}}{\alpha}$ for $-\infty < \alpha < 1, A > 0$. Then, $\eta = \frac{1}{\alpha - 1} < 0$. Under this specification the earnings function is strictly concave with respect to time.
- Power Production Function 2 : consider the case of $F(x) = a - bx^{-\alpha}$ for $-1 < \alpha < \infty, a, b, c > 0$. Then, $\eta = \frac{-1}{\alpha+1} < 0$. Under this specification the earnings function is strictly concave with respect to time.
- Power Production Function 3: consider the case of $F(x) = a - b\exp(-cx)$ with $b, c > 0$. Then, $\eta = 0$.
- Quadratic Production Function: any quadratic production function has $F''' = 0$ and does not induce concavity of earnings with respect to time.

2.2 The Specialization Period

- Specialization happens when the agent devotes her entire human capital to produce human capital stock, i.e. when $I(t) = 1$ for $t \in [\underline{t}, \bar{t}]$.
- In order to analyze some of the properties of specialization periods we assume away $D(t)$ so that $F_{D(t)} = 0$ and rule out depreciation.

Condition 2

(Conditions for the Existence of a Period of Specialization in the Basic Ben-Porath Model with no Depreciation)

$$
\begin{aligned}
F'(H(t))g(t) &> R \quad \forall\ t \in [0, t^*) \\
F'(H(t^*))g(t^*) &= R \\
I(t) &= 1 \quad \forall\ t \in [0, t^*] \\
H(t^*) &= \int_0^{t^*} F(H(\tau))d\tau + H_0
\end{aligned}
\tag{20}
$$

where $H(t^*)$ is the human capital stock accumulated up to time t^*.

Case 1

(No Depreciation and the Cobb-Douglas Production Function for Human Capital: Initial Level of Human Capital) *In this case* $\dot{H} = A(IH)^{\alpha}$ *where* $0 < \alpha < 1, A > 0$. *As argued above, if it exists, specialization happens in the period* $[0, t^*]$. *Thus*

$$\alpha A(H(0))^{\alpha-1} g(0) > R$$
$$\Leftrightarrow$$
$$H(0) < \left[\frac{R}{g(0)\alpha A}\right]^{\frac{1}{\alpha-1}}. \tag{21}$$

As the conditions in (20) *establish, the time spent in specialization is a decreasing function of* $H(0)$. *In this example, actually, the initial human capital needs to be below certain threshold in order for the individual to specialize during one period.*

Case 2

(No Depreciation and the Cobb-Douglas Production Function for Human Capital with Infinite Horizon: Initial Level of Human Capital) In the setting of Case 1 and if the horizon of the problem is infinite: $H(0) < \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$ *because* $g(t) = \frac{R}{r}$.

Case 3

(No Depreciation and the Cobb-Douglas Production Function for Human Capital: the Specialization Period) In the period of specialization $I(t) = 1$. Then,

$$\dot{H} = A\,(H)^{\alpha}\,. \tag{22}$$

The general solution for (22) is

$$H(t) = [(1-\alpha)(At+K)]^{\frac{1}{1-\alpha}} \tag{23}$$

for some constant K. Given an initial condition $H(0) = H_0$, $K = \frac{H_0^{1-\alpha}}{1-\alpha}$ and

$$H(t) = \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\,. \tag{24}$$

At the end of the specialization period, as established in (20):

$$\alpha g(t^*)A\left(H(t^*)\right)^{\alpha-1} = R. \tag{25}$$

If $T \to \infty$, $g(t) = \frac{R}{r}$ *and*

$$t^* = -\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha}\frac{1}{r}. \tag{26}$$

(26) *provides some intuitive results:*

1. *an individual with relatively high initial human capital specializes during a relatively shorter period:* $\frac{\partial t^*}{\partial H_0} < 0$;
2. *a relatively able individual specializes during relatively long period:* $\frac{\partial t^*}{\partial A} > 0$;
3. *a relatively impatient individual specializes for a relatively shorter period:* $\frac{\partial t^*}{\partial r} < 0$.

Case 4

(No Depreciation and the Cobb-Douglas Production for Human Capital: Post-school Earnings) Let $\tau = t - t^*$ define the post-school work experience and write post-school earnings as follows:

$$E(\tau) = R \int_0^{\tau} \dot{H}(l + t^*)dl + RH(t^*) - RIH(\tau + t^*). \qquad (27)$$

Now, from (20) *the following equality holds:*

$$\alpha g(t) A \left(IH(t)\right)^{\alpha - 1} = R$$

$$\Leftrightarrow$$

$$IH(t) = \left[\frac{\alpha g(t) A}{R}\right]^{\frac{1}{1-\alpha}} \qquad (28)$$

Combining (28) *and the law of motion for human capital:*

$$\dot{H} = A\left[\frac{\alpha g(t)A}{R}\right]^{\frac{\alpha}{1-\alpha}}. \tag{29}$$

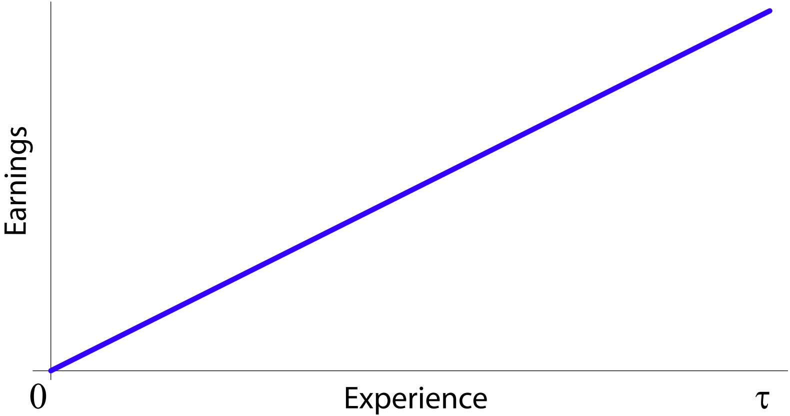
Then,

$$E(\tau) = R\int_0^{\tau} A\left[\frac{\alpha g(l+t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}} dl + RH(t^*) - R\left[\frac{\alpha g\left(\tau+t^*\right)A}{R}\right]^{\frac{1}{1-\alpha}} \tag{30}$$

and if $T\rightarrow\infty$

$$E(\tau) = RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau. \tag{31}$$

Figure 1: Earnings and Experience, Cobb Douglas Technology and No Depreciation



- *When the time horizon is infinite, there is no concern with the reduction in time left for capturing returns to human capital investment and thus $g(t)$ is fixed over time.*
- *When the solution is interior, the optimal choice on $I(t)H(t)$ is constant overtime, which implies the increase in $H(t)$ overtime is also a constant.*
- *This is why $E(t)$ increases at a constant rate as well.*
- *However, with a finite time horizon, the Cobb-Douglas production function with no depreciation implies a strictly concave earning function $E(t)$.*

2.3 The Baseline Model Dynamics under the Cobb-Douglas Specification: A Summary

2.3.1 Human Capital

- At $t = 0$ an initial condition is given.
- At $0 < t < t^*$ the system (20) provides the conditions that human capital satisfies and its expression is given by (24).
- At $t = t^*$ (24) is still a valid expression for human capital. To obtain the exact quantity it suffices to substitute the expression for t^*, (26), into (24).
- At $t > t^*$ (20) and the expression for $\dot{H}$, (29), provide the expression for human capital.

Then,

$$H(t) = \begin{cases} H_0 & t = 0 \\ \left[(1-\alpha)At + H_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, & 0 < t < t^* \\ \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t = t^* \\ A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t - t^*) + \left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}, & t > t^*. \end{cases} \tag{32}$$

2.3.2 Investment

- We focus on the case in which there is an specialization period, i.e. the case in which (21) holds.
- The combination of (28) and (32) gives the following

$$I(t) = \begin{cases} 1 & t = 0 \\ 1 & 0 < t < t^* \\ 1 & t = t^* \\ \frac{\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}{A\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t-t^*)+\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}} & t > t^*. \end{cases} \tag{33}$$

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2.3.3 Earnings

- For earnings we also focus on the case with a specialization period, i.e. the case in which (21) holds.
- Thus, (1), (32), (33) define earnings as follows

$$E(t) = \begin{cases} 0 & t = 0 \\ 0 & 0 < t < t^* \\ 0 & t = t^* \\ RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}(t - t^*) & t > t^*. \end{cases} \tag{34}$$

Figure 2: Dynamics with Variations in a Production Technology Parameter
$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)
for $A = 3, r = .05, H_0 = 1$

(a) Human Capital Investment

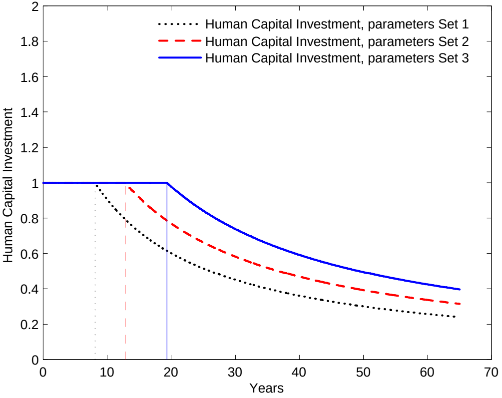


Figure 2: Dynamics with Variations in a Production Technology Parameter

$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)

for $A = 3$, $r = .05$, $H_0 = 1$

(b) Human Capital Stock

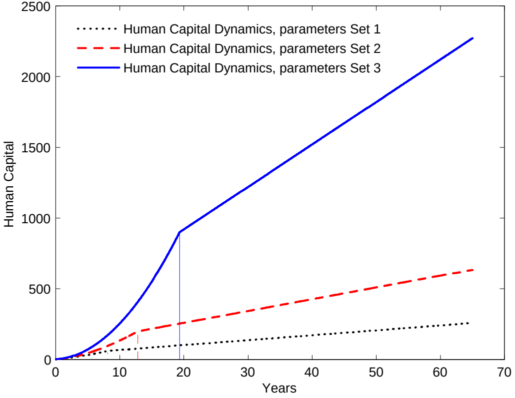


Figure 2: Dynamics with Variations in a Production Technology Parameter

$\alpha = .3$ (dotted); $\alpha = .4$ (dashed); $\alpha = .5$ (solid)
for $A = 3, r = .05, H_0 = 1$

(c) Earnings

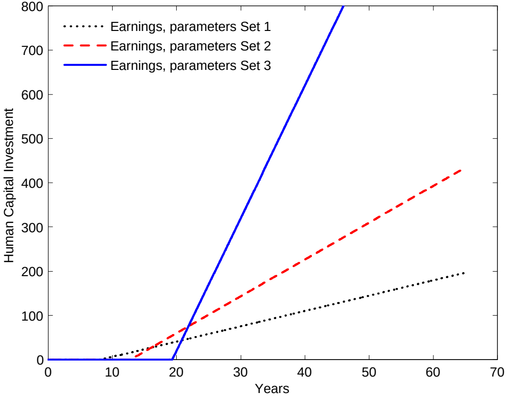
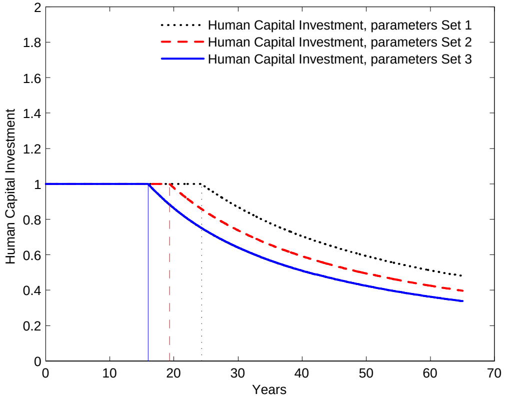


Figure 3: Dynamics with Variations in the Discounting Factor
$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)
for $A = 3, \alpha = .5, H_0 = 1$

(a) Human Capital Investment



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$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)
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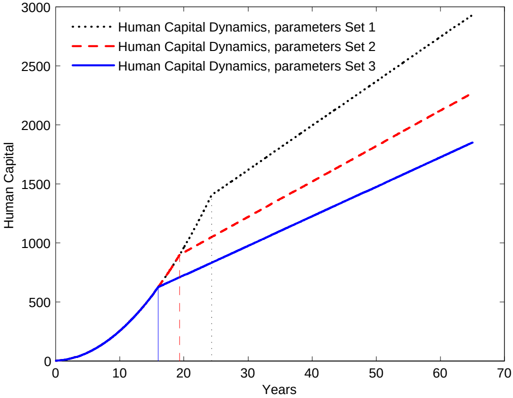
(b) Human Capital Stock



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$r = .04$ (dotted); $r = .05$ (dashed); $r = .06$ (solid)
for $A = 3, \alpha = .5, H_0 = 1$

(c) Earnings

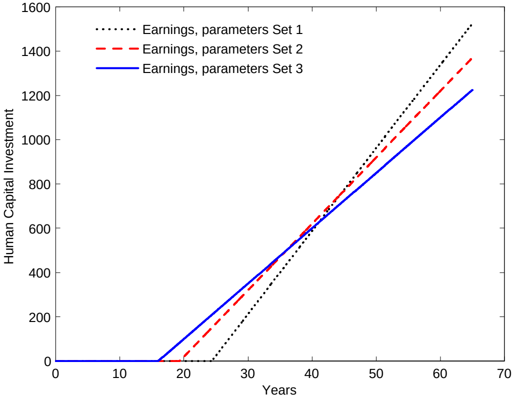

Figure 4: Dynamics with Variations in a Production Technology Parameter

$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)
for $r = .03, \alpha = .5, H_0 = 10$

(a) Human Capital Investment

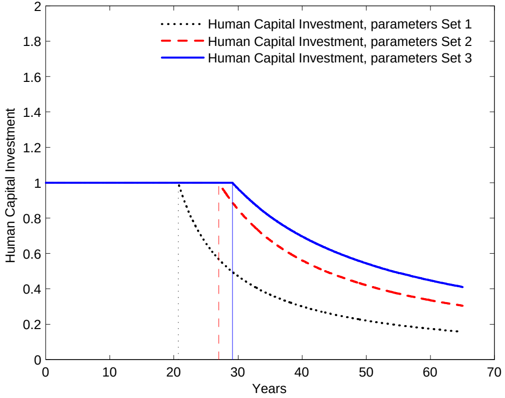

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Figure 4: Dynamics with Variations in a Production Technology Parameter

$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)

for $r = .03, \alpha = .5, H_0 = 10$

(b) Human Capital Stock

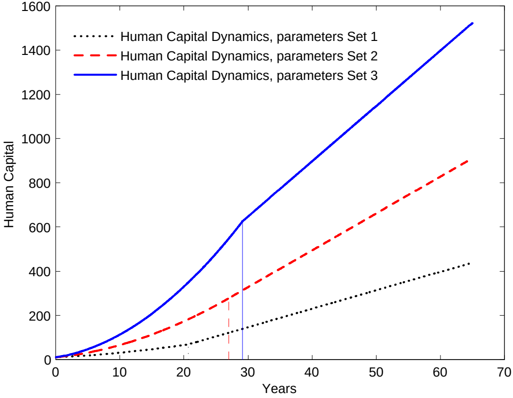


Figure 4: Dynamics with Variations in a Production Technology Parameter

$A = .5$ (dotted); $A = 1.0$ (dashed); $A = 1.5$ (solid)

for $r = .03, \alpha = .5, H_0 = 10$

(c) Earnings

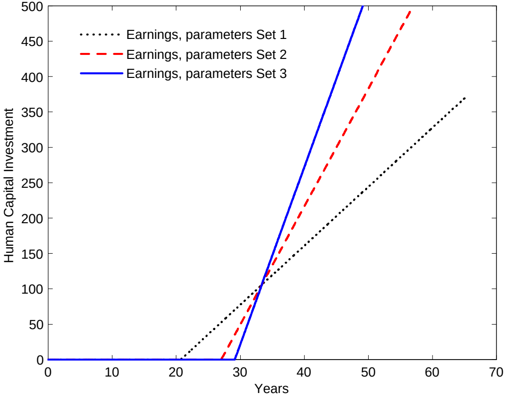


Figure 5: Dynamics with Variations in the Initial Level of Human Capital $H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid) for $r = .025, \alpha = .5, A = .6,$

(a) Human Capital Investment

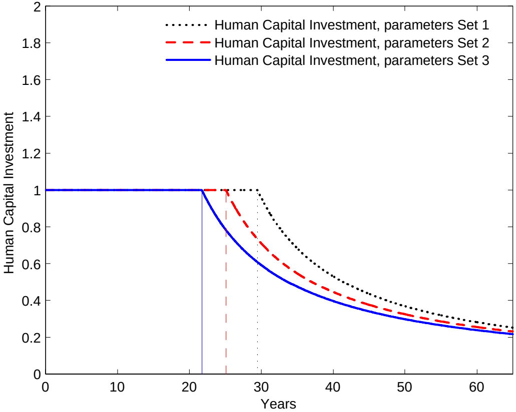


Figure 5: Dynamics with Variations in the Initial Level of Human Capital $H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid) for $r = .025, \alpha = .5, A = .6,$

(b) Human Capital Stock

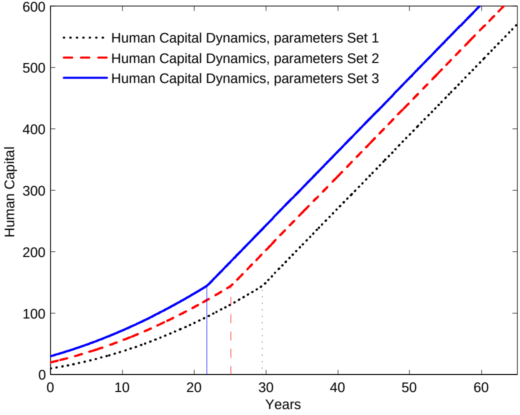
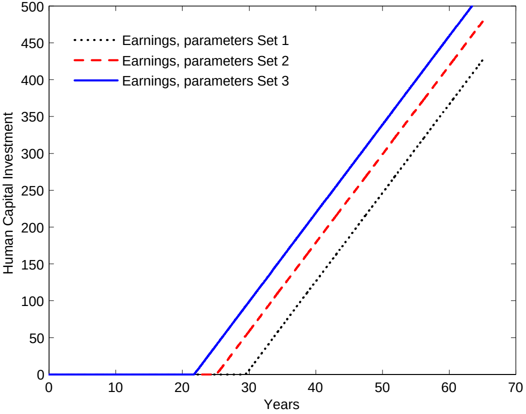


Figure 5: Dynamics with Variations in the Initial Level of Human Capital $H_0 = 10$ (dotted); $H_0 = 20$ (dashed); $H_0 = 30$ (solid) for $r = .025, \alpha = .5, A = .6,$

(c) Earnings



2.4 Rates of Return for the Cobb-Douglas Specification

2.4.1 Return to Schooling

Definition 4

("Internal" Rate of Return to Schooling) φ is the (internal) rate of return to schooling and solves the equation

$$\int_{t^*}^{\infty} \exp(-\varphi t) \left\{ R \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} \right\} dt = \int_{0}^{\infty} \exp(-\varphi t) \{ R H_0 \} \, dt$$

$$\Rightarrow$$

$$\varphi = \frac{\ln \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \ln H_0}{-\frac{H_0^{1-\alpha}}{A(1-\alpha)} + \frac{\alpha}{1-\alpha} \frac{1}{r}}. \tag{35}$$

Question: How does φ vary with H_0? α? A? r? When does $\varphi = r$?

2.4.2 Return to Post-Schooling

- Let $E(\tau)^{NPS}$ and $E(\tau)^{PS}$ denote earnings with and without post-schooling investment, respectively.
- By (34) we can write

$$\begin{aligned} E(\tau)^{NPS} &= RH(t^*) \qquad (36) \\ &= R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\ E(\tau)^{PS} &= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau \end{aligned}$$

- The increment in earnings due to post-schooling at τ is

$$\Delta^{E(\tau)} \equiv E(\tau)^{PS} - E(\tau)^{NPS}. \qquad (37)$$

- Can interpret $\Delta^{E(\tau)}$ as "returns less costs" from post-schooling, with $E(\tau)^{NPS}$ as the costs (i.e. foregone earnings) of post-schooling investments.
- Define the (internal) rate of return to post-schooling as follows.

Definition 5

("Internal" Rate of Return to Post-schooling) ϕ is the (internal) rate of return and solves the equation

$$\int_0^\infty \exp(-\phi\tau)\left[RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau - R\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}\right]d\tau = 0 \tag{38}$$

Using the Laplace transform, (38) implies

$$\begin{aligned} \frac{1}{\phi^2}RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} - \frac{R}{\phi}A\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} &= 0 \\ &\Rightarrow \\ \phi &= \frac{r}{\alpha}. \end{aligned} \tag{39}$$

- The (internal) rate of return to post-schooling investment is a decreasing function of α.

2.5 Earnings Growth and Patience in Finite Horizon

Claim 4

Assume that $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ *(recall from Claim 3 that this is a sufficient condition for* $\ddot{E}(t) < 0$ *in the current context). Then,* $\frac{\partial \dot{E}(\tau)}{\partial r} < 0$.

Proof.

Without loss of generality, assume that R =1 and note that

$$\frac{\partial \dot{E}(\tau)}{\partial r} = F'(\cdot)\frac{\partial IH}{\partial r} - \frac{\partial}{\partial r}\overset{\cdot}{\overbrace{IH}}. \tag{40}$$

From (13) we know that the first order condition of the agent's problem is

$$g(t)F'(\cdot) = 1 \tag{41}$$

which by the implicit function theorem yields

$$\begin{aligned} \frac{\partial IH}{\partial r} & - \frac{\frac{\partial g(t)}{\partial r}F'(\cdot)}{g(t)F''(\cdot)} \\ & < 0 \end{aligned} \tag{42}$$

Inequality follows from strict concavity of $F(\cdot)$ and $g(t) > 0$, $\frac{\partial g(t)}{\partial r} < 0$ (see (45)).

Thus, the first term in (40) is negative. If we show that the second term is negative then we can sign (40) and give meaning to these results. In order to do so we need $\frac{\partial \dot{IH}}{\partial r} > 0$. From (16) note that

$$\frac{\partial \dot{\overbrace{IH}}}{\partial r} = -\frac{\dot{g}}{g}\left[1 - \frac{F'(\cdot)F'''(\cdot)}{F''(\cdot)^2}\right]\frac{\partial IH}{\partial r} + \frac{F'(\cdot)}{F''(\cdot)}\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] \tag{43}$$

We know from $1 - \frac{F'(\cdot)F'''(\cdot)}{F''^2} < 0$ and $\dot{g}, \frac{\partial IH}{\partial r} < 0$ that the first term in (43) is positive. To sign the second term note that
$\dot{g} = rg - 1, -\frac{\dot{g}}{g} = \frac{1}{g} - r$. Then,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1. \tag{44}$$

To sign (44) note that

$$\frac{\partial g}{\partial r} = \frac{\exp(r(t-T))(1-r(t-T))-1}{r^2} \tag{45}$$

$$< 0$$

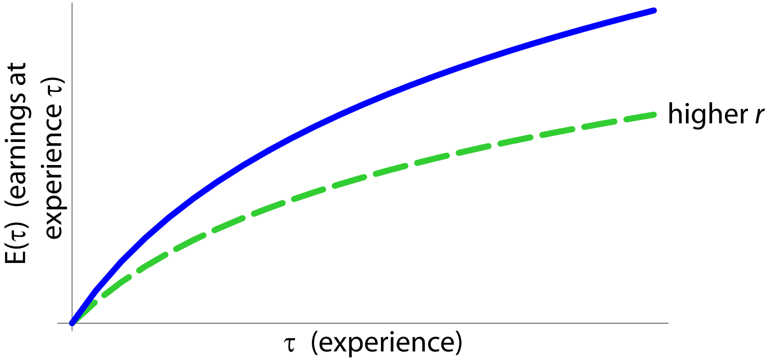
and

$$-\frac{\partial g}{g^2 \partial r} - 1 = \frac{1}{r^2 g^2} \exp(r(t-T))(1+r(t-T)-\exp(r(t-T))). \tag{46}$$

$$< 0$$

which implies that $\frac{\partial \dot{E}}{\partial r} < 0$. $\square$

Figure 6: Earnings Profiles in Finite Horizon for Different Values of r



3. The Haley-Rosen Specification: Finite Horizon and the Autoregression Form

- Analyze the finite horizon case under the specification that Haley (1976) and Rosen (1976) use.
- Specifically, we assume that $\dot{H} = A\left(IH\right)^{\alpha}, \alpha = \frac{1}{2}, \sigma = 0$ and the exact same setting as in Section 2.
- Actually, in Section 2 we rely on an infinite horizon to derive a set of closed form solutions to the individual's problem.

- From (30) we can write

$$
\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^\tau A\left[\frac{1}{2}\frac{g(t^*+l)A}{R}\right]dl - R\left[\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right]^2 \\
&\Rightarrow \\
\dot{E}(\tau) &= \frac{g(t^*+\tau)A^2}{2R}\left(2R - rg(t^*+\tau)\right) \\
&\Rightarrow \\
\ddot{E}(\tau) &= -\frac{A^2}{R}\dot{g}(t^*+\tau)^2
\end{aligned}
\tag{47}
$$

- where the second and third equalities use (11).

- Combining (11) and (47) we obtain a second order ODE with constant coefficients:

$$\ddot{E}(\tau) = 2r\dot{E}(\tau) - A^2R \tag{48}$$

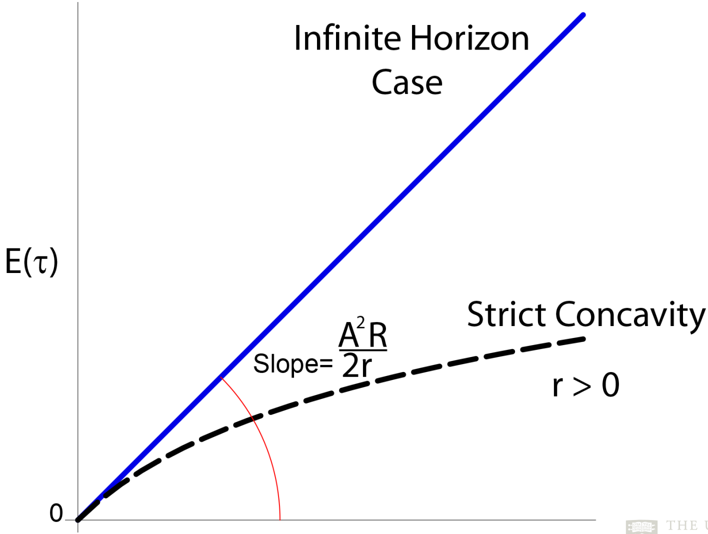
- where the natural initial and terminal conditions that we impose are $E(0) = 0$ and $\dot{E}(T) = 0$ and then we guess and verify that the general solution to (48) is the following.

$$E(\tau) = c_0 + c_1 \exp(-2r\tau) + c_2\tau \tag{49}$$

- So $E(0) = 0$ implies $c_1 + c_0 = 0$ and $\dot{E}(T) = 0$ implies $2rc_1 \exp(2rT) + c_2 = 0$.
- Together with (48), we can solve for $c_0 = \frac{A^2 R}{4r^2 \exp(2rT)}$, $c_1 = -c_0$, and $c_2 = \frac{A^2 R}{2r}$.
- Therefore,

$$E(\tau) = \frac{A^2 R}{4r^2} \exp(-2rT)\left(1 - \exp(2r\tau)\right) + \frac{A^2 R}{2r}\tau. \qquad (50)$$

Figure 7: Post-school Earnings in the Haley-Rosen Specification



- From (47), we know that in the finite horizon case, the earnings function is strictly concave unless $t = T$.
- The intuition behind the linearity of the earnings function in the infinite horizon case is provided in Section 2.2.

3.1 Autoregressive Form

- From (50) it is possible to write

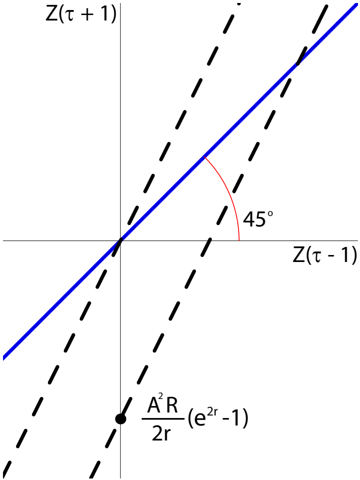
$$E(\tau+1)-E(\tau) = \frac{A^2R}{2r} + \frac{A^2R}{4r^2}\exp(-2rT)\left(\exp(2r\tau) - \exp(2r(\tau+1))\right) \tag{51}$$

- Implies:

$$Z(\tau+1) = \exp(2r)Z(\tau) + \frac{A^2R}{2r}\left(1-\exp(2r)\right) \tag{52}$$

- where $Z(\tau) \equiv E(\tau+1) - E(\tau)$ and we can analyze the growth dynamics of earnings.

Figure 8: Earnings Growth in the Haley-Rosen Representation



- Apparently, the dynamics of the earnings growth are explosive.
- However, note that

$$\frac{\partial\left[E(\tau)-E(\tau-1)\right]}{\partial\tau} = \frac{A^2 R}{2r}\exp(2r(\tau-T))\left[\exp(-2r)-1\right]$$
$$< 0 \tag{53}$$

- Even when the growth dynamics of earnings is explosive, the earnings dynamics, $E(t)$, can converge over time.

3.2 From the Haley-Rosen Specification to the Mincer Equation

- The earnings function in the Haley-Rosen specification actually lead to the Mincer equation.
- To see that take the log of (50) and obtain

$$\ln E(\tau) = \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{\exp(-2rT) - \exp(2r(\tau - T))}{2r\tau}\right]. \tag{54}$$

- Can approximate around τ_0 the second and third terms in (54) to obtain

$$\ln(\tau) \approx \ln(\tau_0) + \frac{1}{\tau_0}(\tau - \tau_0) - \frac{1}{\tau_0^2}\frac{(\tau - \tau_0)^2}{2!}$$

$$\ln\left[1 + \frac{\exp(-2rT) - \exp(2r(\tau - T))}{2r\tau}\right] \approx \xi_0 + \xi_1(\tau - \tau_0) + \xi_2\frac{(\tau - \tau_0)^2}{2!} \tag{55}$$

for appropriate values ξ_0, ξ_1, ξ_2.

- Thus,

$$\ln(\tau) + \ln\left[1 + \frac{\exp(-2rT) - \exp(2r(\tau - T))}{2r\tau}\right] \approx \alpha_0 + \alpha_1(\tau - \tau_0) + \alpha_2(\tau - \tau_0)^2 \tag{56}$$

- $\alpha_0 \equiv \ln(\tau_0) + \xi_0, \alpha_1 \equiv \frac{1}{\tau_0} + \xi_1, \alpha_2 \equiv \frac{-\frac{1}{\tau_0^2} + \xi_2}{2}$.
- This leads to the so called Mincer equation (see Mincer, 1974):

$$\ln E(\tau) = k_0 + k_1\tau + k_2\tau^2 \tag{57}$$

- where $k_0 = \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2, k_2 = \alpha_2$.

Table 1: The Ben-Porath and the Mincer Coefficients

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	-0.0010
0.05	25	60	0.0808	-0.0008
0.05	20	65	0.1002	-0.0013
0.0675	24.70	74.77	0.081	-0.0008
Mincer Coefficients			0.081	-0.0012

Note: The Mincer model or Mincer equation is $\ln(\mathrm{E}) = k_0 + k_1\tau + k_2\tau^2$, where τ is experience.

- Now, if $rT \approx 0$ then $\exp(-rT) \approx 1$ and (54) becomes

$$\ln E(\tau) \approx \ln\left(\frac{A^2 R}{2r}\right) + \ln \tau + \ln\left[1 + \frac{1 - \exp(2r\tau)}{2r\tau}\right] \tag{58}$$

- The Haley-Rosen specification of the Ben-Porath model implies no economic content for the Mincerian rate of return on post-school investment.

- Actually, this model implies that the entire economic content is in the intercept (see (58)).
- (58) implies that, *ceteris paribus*, schooling has no effect on earnings.

Link to
Appendix: Ben-Porath Notes

4. Generalized Ben-Porath Model

- Law of motion for human capital stock in the generalized Ben-Porath model:

$$\dot{H} = AI^{\alpha} H^{\beta} - \sigma H. \tag{59}$$

- The Hamiltonian of the problem is

$$\mathcal{H} = RH(t)\left(1 - I(t)\right) + \mu(t)\left(AI(t)^{\alpha} H(t)^{\beta}\right) \tag{60}$$

- $\mu(t)$: shadow price of human capital.
- The following condition must be satisfied for interior solutions.

Condition 3

(Optimality Conditions for the Life-Cycle Individual's Problem in the Generalized Ben-Porath Model)

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \Leftrightarrow \mu(t) A \alpha I(t)^{\alpha-1} H(t)^{\beta} = RH(t) \tag{61}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu}(t) \Leftrightarrow -R(1 - I(t)) - \beta \mu(t) A I(t)^{\alpha} H(t)^{\beta-1} = \dot{\mu}(t) \tag{62}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H} \Leftrightarrow \dot{H}(t) = A I(t)^{\alpha} H(t)^{\beta} \tag{63}$$

$$Transversality: \lim_{t \to T} \mu(t) H(t) = 0 \tag{64}$$

- Condition 3 is equivalent to the Mangasarian sufficient conditions for a global optimum if $\beta \leq 1$ (see Mangasarian, 1966).

4.1 Specialization

- If $\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} > 0$ with $I(t) = 1$, the agent would specialize.
- Thus the condition that guarantees specialization is as follows.

Condition 4

(Conditions for Specialization in the Generalized Ben-Porath Model)

$$\textit{Conditions for Specialization}: \begin{cases} H > \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta > 1 \\ 1 > \left[\frac{R}{\alpha A \mu}\right], & \beta = 1 \\ H < \left[\frac{R}{\alpha A \mu}\right]^{\frac{1}{\beta-1}}, & \beta < 1. \end{cases} \tag{65}$$

- During the period(s) of specialization (62), (63) become

$$\dot{\mu} = -\beta \mu A H^{\beta - 1} \tag{66}$$

$$\dot{H} = A H^{\beta} \tag{67}$$

- Can solve for the dynamics of human capital stock in this region

$$H(t) = \begin{cases} c_0 \exp(At), & \beta = 1 \\ (At + c_1)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{68}$$

- The initial condition for the human capital stock leads to $c_0 = H_0$ and $c_1 = \frac{H_0^{1-\beta}}{1-\beta}$ which implies that

$$H(t) = \begin{cases} H_0 \exp(At - 1), & \beta = 1 \\ \left(At + \frac{H_0^{\frac{1}{1-\beta}}}{1-\beta}\right)^{\frac{1}{1-\beta}} (1-\beta)^{\frac{1}{1-\beta}}, & \beta \neq 1. \end{cases} \tag{69}$$

- Also, we can solve (66) and find that

$$\mu(t) = \begin{cases} k_0 \exp(-At), & \beta = 1 \\ \frac{k_1}{(At+c_1)^{\frac{\beta}{1-\beta}}}, & \beta \neq 1 \end{cases} \tag{70}$$

for which there is an exact solution given an initial condition $\mu(0) = \mu_0$.

- This is, we can find k_0, k_1 in (70) provided $\mu_0 > 0$ (it is a price).
- In particular, note that $k_0 = \mu_0 > 0$ and $k_1 = \mu_0 c_1^{\frac{\beta}{1-\beta}} > 0$ for $0 < \beta < 1$.

- Let t^* denote the time when specialization ends.
- It must be true that, then, (61) holds with strict equality

$$\mu(t^*)A\alpha H(t^*)^\beta = RH(t^*) \tag{71}$$

which implies that

$$t^* = \frac{1}{A}\left(\ln\left[\frac{A\alpha}{R} + \ln k_0\right]\right) \tag{72}$$

for $\beta = 1$.

- For $\beta \neq 1$, t^* solves

$$\frac{k_1}{(At^* + c_0)^{\frac{\beta}{\beta-1}}} \frac{A\alpha}{R} = \left[At^* (1-\beta)^{\frac{1}{1-\beta}} + H_0^{1-\beta} (1-\beta)^{\frac{\beta}{1-\beta}} \right]^{1-\beta}. \tag{73}$$

- Is the period of specialization unique?

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Claim 5

(Uniqueness of the Specialization Period) If the period of specialization exists, it is unique when either when $\beta = 1$ or when $\beta \in [0, 1]$.

Proof.

In both cases (70) implies that $\dot{\mu}(t) < 0$. Importantly, $\mu(t)$ is the shadow price or value of human capital. Thus, $\dot{I}(t) < 0$ and, if it exists, the period of specialization is unique. $\square$

5. The Basic Sheshinski Specification: Bang-Bang Equilibria

Definition 6

(Law of Motion for Human Capital Stock in the Basic Sheshinski Specification)

$$\dot{H}(t) = AI(t)H(t) - \sigma H(t). \tag{74}$$

Condition 5

(Optimality Conditions for the Life-cycle Individual's Problem in the Basic Sheshinski Specification)

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \Leftrightarrow \mu(t)\exp(rt) = \frac{R}{A} \tag{75}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu}(t) \Leftrightarrow -\exp(-rt)R(1 - I(t)) - \mu(t)\left(AI(t) - \sigma\right) = \dot{\mu}(t) \tag{76}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H}(t) \Leftrightarrow \dot{H}(t) = AI(t)H(t) - \sigma H(t) \tag{77}$$

$$\textit{Transversality}: \lim_{t \to T} \mu(t)H(t) = 0 \tag{78}$$

Claim 6

(Bang-Bang in the Sheshinski Specification) Assume that $\sigma + r < A$ and that there is an initial period of specialization. Then, the solution to the problem is Bang-Bang, i.e., either $I = 0$ or $I = 1$.

Proof.

Define $g(t) = \mu(t) \exp(t)$ and use (76), (78) to obtain

$$\dot{g} = -R + (R - Ag)I + (\sigma + r)g \tag{79}$$

$$g(T) = 0. \tag{80}$$

In the specialization period $I(t) = 1$. If $\sigma + r < A$, $\dot{g}(t) < 0$. Actually, by (75), as $g(t)$ decreases to $\frac{R}{A}$, $I(t)$ switches from it's upper bound 1 to its lower bound 0.

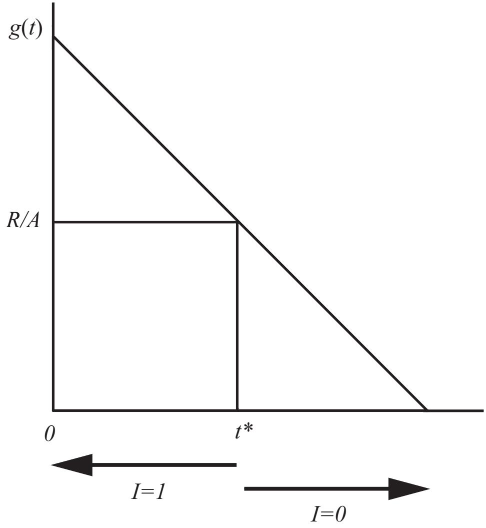
Then, with $I(t) = 0$ we can use $g(T) = 0$ and write

$$\dot{g}(t) = (\sigma + r)g(t) - R$$
$$\Rightarrow$$
$$g(t) = \frac{R}{\sigma + r}\left[1 - \exp((\sigma + r)(t - T))\right]. \tag{81}$$

for which $\dot{g}(t) < 0$ as well.
Therefore, once $I(t)$ reaches zero, it is never positive again.
This formulation has a Bang-Bang solution. $\square$

- Figure 9 is a graphical representation of Claim 6.

Figure 9: Bang-Bang Equilibrium in the Basic Sheshinski Specification



- We can actually solve for t^*, the length of the schooling period, using the fact that $g(t^*) = \frac{R}{A}$ by (75) and $g(t^*) = \frac{R}{\sigma+r}\left[1 - \exp((\sigma + r)(t^* - T))\right]$ by (81):

$$\frac{R}{A} = \frac{R}{\sigma + r}\left[1 - \exp((\sigma + r)(t^* - T))\right]$$

$$\Leftrightarrow$$

$$\text{Length of Schooling Period: } t^* = \frac{1}{\sigma + r} \ln \frac{A - (\sigma + r)}{A} + T. \quad (82)$$

Thus,

1. Longer life horizons imply more schooling, $\frac{\partial t^*}{\partial T} > 0$;
2. Greater depreciation implies less schooling, $\frac{\partial t^*}{\partial \sigma} < 0$;
3. Higher relative impatience implies less schooling, $\frac{\partial t^*}{\partial r} < 0$;
4. Higher productivity implies more schooling, $\frac{\partial t^*}{\partial A} > 0$;
5. Initial human capital does not affect schooling, $\frac{\partial t^*}{\partial H_0} = 0$.

5.1 From the Basic Sheshinski Specification to the Mincer Equation

- Assume that there is a period of specialization.
- From (67) we know that in the period $[0, t^*]$

$$\dot{H}(t) = (A - \sigma)H(t)$$
$$\Rightarrow$$
$$H(t) = H_0 \exp((A - \sigma)t). \tag{83}$$

- At t^*, actually, $I(t) = 0$ so earnings are $Y(t) = RH(t^*)$.
- Then, $t > t^*$

$$\ln Y(t) = \ln(RH_0) + (A - \sigma)t. \tag{84}$$

6. The Modified Sheshinski Specification

Definition 7

(Law of Motion for Human Capital in the Modified Sheshinski Specification)

$$\dot{H} = AI - \sigma H. \tag{85}$$

Condition 6

(Optimality Condition for the Life-cycle Individual's Problem in the Modified Sheshinski Specification)

$$\frac{\partial \mathcal{H}(\cdot)}{\partial I(t)} = 0 \Leftrightarrow \mu \exp(rt) = \frac{RH}{A} \tag{86}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial H(t)} = -\dot{\mu}(t) \Leftrightarrow \dot{\mu} = \mu\sigma - \exp(-rt)R(1-I) \tag{87}$$

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mu(t)} = \dot{H}(t) \Leftrightarrow \dot{H}(t) = AI - \sigma H \tag{88}$$

$$\mathit{Transversality}: \lim_{t \to T} \mu(t)H(t) = 0. \tag{89}$$

6.1 No Depreciation: A Modified Sheshinski Schooling Model

- Define $g(t) = \mu(t)\exp(rt)$ and use (87) to obtain

$$\dot{g} = g(\sigma + r) - R(1 - I). \tag{90}$$

- Let $\sigma = 0$.
- Then $\dot{g} = -R(1 - I) + rg$.
- And $\dot{H} = A$ when $I = 1$.
- So the solution for the human capital trajectory when $I = 1$ is

$$H(t) = At + H_0. \tag{91}$$

- At $t = 0$, $I = 1$ if $g(0) > \frac{R}{A} H_0$.
- Importantly, $I = 1$ implies that $\dot{g}(t) = rg(t) > 0$.
- As t grows, the return for gross investment grows because the payoff period gets closer.
- $I = 1$ cannot be a solution forever because the agent receives no earnings if she invests all of the time during the complete life-cycle.

Claim 7

(Uniqueness of the Period of Specialization in the Modified Sheshinski Specification with no Depreciation) If the specialization period exists, then it is unique.

Proof.

Based on (86), if a specialization period exists and if $g(t) - \frac{RH(t)}{A}$ is strictly decreasing overtime, then the specialization period must occur at the beginning of the life cycle and is unique. In the following we show that $\dot{g}(t) - \frac{R}{A}\dot{H}(t) < 0$.
Given that $\dot{g}(t) = rg(t) - R(1 - I(t))$ and $g(T) = 0$, we have:

$$g(t) = R\int_t^T (1 - I(s))\exp(r(t-s))ds \tag{92}$$

Then taking the derivative with respect to time gives:

$$\dot{g}(t) = R\left[-1 + I(t) + r\int_t^T (1 - I(s))\exp(r(t-s))ds\right]. \tag{93}$$

Together with (85), we have:

$$\begin{aligned}
\dot{g}(t) - \frac{R}{A}\dot{H}(t) &= -R + Rr\int_t^T (1 - I(s))\exp(r(t-s))ds && (94)\\
&\leq -R + Rr\int_t^T \exp(r(t-s))ds && (95)\\
&= -R\exp(r(t-T)) && (96)\\
&< 0, && (97)
\end{aligned}$$

where the first inequality follows from setting $I(\tau) = 0$. $\square$

- To compute the optimal schooling length, t^*, note that (86) holds with strict equality at t^* and (91) is valid so that

$$g(t^*) = \frac{R}{A}\left(At^* + H_0\right). \tag{98}$$

- (92) is also valid for t^*.
- Then,

$$\left(1 - \exp(r(t^* - T))\right) = \frac{r}{A}\left(At^* + H_0\right) \tag{99}$$

- and thus $\frac{\partial t^*}{\partial H_0} < 0$, $\frac{\partial t^*}{\partial A} > 0$ and $\frac{\partial t^*}{\partial r} < 0$ as in the model of Section 2.

6.2 Depreciation

- Let us give some conditions under which human capital investment would have different episodes over the life cycle.
- First assume that $g(0) > \frac{H_0 R}{A}$ so that there is a specialization period to begin with.
- We can solve (88) and (90) to obtain

$$H(t) = \left[H_0 - \frac{A}{\sigma}\right] \exp(-\sigma t) + \frac{A}{\sigma}$$
$$g(t) = g_0 \exp((r + \sigma)t)$$

  with $g_0 > 0$.
- Once the solution becomes interior, $g(t) = \frac{R}{A} H(t)$ by (86).
- Assume that $\sigma < \frac{A}{H_0}$ so that $\dot{H}(0) > 0$.

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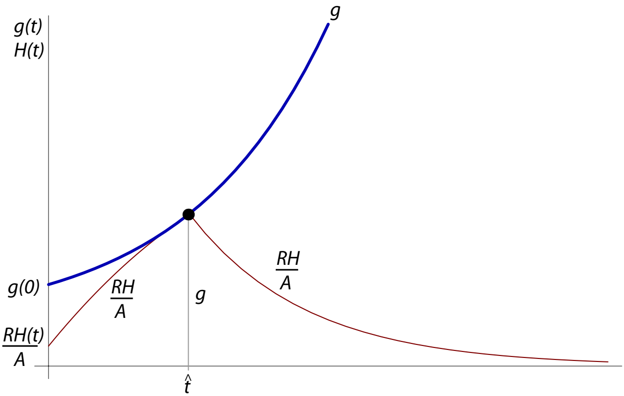
Figure 10: Return to Gross Investment in Human Capital in the Modified Sheshinski Specification

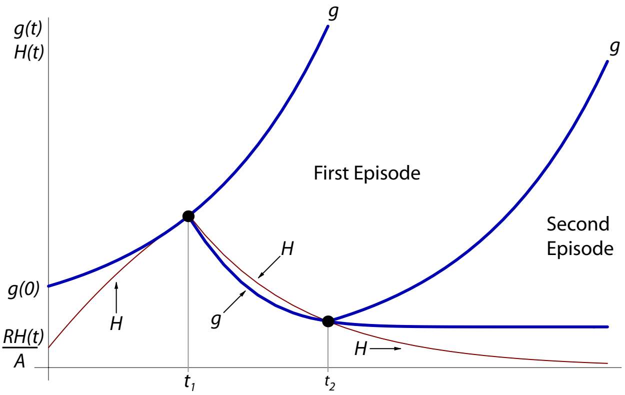


- Let t_1 denote the time in which the first period of specialization ends.
- If the solution "bangs-out" to $I = 0$ we can use (90) and (85) to get

$$\begin{aligned} \dot{g} &= (\sigma + r)g - R \\ H(t) &= H(t_1)\exp(-\sigma(t - t_1)) \end{aligned} \tag{100}$$

  for $t_1 < t < t_2$.
- Likewise, we can define a period t_2 in which the solution "bangs-in" again and so on.

Figure 11: Human Capital Investment Episodes in the Modified Sheshinski Specification



- In $t < t_1$, $I = 1$ implies $\dot{g} > 0$.
- g needs to decrease for the problem to respect the transversality condition.
- Thus, in the neighborhood of t_1 it has to be that $\dot{g}(t_1) < \frac{R\dot{H}(t_1)}{A}$ (see Figure (11)).
- If we take the expression from the right of $g(t_1)$ this requires

$$\begin{aligned} -R(\sigma + r)g(t_1) &< \frac{R\dot{H}(t_1)}{A} \\ &= \frac{-\sigma RH(t_1)}{A} \\ &= -\sigma g(t_1) \\ &\Leftrightarrow \\ g(t_1) &< \frac{R}{r}. \end{aligned} \tag{101}$$

- To wrap up this section, note that we have an initial period of specialization if $g_0 > \frac{RH_0}{A}$.

Link to
Appendix: Generalized Ben-Porath Model

7. Literature Extending the Ben-Porath Framework

- Grossman (1972): seminal theory paper studying people's life cycle investment decisions on health.
- In addition to choosing schooling and on-the-job training as in the Ben-Porath model, Keane and Wolpin (1997) also models individuals' choices on occupation.
- Heckman et al. (1998) develops a dynamic general equilibrium model with heterogeneous agents to explain the rising wage inequality in the U.S. The Ben-Porath model is extended in several ways.

- Gibbons and Waldman (1999) aims at explaining a variety of empirical evidence on firms' wage and promotion dynamics.
- The paper studies why workers receiving large wage increases early at a given job level are promoted faster.
- They also claim that using the Ben-Porath model alone is not sufficient.
- Two other principle elements are taken into account.
- In particular, they model how firms assign different jobs to different workers by using the idea of comparative advantage.
- They also include the component of learning, given that firms do not have perfect information on workers' productivity.
- By using the Ben-Porath model with two additional principal ingredients, the authors conclude that they are able to explain the main findings in the empirical literature on wage and promotion dynamics inside firms.

- Leibowitz (1974) is a pioneering study investigating the effect of family investments on child's future outcomes including ability, schooling and earnings.
- Following Ben-Porath, Leibowitz (1974) applies a Cobb-Douglas human capital production technology, which assumes that the inputs, including the current level of human capital stock and investments, are complementary.

- In Cunha and Heckman (2007) and Cunha et al. (2010), empirical evidence is provided to show that the human capital formation process is governed by a multistage technology.
- Eckstein and Weiss (2004) studies the mechanism causing the differences in wage growth patterns between natives and immigrants in Israel from 1990-2000.
- In Manning and Swaffield (2008), the authors seek to solve the puzzle that whereas the average earnings of the males and females are similar when they enter the labor market, the growth of the earnings for males is much faster than for females in the first ten years after labor market entry. The Ben-Porath model provides an answer to justify the puzzle.

- Huggett et al. (2011) documents that the mean of individuals' earnings is hump-shaped over the life cycle and the dispersion of individuals' earnings is increasing with age.
- The authors justify the hump-shaped mean earnings profile simply by using the Ben-Porath model.

- The estimated Ben-Porath framework with a risky human capital production technology shows that the differences in individuals' lifetime earnings are mainly due to the differences in individuals' initial conditions at age 23, rather than the idiosyncratic shocks experienced over the rest of their lives.

Appendix 1: Optimal Control

Problem 8

(Basic Formulation)

$$\max_{x(t),u(t)} \int_0^T e^{-\rho t} f(x(t), u(t), t)\,\mathrm{d}t \qquad (102)$$

s.t.

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)) \\ x(0) &= x_0 \\ x(T) &= T \end{aligned}$$

where $0 \leq t \geq T$ and $\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

- In this case, $x(t)$ is a vector of state variables, $u(t)$ is a vector of control variables and $\dot{x}(t)$ is the law of motion.
- The terminal condition may also be left free with $T \to \infty$; however, we only deal with finite horizon problems in this document and thus, focus on that case.
- Typically, the objective function and law of motion are assumed to be continuous, twice differentiable, strictly increasing and concave in their arguments and to satisfy the Inada conditions.

- A function $y(x)$ satisfies the Inada conditions if:

$$\lim_{x\to\infty} \frac{\partial y(x)}{\partial x} = 0$$
$$\lim_{x\to 0} \frac{\partial y(x)}{\partial x} = \infty$$

- where in discrete time we would use a Lagrangian to solve the problem, in continuous time we use a Hamiltonian function.

- We can choose to set up the Hamiltonian as a *current value* function or a *present value* function.
- Both result in the same optimal paths; however, a present value Hamiltonian considers values discounted to the present value, while the current value Hamiltonian does not.
- Because we use the present value formulation throughout this document, we will provide an overview of the present value formulation here.

- The present value Hamiltonian is

$$H(f(\cdot), u(\cdot), \Lambda, t) = e^{-\rho t} f(x(t), u(t), t) + \lambda g(x(t), u(t)) \quad (103)$$

- where λ is called the *co-state variable* and is analogous to the Lagrangian multiplier.
- The necessary conditions are the following:

$$\frac{\partial H(\cdot)}{\partial u(t)} = 0$$

$$\frac{\partial H(\cdot)}{\partial x(t)} = -\dot{\lambda}(t)$$

$$\lambda(T) \geq 0, \lambda(T) x^*(T) = 0 \quad (104)$$

- where (104) is called transversality condition.

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- (Mangasarian, 1966) proves that if $(x*(t), u*(t))$ is an admissible pair for (102) and if $H(\cdot)$ is a concave function over an open convex set of all the admissible values of all x, u, then there is a global maximum of $\int_0^T f(x(t), u(t), t)\,\mathrm{d}t$ at $(x*(t), u*(t))$.
- If $H(\cdot)$ is strictly concave, then $(x*(t), u*(t))$ yields the unique global maximum of $\int_0^T f(x(t), u(t), t)\,\mathrm{d}t$.

Return to text

Appendix: Ben-Porath Notes

Notes on Ben-Porath Human Capital Model

- Perfect Capital Markets
- No Nonmarket Benefits of Human Capital
- Fixed Labor Supply
- H is human capital
- $I \in [0, 1]$ is investment time
- D is goods input
- F is a strictly concave function in two normal inputs

Human Capital Production Function

- $\dot{H}(t) = F(I(t), H(t), D(t)) - \sigma H(t)$
- $F(I(t), H(t), D(t)) = F(I(t)H(t), D(t)) \quad$ (neutrality)
- R is rental rate of human capital.
- Potential earnings: $Y(t) = RH(t)$.
- Observed earnings:

$$E(t) = RH(t) - \underbrace{RI(t)H(t)}_{\substack{\text{earnings} \\ \text{forgone}}} - \underbrace{P_D D(t)}_{\substack{\text{direct goods} \\ \text{costs}}}$$

- Consumer problem (max with respect to $I(t), D(t)$):

$$\int_0^T e^{-rt} E(t)dt \qquad \text{given } H(0) = H_0$$

- Formal solution (Hamiltonian): Flow of value from the optimal lifetime program

$$\underbrace{e^{-rt}[RH(t) - RI(t)H(t) - P_D D(t)]}_{\text{current flow}} \quad \underbrace{+\mu(t)}_{\substack{\text{shadow price} \\ \text{of human capital}}} \quad [\dot{H}]$$

- FOC Conditions (for interior solution):

$$
\begin{aligned}
I(t): \quad Re^{-rt}H(t) &= \mu(t)F_1 H(t) \\
D(t): \quad e^{-rt}P_D &= \mu(t)F_2 \\
\dot{\mu}(t) &= -e^{-rt}[R - RI(t)] - \mu(t)F_1 I(t) + \mu(t)\sigma
\end{aligned}
$$

- Use FOC for investment to obtain:

$$\dot{\mu}(t) = -e^{-rt}R + \mu(t)\sigma.$$

- Define $g(t) = \mu(t)e^{+rt}$
  $\dot{g}(t) = \dot{\mu}e^{+rt} + r\mu(t)e^{+rt}$
  $\dot{g}(t) = (\sigma + r)g(t) - R.$

- Transversality: $\lim_{t \to T} \mu(t) H(t) = 0$

$$\therefore \mu(T) = 0 \Longrightarrow g(T) = 0$$

$$g(t) = \frac{R\left(1 - e^{(\sigma + r)(t - T)}\right)}{\sigma + r}.$$

- Note that g(t) is a discount factor that adjusts for exponential depreciation of gross investment.
- $\dot{H}(t) + \sigma H(t) = F(IH(t), D(t))$.

- $0 < I(t) < 1$, we can set up the problem in a "myopic" way.
- Gross "output" is $F(I(t)H(t), D(t))$.
- Returns on gross output: $g(t)$.
- Costs: $P_D D(t) + RI(t)H(t)$.
- Note: these are costs and returns as of period t.

- The agent's problem is:

$$\max_{I(t),D(t)} [g(t)F(I(t)H(t), D(t)) - P_D D(t) - RI(t)H(t) = 0]$$

FOC:

- $g(t)F_1(I(t)H(t), D(t))H(t) = RH(t)$
- $g(t)F_2(I(t)H(t), D(t)) - P_D = 0.$

Demand functions are inverted first order conditions:

- $I(t)H(t) = I(t)H\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$
- $D(t) = D\left(\frac{R}{g(t)}, \frac{P_D}{g(t)}\right)$

From normality of inputs, since $\dot{g}(t) < 0$, we have:

- $\dot{IH}(t) < 0$, $\dot{D}(t) < 0$.

- Then, if $\sigma = 0$,
  $\dot{E} = RF(I(t)H(t), D(t)) - R\,I\dot{H}(t) - P_D\dot{D}(t) > 0.$
- Otherwise earnings can rise and then fall over the life cycle. ($\sigma \neq 0$).
- What about $\ddot{E}(t)$? Ben Porath chose a Cobb-Douglas form for $F(I(t)H(t), D(t))$ and proves $\ddot{E}(t) < 0$.
- $\therefore$ Earnings increase at a decreasing rate over the life cycle.
- To simplify derivations, let $F_2 \equiv 0$ (i.e. ignore $D(t)$).

- First order condition for investment is:
  $g(t)F'(IH) = R.$
  $\dot{g} = (\sigma + r)g(t) - R$
- Differentiate the first order condition for investment.
- Set $R = 1$ (for convenience)
  (Note that $\dfrac{\dot{g}}{g} = \sigma + r - \dfrac{1}{g}$)
  $\dot{g}(t)F'(I(t)H(t)) + g(t)F''(I(t)H(t))I(t)\dot{H}(t) = 0.$
  Thus $\quad \dot{IH}(t) = -\left(\dfrac{\dot{g}(t)}{g(t)}\right)\left[\dfrac{F'}{F''}\right].$

- To simplify notation, drop "t" argument for $I(t), H(t), g(t)$ unless it clarifies matters to keep it explicit
- Then $I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} - \frac{\dot{g}}{g}\left[I\dot{H} - \frac{F'F'''}{(F'')^2}I\dot{H}\right]$.
- Note that $\ddot{g} = (\sigma + r)\dot{g}$.
- $\therefore \frac{\ddot{g}}{\dot{g}} = (\sigma + r)$ and $\frac{\ddot{g}}{g} = (\sigma + r)\frac{\dot{g}}{g}$ $(\dot{g} \neq 0)$.
- Thus, substituting for $I\dot{H}$ we have

$$I\ddot{H} = -\left[\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right]\frac{F'}{F''} + \left(\frac{\dot{g}}{g}\right)^2\left[1 - \frac{F'F'''}{(F'')^2}\right]\left[\frac{F'}{F''}\right].$$

- Earnings growth is given by (recall $R = 1$)
- $\dot{E} = F(I\!H) - I\dot{H} - \sigma H$
- $\ddot{E} = F'(I\!H)I\dot{H} - I\ddot{H} - \sigma \dot{H}$
- Since $F' = \dfrac{1}{\mathrm{g}}$ we have that

  $$\ddot{E} = \frac{1}{g} I\dot{H} - I\ddot{H} - \sigma \dot{H}$$

- Set $\sigma = 0$ for the moment and use the expression for $I\dot{H}$ given above (including $I\dot{H}$).

Thus

- $\ddot{E} = I\dot{H}\left[\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{(F'')^2}\right)\right] + \left(\frac{\ddot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right)\frac{F'}{F''}.$
- Use $I\dot{H} = -\frac{\dot{g}}{g}\frac{F'}{F''}$ and $\frac{\ddot{g}}{g} = (\sigma + r)\frac{\dot{g}}{g}$ to conclude that

$$\ddot{E} = -\frac{\dot{g}}{g}\left[\frac{F'}{F''}\right]\left\{\frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F''}{(F'')^2}\right)\right\}$$

$$+ \left((\sigma + r)\frac{\dot{g}}{g} - \left(\frac{\dot{g}}{g}\right)^2\right)\frac{F'}{F''}$$

$$= -\frac{\dot{g}}{g}\left(\frac{F'}{F''}\right)\left\{\begin{array}{c} \frac{1}{g} + \frac{\dot{g}}{g}\left(1 - \frac{F'F'''}{(F'')^2}\right) \\ -\frac{g(\sigma + r) - \dot{g}}{g} \end{array}\right\}$$

but $\dot{g} = (\sigma + r)g - 1 \qquad (\sigma + r)g - \dot{g} = 1.$

Thus

$$\begin{aligned}
\ddot{E} &= \left(-\frac{\dot{g}}{g}\frac{F'}{F''}\right)\left(\frac{\dot{g}}{g}\right)\left(1-\frac{F'\,F'''}{(F'')^2}\right) \\
&= \underbrace{-\left(\frac{\dot{g}}{g}\right)^2\frac{F'}{F''}}_{\substack{+ \\ \text{(by concavity)}}} \cdot \underbrace{\left(1-\frac{\mathrm{F}'\mathrm{F}'''}{(F'')^2}\right)}_{\substack{\text{Term depends on the} \\ \text{sign of } F'''}} .
\end{aligned}$$

- Define $\eta = 1 - \dfrac{F'F'''}{(F'')^2}$.
- Necessary condition for concavity of earnings profiles with age is $F''' > 0$;
- Stronger condition is $-\eta > 0$.

- Note: if $F(x) = \dfrac{Ax^{\alpha}}{\alpha}$, $-\infty < \alpha < 1$, $A > 0$, $F'(x) = Ax^{\alpha-1}$
- $F''(x) = (\alpha - 1)Ax^{\alpha-2}$
- $F'''(x) = (\alpha - 1)(\alpha - 2)Ax^{\alpha-3}$
- $\eta = \frac{\alpha-2}{\alpha-1} < 0$. Thus $\ddot{E}$ is negative (concavity).
- If $F(x) = a - be^{-cx}$, for $b, c > 0$, $\eta = 0$ and $\ddot{E}$ negative.
- Obviously fails with quadratic technologies.

Period of Specialization

- Period of specialization is associated with full time investment.
- Assume $F_2 \equiv 0$ (ignore D).
- Suppose that at time t

$$F'(H_0)g(t) > R.$$

- Then it pays to specialize.
- How to solve? Initially assume $\sigma = 0$.
- Note that marginal returns to investment decline with capital stock growth ($F' \downarrow$) and with time $\dot{g} < 0$.

- Then there is at *most* one period of specialization: $[0,\ t^*]$.
- This is "schooling" in the Ben-Porath model.
- t^* is characterized by

$$
\begin{aligned}
F'(H(t^*))g(t^*) &= R \\
I(t^*) &= 1 \text{ (at the endpoint of the interval)} \\
H(t^*) &= \int_0^{t^*} F\left(H(\tau)\right) d\tau + H_0.
\end{aligned}
$$

- Note that anything that lowers $g(t)$ (and not R) lowers t^*.
- Thus the higher r, the lower t^*.
- Note, also, that the higher H_0, the lower t^*, since it takes less time to acquire $H(t^*)$.

- Now to get $H(\tau)$, notice that $\dot{H} = F(H)$ in the period of specialization.
- Solve jointly to get t^*.
- Now, if $\sigma > 0$, we get the same condition for specialization but could get cycling in the model. (Initially, high σ knocks off capital makes specialization in investment productive again.)
- Let $\sigma = 0$, thus no cycling possible in the model.

Cobb-Douglas

Cobb-Douglas example:

$$\dot{H} = A(IH)^{\alpha} - \sigma H, \quad 0 < \alpha < 1, \quad A > 0$$

A period of specialization arises if

$$g(t_0)\alpha A(H_0)^{\alpha-1} > R.$$

Then if

$$(H_0)^{\alpha-1} > \left[\frac{R}{g(t_0)\alpha A}\right]$$

$$\text{or} \quad H_0 < \left[\frac{R}{g(t_0)\alpha A}\right]^{\frac{1}{\alpha-1}},$$

the agent will specialize.
If $T \to \infty$, the condition simplifies to

$$H_0 < \left(\frac{r}{\alpha A}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$$

$$\text{since } g(t) = \frac{R}{r}$$

If the condition required for specialization is satisfied, we obtain:

$$\dot{H} = A(IH)^{\alpha}$$

$$\therefore \qquad \frac{\dot{H}}{H^{\alpha}} = A$$

$$\begin{aligned} H(t)^{1-\alpha} &= (1-\alpha)At + (1-\alpha)K_0 \\ H(t) &= [(1-\alpha)At + (1-\alpha)K_0]^{\frac{1}{1-\alpha}} \\ [K_0(1-\alpha)]^{\frac{1}{1-\alpha}} &= H_0 \\ K_0(1-\alpha) &= H_0^{1-\alpha} \\ K_0 &= \frac{H_0^{1-\alpha}}{(1-\alpha)} \end{aligned}$$

Therefore, we have that during the period of specialization (schooling) human capital is accumulating via the following growth process:

$$
\begin{aligned}
H(t) &= [A(1-\alpha)t + K_0(1-\alpha)]^{\frac{1}{1-\alpha}} \\
&= [A(1-\alpha)t + H_0^{1-\alpha}]^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

At the end of the period of specialization we have that

$$
\alpha g(t^*) A(H(t^*))^{\alpha-1} = R.
$$

Let $T \to \infty$, then $g(t^*) = R/r$ and t^* is defined by solving

$$
\alpha \frac{R}{r} A \left(A(1-\alpha)t^* + H_0^{1-\alpha} \right)^{-1} = R.
$$

Thus,

$$\left(\frac{\alpha A}{r}\right) = A\left(1-\alpha\right)t^{*} + H_0^{1-\alpha}$$

$$\text{Schooling:} \qquad t^{*} = -\frac{H_0^{1-\alpha}}{A\left(1-\alpha\right)} + \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r}$$

Higher A, higher t^{*} "ability to learn."
Higher H_0, lower t^{*} "ability to earn."

Define post school work experience as $\tau = t - t^*$. Then

$$E(\tau) = R \int_0^{\tau} \dot{H}(\ell + t^*) \, d\ell + RH(t^*) - RIH(\tau + t^*).$$

At school leaving age and beyond we have

$$\alpha g(t) \, A(IH(t))^{\alpha - 1} = R.$$

Thus, we have

$$[IH(t)]^{\alpha-1} = \frac{R}{\alpha g(t) A}$$

$$IH(t) = \left[\frac{\alpha g(t) A}{R}\right]^{\dfrac{1}{1-\alpha}}.$$

Thus,

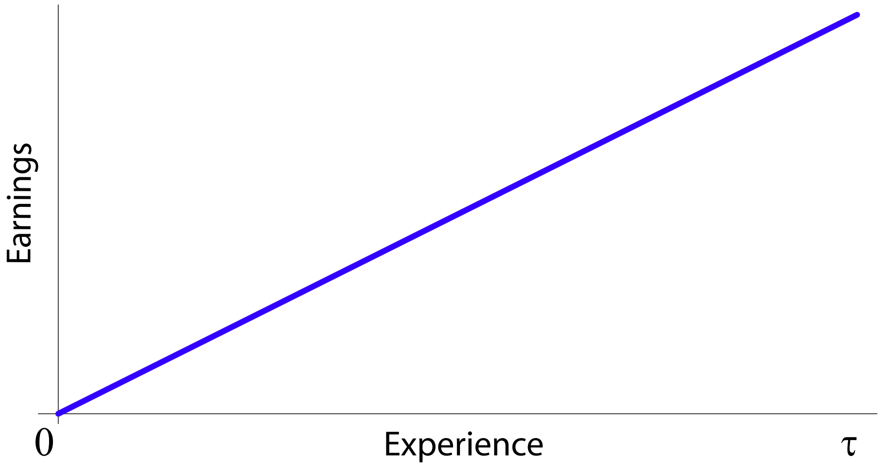
$$\dot{H} = A\left[\frac{\alpha g(t)A}{R}\right]^{\dfrac{\alpha}{1-\alpha}}.$$

Earnings are given by

$$E(\tau) = R\int_0^\tau A\left[\frac{\alpha g(\ell + t^*)A}{R}\right]^{\frac{\alpha}{1-\alpha}}\, d\ell + RH(t^*)$$
$$- R\left[\frac{\alpha g(\tau + t^*)A}{r}\right]^{\frac{1}{1-\alpha}}.$$

Let $T \to \infty$, then $g(t) = \dfrac{R}{r}$

$$E(\tau) = RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau + R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} - R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}$$
$$= RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}\tau.$$


Earnings
0
Experience
τ

$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$\begin{aligned}
t = t_0 &\Rightarrow H(t) = H_0 \\
t_0 < t < t^* &\Rightarrow H(t) = \left(A(1-\alpha)t + H_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \\
t = t^* &\Rightarrow H(t) = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\
t^* < t &\Rightarrow H(t) = \left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}}(t - t^*) + H(t^*)
\end{aligned}$$

$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

$$\begin{aligned} t = t_0 &\Rightarrow I(t) = 1 \quad \text{if} \quad F'(H_0)g(t) > R \\ t_0 < t \leq t^* &\Rightarrow I(t) = 1 \\ t^* < t &\Rightarrow I(t) = \frac{\left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}}(t-t^*) + H(t^*)} \\ I(t) &= \left(\left(\frac{\alpha A}{r}\right)^{-1}(t-t^*) + 1\right)^{-1} \end{aligned}$$

$$t_0 < t < T, \qquad T \to \infty, \qquad t^* = \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r} - \frac{H_0^{1-\alpha}}{A(1-\alpha)}$$

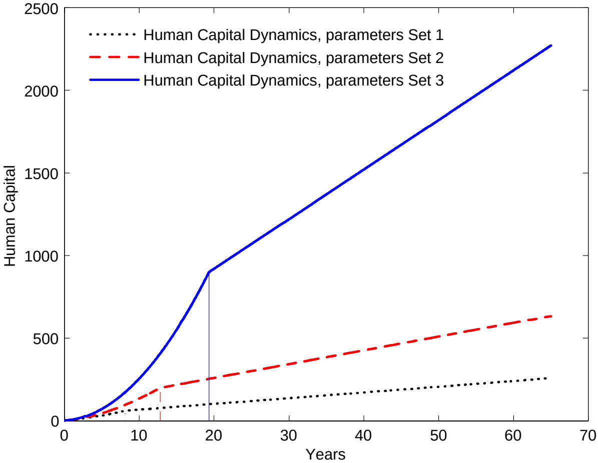
$E(t) = RH(t) \cdot (1 - I(t))$, so

$$t_0 < t \leq t^* \Rightarrow \quad I(t) \quad = 1 \Rightarrow E(t) = 0$$

$$\begin{aligned} t^* < t \Rightarrow \quad E(t) &= RH(t) - RH(t)I(t) \\ &= RH(t) - \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \\ &= R(A(1-\alpha)t + H_0^{1-\alpha})^{\frac{1}{1-\alpha}} - \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

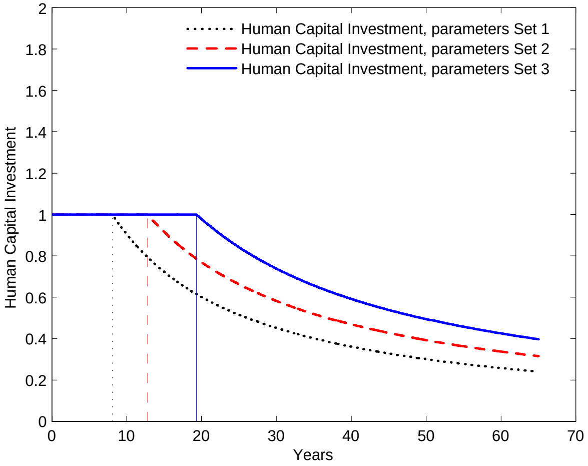
Human capital dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



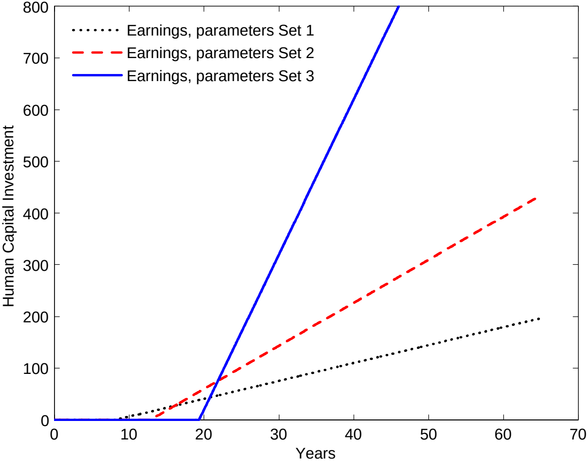
Human investment dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$, $R = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



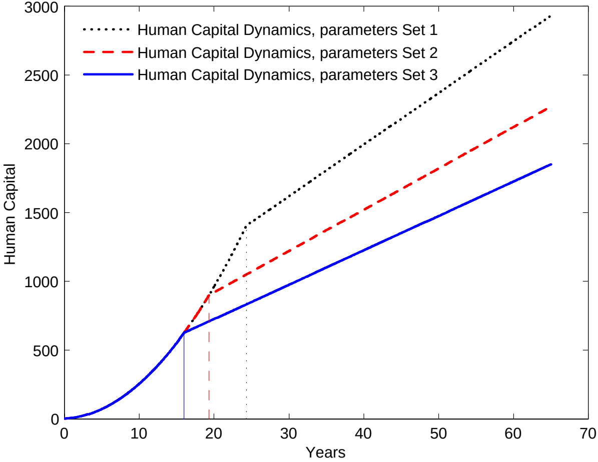
Earnings dynamics, varying α ($A = 3$, $r = 0.05$, $H_0 = 1$)

$\alpha = 0.3$ (dotted line), $\alpha = 0.4$ (dashed line), $\alpha = 0.5$ (solid line)



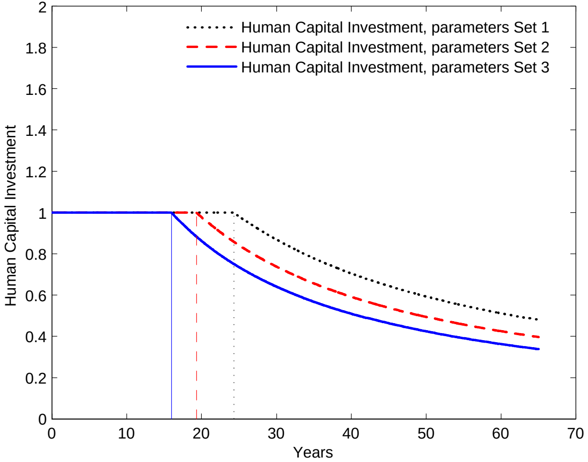
Human capital dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



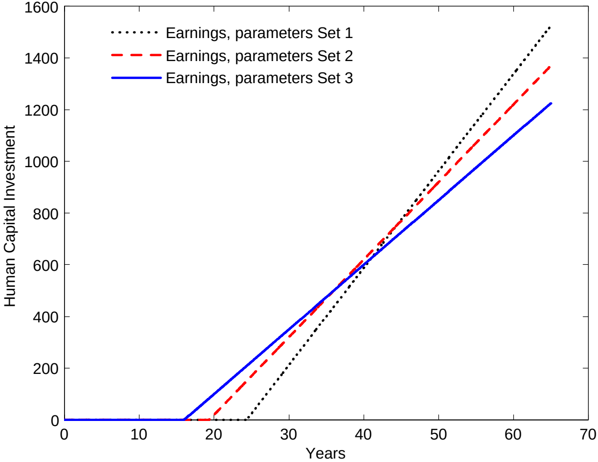
Human investment dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



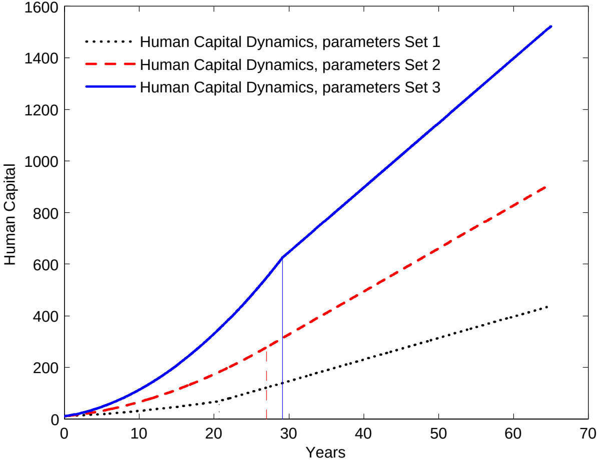
Earnings dynamics, varying r ($A = 3$, $H_0 = 1$, $\alpha = 0.5$, $R = 1$)

$r = 0.04$ (dotted line), $r = 0.05$ (dashed line), $r = 0.06$ (solid line)



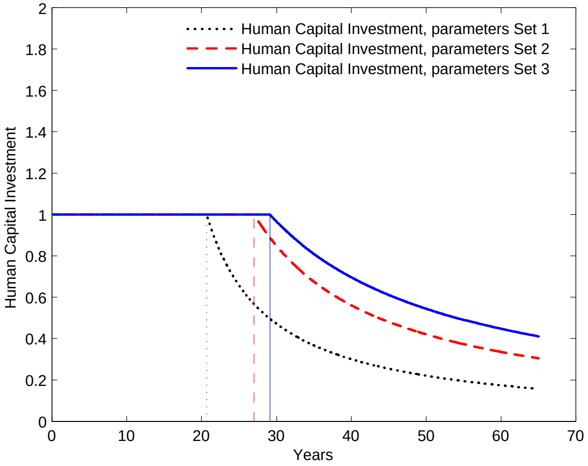
Human capital dynamics, varying A ($r = 0.03$, $H_0 = 10$, $\alpha = 0.5$)

$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)

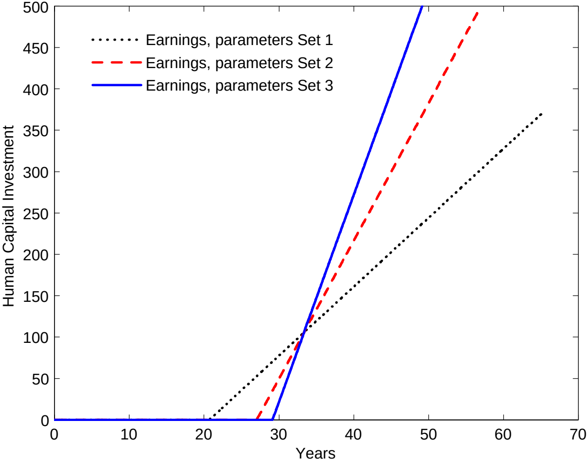


Human investment dynamics, varying A ($r = 0.03$, $H_0 = 10$, $\alpha = 0.5$)

$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)

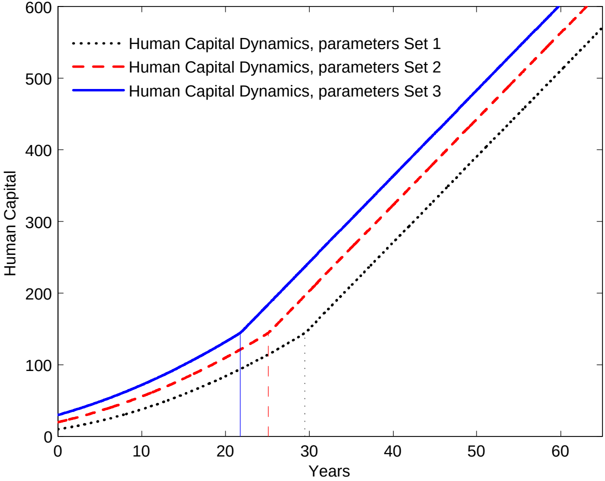


$A = 0.5$ (dotted line), $A = 1.0$ (dashed line), $A = 1.5$ (solid line)



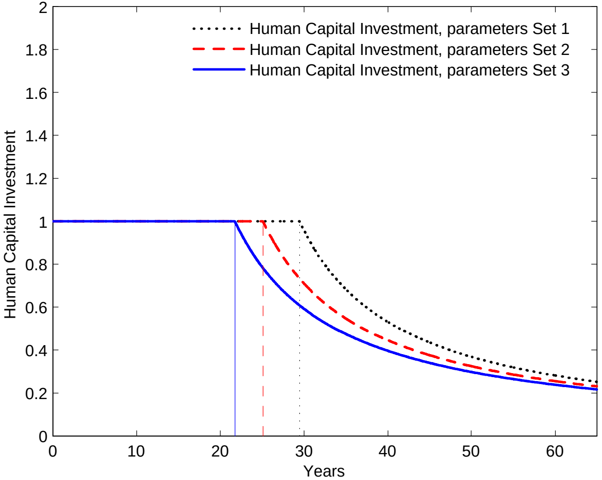
Human capital dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



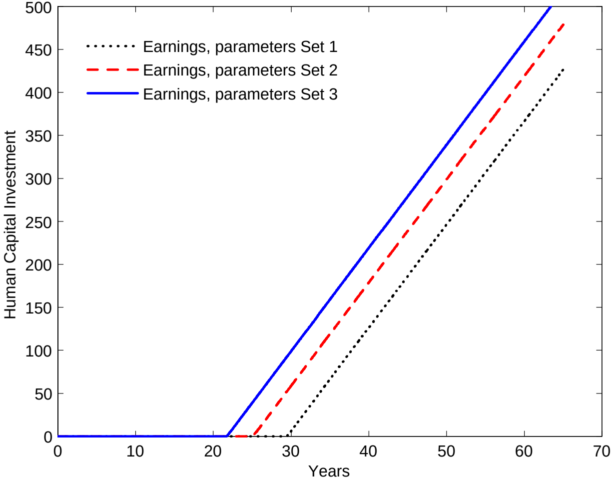
Earnings dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



Human investment dynamics, varying H_0 ($A = 0.6$, $r = 0.025$, $\alpha = 0.5$, $R = 1.0$)

$H_0 = 10$ (dotted line), $H_0 = 20$ (dashed line), $H_0 = 30$ (solid line)



Finite Horizon

Finite Horizon Ben Porath Model in Level and Autogressive Form ($\alpha = 1/2$)

- $\dot{H} = A(IH)^{\alpha}$
- $\alpha = 1/2$ (Haley, 1976; Rosen, 1976)
- $\sigma = 0$
- $R =$ rental rate

$$\dot{E}(\tau) = \frac{A^2}{2} g(\tau + t^*) - 2R \left[\frac{A}{2} \frac{g(\tau + t^*)}{R} \right] \left[\frac{A}{2R} \dot{g}(\tau + t^*) \right]$$

$$\dot{g} = rg - R$$

- Thus,

$$\dot{E}(\tau) = \left[\frac{A^2}{2R} \right] g[2R - rg]$$

$$\ddot{E}(\tau) = \frac{-A^2}{R} (\dot{g})^2.$$

- Using $\dot{g} = rg - R$,

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2R}\left(\frac{\dot{g}+R}{r}\right)\left(2R - r\frac{(\dot{g}+R)}{r}\right)\\ &= \frac{A^2}{2Rr}(R^2 - (\dot{g})^2).\end{aligned}$$

- Thus,

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2Rr}R^2 - \frac{1}{2r}\frac{A^2}{R}(\dot{g})^2\\ &= \frac{A^2}{2Rr}R^2 + \frac{1}{2r}\ddot{E}(\tau).\end{aligned}$$

- Thus,

$$\ddot{E}(\tau) = 2r\dot{E}(\tau) - A^2 R. \tag{105}$$

- This is a standard ordinary differential equation with constant coefficients. The solution is of the form

$$E(\tau) = c_1 e^{2r\tau} + c_2\tau + c_0.$$

- We can pin this equation down knowing that

$$\begin{aligned} E(0) &= 0 \qquad && (\text{so } c_1 + c_0 = 0) \\ \dot{E}(T) &= 0 \qquad && (\text{so } 2rc_1 e^{2rT} + c_2 = 0). \end{aligned}$$

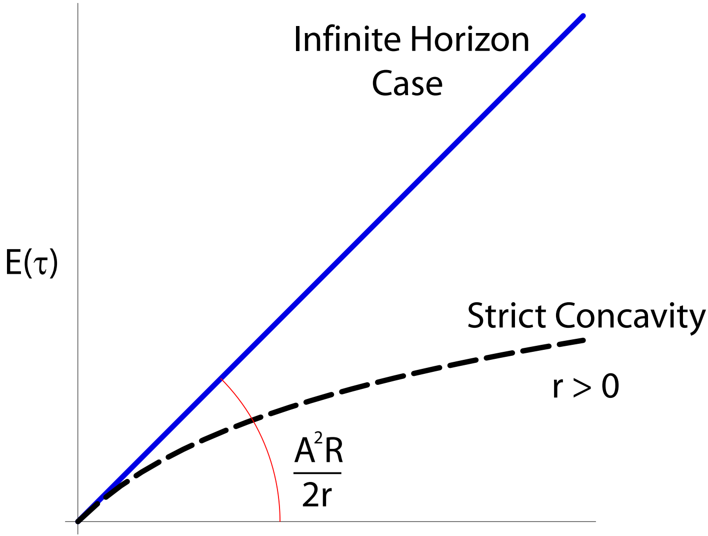
- Finally, optimality produces (105) above to get c_0.
- Set

$$c_1 = -c_0$$
$$c_2 = \frac{A^2 R}{2r} e^{2rT},$$

using $E(T) = 0$ and (105).

- Thus

$$E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT}(1 - e^{2r\tau}) + \left(\frac{A^2 R}{2r}\right)\tau. \tag{106}$$


Infinite Horizon
Case
E(τ)
Strict Concavity
r > 0
$\frac{A^2R}{2r}$

- This, in its essential form, is the equation that Brown (JPE, 1976) fits; from the τ term, one can identify $\dfrac{A^2R}{2r}$.
- From the exponential (in τ) one can pick up r and A^2R, but his estimates are poor, $r \to 0$.
- But from Brown, $T \to \infty$ is a good approximation. (His sample is young). Thus

$$E(\tau) \doteq \frac{RA^2}{2}\tau.$$

- Thus "r" is not identified.

- Write this as an autoregression:

$$E(\tau+1) - E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT} \left(e^{2r\tau} - e^{2r(\tau+1)}\right) + \frac{A^2 R}{2r}$$

$$E(\tau) - E(\tau-1) = \frac{A^2 R}{(2r)^2} e^{-2rT} \left(e^{2r(\tau-1)} - e^{2r\tau}\right) + \frac{A^2 R}{2r}.$$

- Multiply second equation by e^{2r}:

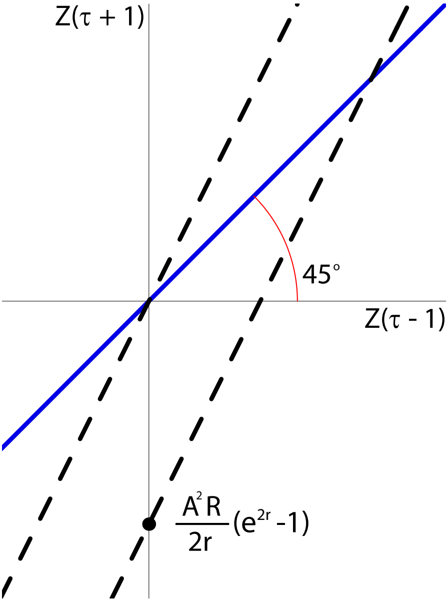
$$e^{2r}[E(\tau) - E(\tau - 1)] = \frac{A^2 R}{2r^2} e^{-2rT}(e^{2r\tau} - e^{2r(\tau+1)}) + e^{2r}\frac{(A^2 R)}{2r}$$
$$= E(\tau + 1) - E(\tau) - (1 - e^{2r\tau})\frac{A^2 R}{2r}.$$

- Thus

$$E(\tau + 1) - E(\tau) = e^{2r}[E(\tau) - E(\tau - 1)] - (e^{2r} - 1)\frac{A^2 R}{2r}.$$

- Let

$$
\begin{aligned}
Z(\tau+1) &= E(\tau+1) - E(\tau) \\
Z(\tau) &= E(\tau) - E(\tau-1) \\
Z(\tau+1) &= e^{2r} Z(\tau) - (e^{2r}-1)\left(\frac{A^2 R}{2r}\right).
\end{aligned}
$$


$Z(\tau + 1)$
45°
$Z(\tau - 1)$
$\frac{A^2 R}{2r}(e^{2r} - 1)$

- Apparently explosive, it actually converges. Observe:

$$
\begin{aligned}
E(\tau) - E(\tau - 1) &= \frac{A^2 R}{(2r)^2} e^{-2rT}\left(e^{2r(\tau-1)} - e^{2r\tau}\right) + \frac{A^2 R}{2r} \\
&= \frac{A^2 R}{2r}\left[1 + \frac{e^{-2rT}}{2r} e^{2r\tau}\left(1 - e^{2r}\right)\right]
\end{aligned}
$$

$$
\frac{\partial[E(\tau) - E(\tau - 1)]}{\partial\tau} = \frac{A^2 R}{2r}(e^{-2rT} e^{2r\tau}\left(1 - e^{2r}\right) < 0
$$

- Increments are actually decreasing.
- Let $b = e^{2r}$.

$$c = -\left(\frac{e^{2r}-1}{2r}\right)\frac{A^2R}{2r} = \left(\frac{1-e^{2r}}{2r}\right)A^2R$$

$$Z(T) = \underbrace{(b)^T(Z_0)}_{\text{growing}} + \underbrace{c\sum_{j=0}^{T-1} b^j}_{\text{declining}},$$

but converges to a constant (even though autoregression is "explosive").

Deriving Mincer from Ben Porath

Using (106), we obtain

$$E(\tau) = \left(\frac{A^2 R}{2r}\right)\left[\tau + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r}\right].$$

In logs,

$$\begin{aligned}\ln \mathrm{E}(\tau) &= \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{e^{-2rT} - e^{2r(\tau-T)}}{2r\tau}\right] \\ &= \ln\left[\frac{A^2 R}{2r}\right] + \ln\tau + \ln\left[1 + \frac{e^{-rT}(1 - e^{2r\tau})}{2r\tau}\right].\end{aligned}$$

The Taylor Expansions

$$\ln(\tau) \doteq \ln(\tau_0) + \frac{1}{\tau_0}(\tau - \tau_0) - \frac{1}{\tau_0^2}\frac{(\tau-\tau_0)^2}{2!}$$

$$\ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right) \doteq \xi_0 + \xi_1(\tau - \tau_0) + \xi_2 \frac{(\tau-\tau_0)^2}{2!}$$

$$\xi_0 \equiv \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau_0 - T)}}{2r\tau_0}\right)$$

$$\xi_1 \equiv -\left(\frac{e^{-2rT} + e^{2r(\tau_0 - T)}(2r\tau_0 - 1)}{\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)}\right)$$

$$\xi_2 \equiv \left[\begin{array}{c} \frac{\left(e^{-2rT} + e^{2r(\tau_0 - T)}(2r\tau_0 - 1)\right)}{\left(\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)\right)^2}\left(4r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}(2r\tau_0 + 1)\right) \\ -\left(\frac{(2r)^2\tau_0 e^{2r(\tau_0 - T)}}{\left(\tau_0\left(2r\tau_0 + e^{-2rT} - e^{2r(\tau_0 - T)}\right)\right)}\right) \end{array}\right]$$

Adding the terms together:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right) \doteq \alpha_0 + \alpha_1(\tau - \tau_0) + \alpha_2(\tau - \tau_0)^2$$

$$\alpha_0 \equiv \ln(\tau_0) + \xi_0$$

$$\alpha_1 \equiv \xi_1 + \frac{1}{\tau_0}$$

$$\alpha_2 \equiv \left(-\frac{1}{\tau_0^2} + \xi_2\right) \Big/ 2$$

To obtain Mincer Equations:

$$\ln(\tau) + \ln\left(1 + \frac{e^{-2rT} - e^{2r(\tau - T)}}{2r\tau}\right) \doteq k_0 + k_1\tau + k_2\tau^2$$

$$\begin{aligned}
k_0 &\equiv \alpha_0 - \tau_0\alpha_1 + \alpha_2\tau_0^2 \\
k_1 &\equiv \alpha_1 - 2\alpha_2\tau_0 \\
k_2 &\equiv \alpha_2
\end{aligned}$$

Mincer Obtained:

- Mincer coefficients

$$\hat{k}_1 = 0.081$$
$$\hat{k}_2 = -0.0012$$

- Using $r = 0.0225$, $\tau_0 = 29.54$, $T = 41.43$,

$$k_1 = 0.081$$
$$k_2 = -0.0010$$

Parameters			Ben Porath Coefficients	
r	τ_0	T	k_1	k_2
0.0225	29.54	41.43	0.081	-0.0010
0.05	25	60	0.0808	-0.0008
0.05	20	65	0.1002	-0.0013
0.0675	24.70	74.77	0.081	-0.0008
Mincer Coefficients			0.081	-0.0012

Model: $\ln(\text{Earnings}) = k_0 + k_1\tau + k_2\tau^2$

- Suppose

$$rT \doteq 0 \quad \text{and} \quad e^{-rT} = 1\,.$$

$$\ln \mathrm{E}(\tau) \doteq \ln\left(\frac{A^2 R}{2r}\right) + \ln\tau + \ln\left[1 + \frac{1 - e^{2r\tau}}{2r\tau}\right]$$

Conclusion

- There may be no economic content in Mincer's "rate of return" on post-school investment.
- All of the economic content is in the intercept term.
- Note, however, holding experience constant, there should be no effect of schooling on the earnings function.
- Mincer finds an effect. This would seem to argue against the Ben-Porath model.
- Not necessarily. Look at equation

$$t^* = \frac{1}{r} - \frac{1}{2}\frac{H_0^{1/2}}{A} \quad \text{for } \alpha = 1/2 \text{ and } T \text{ ``big.''}$$

- Suppose A is randomly distributed in the population.
- Then, we have that if H_0 is distributed independently of A, the coefficient on t^* (length of schooling) is

$$E\left[\left(-\frac{1}{2}\frac{H_0^{1/2}}{A}\right)(2\ln A)\right] > 0.$$

- Thus, the coefficient on schooling is

$$-E\left(H_0^{1/2}\right)E\left(\frac{\ln A}{A}\right).$$

If A is Pareto;

$$F(A) = \left(\frac{\alpha}{A_0}\right)\left(\frac{A_0}{A}\right)^{\alpha+1}, \quad A_0 > 0,\ \alpha > 0.$$

Integrate by parts to reach

$$\begin{aligned} E\left(\frac{\ln A}{A}\right) &= -\frac{(A_0)^{\alpha+1}\alpha}{A_0}(\ln A_0)\, A_0^{-(\alpha+1)} - \frac{1}{\alpha+1} \\ &= -\frac{\alpha \ln A_0}{A_0} - \frac{1}{\alpha+1} \end{aligned}$$

Therefore, the coefficient on schooling is

$$E\left(H_0\right)^{1/2}\left[\frac{1}{\alpha+1}+\frac{\alpha \ln A_0}{A_0}\right]>0.$$

Since units of H_0 are arbitrary, we are done.
Therefore, positive coefficient on schooling solely as a consequence of *not* including ability measures.

Rate of Return to Post-School Investment

Let $T \to \infty$. Without post-school investment, person makes

$$R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}.$$

Increment in earnings at post-school age τ is simply

$$\underbrace{RA\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}\tau}_{\text{Earnings (above schooling earnings) at } \tau} - \underbrace{R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}}_{\text{Costs}}.$$

- ϕ is that rate that equates returns and costs. Thus, solve for ϕ.

$$\int_0^\infty e^{-\phi\tau}\left[RA\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}\tau - R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}}\right] d\tau = 0$$

- Use the Laplace transform.
- Then

$$\frac{1}{\phi^2} RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}} - \frac{1}{\phi} R\left[\frac{\alpha A}{r}\right]^{\frac{1}{1-\alpha}} = 0$$

$$\phi = \frac{r}{\alpha}.$$

- Therefore the rate of return to post-schooling investment is r/α.
- Smaller α, higher ϕ.
- Thus, the lower the productivity (i.e., α), the higher ϕ.

Rate of Return to Schooling (Holding Post-School Investment Fixed)

Person without schooling can earn RH_0. With schooling can earn $RA\left[\frac{\alpha A}{r}\right]^{\frac{\alpha}{1-\alpha}}$. (Assuming no post school investment.)

Recall that (for $T \to \infty$), optimal schooling is given by

$$t^* = \frac{1}{r} - \frac{1}{2}\frac{H_0^{1/2}}{A}.$$

During this period (before t^*), under our assumptions, there are no earnings.

Then the rate of return is given by comparing

$$\int_{t^*}^{\infty} e^{-\phi t} \left[R \left(\frac{\alpha A}{r} \right)^{\frac{1}{1-\alpha}} \right] dt \text{ with } \int_{0}^{\infty} e^{-\phi t} R H_0 \, dt.$$

Solve for ϕ:

$$\begin{aligned} \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} e^{-\phi t^*} &= H_0 \\ \ln \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \phi t^* &= \ln H_0 \\ \phi = \frac{\ln \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \ln H_0}{t^*} &= \frac{\ln \left[\frac{\alpha A}{r} \right]^{\frac{1}{1-\alpha}} - \ln H_0}{\frac{1}{r} - \frac{1}{2} \frac{H_0^{1/2}}{A}} \end{aligned}$$

Has no simple relationship to the rate of return to investment.

Growth of Earnings

- Keep time argument implicit unless being explicit helps.
- E, H, IH all depend on t.
- Growth of earnings:

$$\dot{E} = f(IH) - (\dot{IH})$$

$$\frac{\partial \dot{E}}{\partial r} = \ ?$$

- FOC:

$$g(t)\, f'(IH) = 1$$

$$g(t) = \frac{1 - e^{r(t-T)}}{r}$$

- Totally differentiate FOC with respect to t:

$$\dot{g}f'(IH) + gf''(IH)(\dot{IH}) = 0$$

$$-\left(\frac{\dot{g}}{g}\frac{f'}{f''}\right) = (\dot{IH})$$

- First note that

$$\frac{\partial \dot{E}}{\partial r} = f'\left(\frac{\partial IH}{\partial r}\right) - \frac{\partial}{\partial r}\left[(\dot{IH})\right].$$

- Now observe further that

$$\frac{\partial(IH)}{\partial r} < 0.$$

- Thus the first term is negative.
- Observe that we can show that

$$\frac{\partial(I\dot{H})}{\partial r} > 0$$

if concavity on earnings is satisfied ($\ddot{E} < 0$).

- Intuition: the time rate of decrease in IH is slowed down ($r \uparrow \Rightarrow IH \downarrow$; the function is "less concave").

- If we can establish this, we know that the contribution of the second term is negative and

$$\frac{\partial \dot{E}}{\partial r} < 0.$$

- To show this, observe that

$$\frac{\partial[\dot{IH}]}{\partial r} = \left[-\frac{\dot{g}}{g}\right]\left[1 - \frac{f' f'''}{(f'')^2}\right]\frac{\partial(IH)}{\partial r} + \left(\frac{f'}{f''}\right)\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right].$$

- From the earlier notes, concavity of earnings function in experience ($\ddot{E} < 0$)

$$\left[1 - \frac{f' f'''}{(f'')^2}\right] < 0.$$

- The first term is positive, since $\dot{g} < 0$ and

$$\frac{\partial(IH)}{\partial r} < 0.$$

- To investigate the second term, we determine that

$$\dot{g} = rg - 1\,, \qquad \frac{\dot{g}}{g} = r - \frac{1}{g}\,, \qquad -\frac{\dot{g}}{g} = \frac{1}{g} - r\,.$$

- Now,

$$\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] = -\frac{1}{g^2}\frac{\partial g}{\partial r} - 1.$$

- This term is negative. Why?

- $$\frac{\partial g}{\partial r} = \frac{-(t-T)e^{r(t-T)}}{r} - \frac{1-e^{r(t-T)}}{r^2}$$
$$= \frac{1}{r^2}\left[e^{r(t-T)}(1-r(t-T))-1\right]$$

- Now observe that
$$e^{r(T-t)} > 1 + r(T-t) \qquad \text{for } T \geq t\,.$$

- Thus
$$\frac{\partial g}{\partial r} < 0.$$

- Consider next that

$$\frac{-\partial g}{g^2 \partial r} - 1 = \frac{1}{r^2}\left[\frac{1 - e^{r(t-T)}\left(1 - r(t-T)\right)}{g^2}\right] - 1$$

$$= \frac{1}{g^2 r^2}\left[1 - e^{r(t-T)}\left(1 - r(t-T)\right) - \left(1 - e^{r(t-T)}\right)^2\right]$$

$$= \frac{1}{(rg)^2}\left[1 - e^{r(t-T)}\left(1 - r(t-T)\right) - 1 + 2e^{r(t-T)} - e^{2r(t-T)}\right]$$

$$= \frac{1}{(rg)^2}\left[e^{r(t-T)}\right]\left[1 + r(t-T) - e^{r(t-T)}\right].$$

- This expression is clearly negative.
- Set $x \equiv T - t$:

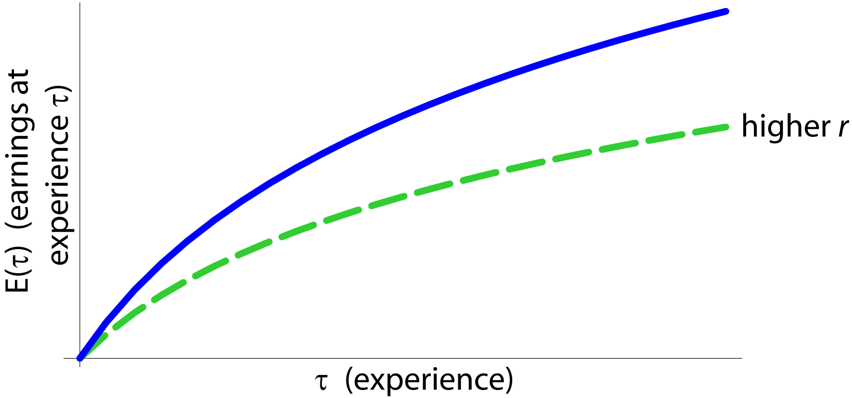
1. $$1 - rx - e^{-rx} = 0 \quad \text{when } x = 0.$$

2. $$\frac{\partial}{\partial x}\left(1 - rx - e^{-rx}\right) = -r + re^{-rx} < 0.$$

- Thus from concavity ($f'' < 0$),

$$\left(\frac{f'}{f''}\right)\frac{\partial}{\partial r}\left[-\frac{\dot{g}}{g}\right] > 0.$$

- Now the proposition is proved for $\sigma = 0$ with $\ddot{E} < 0$ everywhere. *Q.E.D.*


E(τ) (earnings at experience τ)
higher r
τ (experience)

Appendix: Haley-Rosen: Let $\alpha = 1/2$.

$$E(\tau) = RH(t^*) + R\int_0^{\tau} A\left(\frac{1}{2}\frac{g(t^*+\ell)A}{R}\right)\, d\ell - R\left[\frac{1}{2}\frac{g(\tau+t^*)A}{R}\right]^2.$$

This can be written as a simple autoregression in earnings:

$$\begin{aligned}
\dot{E}(\tau) &= R\left[A\left(\frac{1}{2}\frac{g(t^*+\tau)A}{R}\right) - 2R\left[\frac{1}{2}\frac{g(\tau+t^*)A}{R}\right]\frac{A}{2R}\dot{g}(\tau+t^*)\right] \\
&= \frac{1}{2}A^2[g(t^*+\tau)(R-\dot{g}(t^*+\tau)].
\end{aligned}$$

$$\dot{g} = rg - R$$

Thus

$$\dot{E}(\tau) = \frac{A^2}{2R}\left[g(t^* + \tau)\left(R - \dot{g}(t^* + \tau)\right)\right]$$

$$\begin{aligned}\dot{g} &= rg - R \\ \ddot{g} &= r\dot{g}.\end{aligned}$$

Haley-Rosen: $\alpha = \beta = 1/2$

$$
\begin{aligned}
E(\tau) &= RH(t^*) + R\int_0^\tau A\left(\frac{1}{2}\frac{g(t^*+\ell)A}{R}\right) d\ell - R\left[\frac{A}{2}\frac{g(\tau+t^*)}{R}\right]^2 \\
\dot{E}(\tau) &= \frac{A^2}{2}g(\tau+\tau^*) - 2R\left[\frac{A}{2}\frac{g(\tau+t^*)}{R}\right]\left[\frac{A}{2R}\dot{g}\right] \\
&= \frac{A^2}{2}g(\tau+t^*) - \frac{1}{2}\frac{A^2}{R}g\dot{g} \\
&= \frac{1}{2}A^2 g\left[1-\frac{\dot{g}}{R}\right] \qquad \text{use: } \dot{g} = rg - R \\
&= \frac{1}{2}\frac{A^2}{R}g[R-\dot{g}] = \frac{A^2}{2R}g[R-rg+R] \\
&= \frac{A^2}{2R}g[2R-rg]
\end{aligned}
$$

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$$
\begin{aligned}
\ddot{E}(\tau) &= \frac{A^2}{2R}[\dot{g}(2R - rg) + g(-r\dot{g})] \\
&= \frac{A^2}{2R}\dot{g}[2R - 2rg] = \frac{A^2}{R}\dot{g}(R - rg) \\
&= -\frac{A^2}{2}(\dot{g})^2.
\end{aligned}
$$

Notice that $\dot{E}(\tau)$ can be written as

$$
\begin{aligned}
\dot{E}(\tau) &= \frac{A^2}{2R}\left(\frac{\dot{g}+R}{r}\right)\left(2R - r\frac{(\dot{g}+R)}{r}\right) \\
&= \frac{A^2}{2R}\left(\frac{\dot{g}+R}{r}\right)(2R - \dot{g} - R) \\
&= \frac{A^2}{2R}\left(\frac{\dot{g}+R}{r}\right)(R - \dot{g}) = \frac{A^2}{2Rr}(R^2 - (\dot{g})^2).
\end{aligned}
$$

Thus we conclude that

$$\begin{aligned}\dot{E}(\tau) &= \frac{A^2}{2Rr}R^2 - \frac{1}{2r}\frac{A^2}{R}(\dot{g})^2 \\ &= \frac{A^2}{2Rr}R^2 + \frac{1}{2r}\ddot{E}\end{aligned}$$

so that

$$\ddot{E}(\tau) - 2r\dot{E}(\tau) + A^2R = 0.$$

Integrate once to reach

$$\dot{E}(\tau) - 2rE(\tau) + A^2R\tau + c_0 = 0$$

where c_0 is a constant of integration.

Then "reduced equation" is

$$\dot{E}(\tau) = 2rE(\tau)$$

so that

$$E(\tau) = c_1 e^{2r\tau},$$

c_1 is constant of integration.

The general solution is thus:

$$E(\tau) = c_0 + c_2\tau + c_1 e^{2r\tau}.$$

For a period of specialization, $E(0) = 0$ so that $c_1 + c_0 = 0$.

$$\dot{E}(\tau) = 2rc_1 e^{2r\tau} + c_2$$

so that at $\tau = 0$,

$$(2rc_1 e^{2r\tau} + c_2) - 2r[c_1 e^{2r\tau} + c_0 + c_2\tau] + A^2 R\tau + c_0 = 0.$$

Thus we conclude that

$$c_2 = \frac{A^2 R}{2r}$$

To this point, the equation looks like

$$E(\tau) = c_0(1 - e^{2r\tau}) + \frac{A^2 R}{2r}\tau.$$

Now there is no investment at the end of life.

$$\dot{E}(\tau) = 0.$$

Thus

$$\dot{E}(T) = 0 = -2rc_0 e^{2rT} + \frac{A^2 R}{2r}$$

so $c_0 = \frac{A^2 R}{(2r)^2} e^{-2rT}$. Thus

$$E(\tau) = \frac{A^2 R}{(2r)^2} e^{-2rT}(1 - e^{2r\tau}) + \frac{A^2 R}{2r}\tau.$$

Return to text

Appendix: Generalized Ben-Porath Model

Table 1. Estimates of the human capital production function (males)[a].

Source	α	β	γ	A	r	σ	Restricted schooling and OJT model?	Labor supply	Synthetic cohorts?
Heckman (1976) two models	0.99 (.003)	−6.69 (.043)	–	45.49 (3.034)	0.10 (imposed)	0.0016 (0.00025)	No	Yes	Yes
US Bureau of the Census (1960) males	0.67 (0.052)	0 (imposed)	–	0.14×10^{-2} (0.04×10^{-2})	0.10 (imposed)	0 (constrained)	No	Yes	Yes
Heckman (1976)	0.812 (0.0225)	α[b] (restricted)	–	1.53 (1.62)	0.176 (0.275)	0.089 (0.068)	No	No	Yes
US Bureau of the Census (1960) males	0.52 (0.07)	α[b] (restricted)	–	17.3 (25.2)	0.196 (0.613)	0.037 (0.90)	No	Yes	Yes
Haley (1976) CPS (1956–1966) aggregates	0.578 (0.012)	α[b] (restricted)	–	0.019–0.04	0.04–0.069 (0.004) (0.003)	0.005–0.04 (0.014) (0.008)	No[c]	No	Yes
Brown (1976)[d] NLS young men	0.56–0.89	α[b] (restricted)	–	[f]	0.33–0.15	0 (imposed)	No[e]	No	No
Rosen (1976) US Census 1960 and 1970	0.5	1 (imposed)	–	$r+\varepsilon$ ($\varepsilon > 0$) (see next column)	0.0725 (highschool) 0.0875 (college)	176 (0.275)	No	No	Yes

[a] $H_{t+1} = (1-\sigma)H_t + AI_t^{\alpha}H_t^{\beta}D_t^{\gamma}$.
r, interest rate; standard errors are given in parentheses.
[b] $\alpha = \beta$.
[c] All schooling groups.
[d] Brown makes alternative assumptions about the rate of growth of the price of labor services. See also Rosen.
[e] Only highschool graduates.
[f] Not reported.

Source: Browning, Hansen and Heckman (1999)

Table 2 (references cited)

- Heckman, J.J. (1976), “A life-cycle model of earnings, learning, and consumption”, *Journal of Political Economy* 84(4, pt. 2): S11–S44.
- US Bureau of the Census (1960), 1960 Census Public Use Sample (United States Government Printing Office, Washington, DC).
- Haley, W.J. (1976), “Estimation of the earnings profile from optimal human capital accumulation”, *Econometrica* 44: 1223–38.
- Brown, C. (1976), “A model of optimal human-capital accumulation and the wages of young high school graduates”, *Journal of Political Economy* 84(2): 299–316.
- Rosen, S. (1976), “A theory of life earnings”, *Journal of Political Economy* 84(Suppl.): 345–382.

Table 2. Estimated parameters for human capital production function.

Parameter	Estimated value			
	Males		Females	
	Highschool ($S=1$)	College ($S=2$)	Highschool ($S=1$)	College ($S=2$)
α	0.945(0.017)	0.939(0.026)	0.967	0.968
β	0.832(0.253)	0.871(0.343)	0.810	1.000
$A(1)$	0.081(0.045)	0.081(0.072)	0.079	0.057
$H_0(1)^*$	9.530(0.309)	13.622(0.977)	6.696	8.347
$A(2)$	0.085(0.053)	0.082(0.074)	0.082	0.057
$H_0(2)^*$	12.074(0.403)	14.759(0.931)	7.806	9.453
$A(3)$	0.087(0.056)	0.082(0.077)	0.084	0.058
$H_0(3)^*$	13.525(0.477)	15.614(0.909)	8.777	11.563
$A(4)$	0.086(0.054)	0.084(0.083)	0.086	0.058
$H_0(4)^*$	12.650(0.534)	18.429(1.095)	9.689	13.061

Source: Heckman, Lochner and Taber (1998)

Table 2 (notes)

- Human capital production function:
  $H_{a+1}^S = A^S(\theta)(I_a^S)^{\alpha_S}(H_a^S)^{\beta_S} + (1-\sigma)(H_a^S)^{\beta_S}$, with $S = 1, 2$.
  Standard errors are given in parentheses.
- Heckman, Lochner and Taber (1999) do not report the standard errors for females.
- Initial human capital for person of ability quantile using ability levels for NLSY.

- Convention: $H(t), I(t)$ written as I, H unless it clarifies matters not doing so.
- Consider a more general Ben-Porath Model

$$\dot{H} = AI^{\alpha}H^{\beta} - \sigma H$$

  Neutrality: $\alpha = \beta$.
- For simplicity assume no discounting $(r = 0)$
- No depreciation $\sigma = 0$
- Finite life $= T$
- Rental rate $= R$ (efficiency units, price of human capital)
- Initial endowment $= H_0$

- Problem ($0 \leq I \leq 1$; $0 < \alpha < 1$ for smooth problems):

$$\max \int_0^T [RH(t) - RI(t)H(t)]dt$$

such that $\dot{H} = AI^{\alpha}H^{\beta}$ and $H(0) = H_0$.

- Hamiltonian for problem:
Maximized Hamilton must be concave in state variable:

$$\mathcal{H} = RH(t)(1 - I(t)) + \mu(AI^{\alpha}H^{\beta})$$

$\beta \leq 1$ needed for Mangasarian sufficient conditions.

- FOC:

$$\mu A \alpha I^{\alpha-1} H^{\beta} \geq RH \qquad (*)$$

- Let "$\dot{}$" denote time rate of change.

$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial H} = -R(1 - I) - \beta \mu A I^{\alpha} H^{\beta-1}$$

- Rate of change of the shadow value of human capital declines with increases in the human capital stock.
- $\mu(T)H(T) = 0$ (transversality)

$$\mu(t) = \int_{t}^{T} \left[R(1 - I(u)) + \beta(\mu(u)) A I^{\alpha-1}(u) H^{\beta-1}(u) \right] du$$

- Now for the case with strict inequality in $(*)$, we have $I = 1$ (period of specialization associated with schooling or no earnings).

$$\begin{aligned} \alpha\mu A H^{\beta} &> RH \\ H^{\beta-1} &> \frac{R}{\alpha\mu A} \end{aligned}$$

- If $\beta > 1$, we get specialization in investment (no work) if

$$H > \left[\frac{R}{\alpha\mu A}\right]^{\frac{1}{\beta-1}}.$$

- Specialization at $t = 0$ requires

$$H_0 > \left[\frac{R}{\alpha\mu(0)A}\right]^{\frac{1}{\beta-1}} = \left(\frac{R}{\alpha A}\right)^{\frac{1}{\beta-1}} (\mu_0)^{\frac{1}{1-\beta}}$$

- If $\beta < 1$, specialization at $t = 0$ requires

$$H_0 < \left[\frac{R}{\alpha\mu(0)A}\right]^{\frac{1}{\beta-1}} = \left(\frac{R}{\alpha A}\right)^{\frac{1}{\beta-1}} (\mu_0)^{\frac{1}{1-\beta}} .$$

- When $\beta = 1$, specialization at $t = 0$ requires $\mu_0 > \dfrac{R}{\alpha A}$.

- Person just specializing ($I = 1$ is the interior solution) if

$$\alpha\mu A H^{\beta} = RH \qquad (I = 1)$$

$$\mu = \left(\frac{R}{\alpha A}\right) H^{1-\beta}$$

- In a period of specialization, $I = 1$

$$\dot{\mu} = -\beta\mu A H^{\beta-1}$$

$$\dot{H} = A H^{\beta}$$

- Then,

$$\frac{\dot{H}}{H^{\beta}} = A \qquad \text{or} \qquad \frac{dH}{H^{\beta}} = A dt$$

$$\frac{[H(t)]^{1-\beta}}{1-\beta} = At + c_0 \quad , \quad \beta \neq 1$$

$$H(t) = (At + c_0)^{\frac{1}{1-\beta}}(1-\beta)^{\frac{1}{1-\beta}}$$

(making t dependence explicit)

$$H(0) = H_0 = c_0^{\frac{1}{1-\beta}}(1-\beta)^{\frac{1}{1-\beta}}$$

$$\left(\frac{H_0}{(1-\beta)^{\frac{1}{1-\beta}}}\right)^{1-\beta} = c(0).$$

- When $\beta = 1$,

$$\ln H(t) = At + c_0$$
$$H(t) = e^{At + c_0}$$
$$H(0) = H_0 = e^{c_0} \qquad \ln H_0 = c_0.$$

- When $\beta \neq 1$,

$$\dot{\mu} = -\beta \mu A [H]^{\beta - 1} = -\beta \mu A \left[\frac{1}{(At + c_0)(1 - \beta)} \right]$$

$$\frac{\dot{\mu}}{\mu} = \frac{-\beta}{1 - \beta} \cdot \frac{A}{At + c_0} \qquad c_0 \geq 0$$

$$\ln \mu(t) = -\left(\frac{\beta}{1 - \beta} \right) \cdot \ln(At + c_0) + c_1$$

$$\mu(t) = e^{c_1} e^{-\frac{\beta}{1-\beta} \ln(At + c_0)} = \frac{e^{c_1}}{(At + c_0)^{\beta / 1 - \beta}}$$

- At $t = 0$,

$$\mu(0) = \frac{e^{c_1}}{c_0^{\beta/1-\beta}} = \frac{e^{c_1}}{\left(\frac{H(0)}{(1-\beta)^{\frac{1}{1-\beta}}}\right)} = \frac{e^{c_1}}{H(0)^{\beta}}(1-\beta)^{\beta/1-\beta}$$

$$\frac{\mu(0)H(0)^{\beta}}{(1-\beta)^{\beta/1-\beta}} = e^{c_1}$$

$$c_1 = \ln\left[\frac{\mu(0)[H(0)]^{\beta}}{(1-\beta)^{\beta/1-\beta}}\right].$$

- When $\beta = 1$,

$$\frac{\dot{\mu}}{\mu} = -A$$

$$\ln \mu(t) = -At + c_1^*$$

$$\mu(t) = e^{c_1^*} e^{-At}$$

$$\mu(0) = e^{c_1^*} \qquad \ln \mu(0) = c_1^*$$

- Now at end of period of specialization, we must have

$$\mu(t^*)A\alpha[H(t)]^{\beta} = RH(t).$$

- Thus for $\beta = 1$, specialization ends (schooling ends) when

$$\mu(t^*)A\alpha = R \qquad \mu(t^*) = \left(\frac{R}{A\alpha}\right)$$

$$e^{c_1^*}e^{-At^*} = \left(\frac{R}{A\alpha}\right) \qquad \frac{e^{c_1^*}A\alpha}{R} = e^{At^*}$$

$$c_1^* + \ln\left(\frac{A\alpha}{R}\right) = At^* \qquad \frac{1}{A}\left[c_1^* + \ln\left(\frac{A\alpha}{R}\right)\right] = t^*.$$

- When $\beta \neq 1$

$$\mu(t^*)\frac{A\alpha}{R} = [H(t^*)]^{1-\beta} \qquad H(t^*) = \left[\frac{\mu(t^*)A\alpha}{R}\right]^{\frac{1}{1-\beta}}.$$

- Substituting from above we find that we get

$$\frac{e^{c_1}}{(At+c_0)^{\beta/1-\beta}}\frac{A\alpha}{R} = (At+c_0)(1-\beta).$$

- The Ben Porath case is $\alpha = \beta$.
- Therefore, $\dot{\mu} = -R$ (trivial dynamics).

$$\mu(t) = -Rt + c_1, \quad \mu(T) = 0 \Rightarrow c_1 = RT$$
$$\mu(t) = -R(T - t)$$

- General case:

$$\dot{\mu} = R\left[-1 + \underbrace{R^{\frac{1}{\alpha-1}}\left(\frac{1}{A}\right)^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\mu^{\frac{1}{1-\alpha}}H^{\frac{\beta-1}{1-\alpha}}}_{I}(1-\beta/\alpha)\right]$$

$$= R[-1 + I \underbrace{(1-\beta/\alpha)}_{\text{adjustment to } I}]$$

- Therefore, we have that if $\beta/\alpha > 1$, $\dot{\mu} < 0$.
- If $\beta < 0$, $\dot{\mu}$ might be > 0.
- Assume for the moment that $\beta \geq 0$. Then what do we have?
- $\dot{\mu} = -R$ during period of specialization.

- At the end of the period of specialization (if one occurs), we have that

$$I = 1 = \left(\frac{R}{A\alpha}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}}. \qquad (**)$$

- Assume that $\dot{\mu} < 0$ for $0 < \alpha < 1$.
- As t increases, right hand side of $(**)$ decreases if $\dfrac{\beta-1}{1-\alpha} < 0$, i.e., $\beta < 1$.

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$$\frac{\beta}{1-\beta}+1=\frac{\beta+1-\beta}{1-\beta}=\frac{1}{1-\beta}$$

$$\left[\frac{e^{c_1}}{1-\beta}\frac{A\alpha}{R}\right]=(At+c_0)^{\frac{1}{1-\beta}}$$

$$\frac{1}{A}\left[\frac{e^{c_1}}{1-\beta}\frac{A\alpha}{R}\right]^{1-\beta}-c_0=t^*$$

- End of first specialization period. (This is associated with schooling.)

- Question: Is there more than one period of specialization?
- Look ahead to interior segment. In the interior we get:

$$\mu A \alpha I^{\alpha-1} = RH^{1-\beta}$$

$$I^{\alpha-1} = \left(\frac{R}{\mu A \alpha}\right) H^{1-\beta}$$

$$I = \left(\frac{R}{\mu A \alpha}\right)^{\frac{1}{\alpha-1}} H^{\frac{1-\beta}{\alpha-1}} = \left(\frac{R}{A \alpha}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}}$$

- Substitute into costate (shadow price) equation:

$$\dot{\mu} = -R(1 - I) - \beta\mu A I^{\alpha} H^{\beta-1}$$

$$= -(R) + R^{\frac{\alpha}{\alpha-1}} \left(\frac{1}{A}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\frac{\beta-1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}} (\alpha - \beta)$$

  for $\beta > 0$, $\dot{\mu} < 0$.

- Then $I \downarrow$ monotonically over the life cycle when $\beta < 1$.
- $\beta = 1$, obviously $\mu(t) \downarrow \Rightarrow I(t) \downarrow$ monotonically.
- Therefore, we have at most one period of specialization, and it is early on (beginning of life).

- Take $\beta \neq 1$. For a person who specializes, the lifecycle is as follows:
  - $[0, t^*]$ school
  - $[t^*, T]$ work
- Then we solve from t^* on

$$\dot{\mu} = -R + R^{\alpha/\alpha-1}\left(\frac{1}{A}\right)^{\frac{1}{\alpha-1}} \mu^{\frac{1}{1-\alpha}} H^{\beta-1/1-\alpha}(\alpha)^{\alpha/1-\alpha}(\alpha-\beta)$$

$$\dot{H} = A\left[\frac{R}{A\alpha}\right]^{\frac{\alpha}{\alpha-1}} \mu^{\frac{\alpha}{\alpha-1}} H^{\frac{\alpha(\beta-1)}{1-\alpha}} H^{\beta}$$

$$= A\left[\frac{R}{A\alpha}\right]^{\frac{\alpha}{\alpha-1}} \mu^{\frac{\alpha}{1-\alpha}} H^{(\beta-\alpha)/(1-\alpha)}$$

for (μ, H) jointly. (This is a "split endpoint" problem.)

$$\mu(t) = \int_t^T \dot{\mu}(t)dt + c(3)$$

- Impose condition that $\mu(T) = 0$ for $t > t^* \Rightarrow c(3) = 0$.

$$H(t) = \int_{t^*}^{t} \dot{H}(\tau)d\tau + H(t^*)$$

- $\mu(t)$ and $H(t)$ must be solved jointly.
- Substitute for (t^*) above and enforce condition on $\mu(0)$. (Thus, $H(t^*)$ depends on $\mu(0)$ and $H(0)$, but $\mu(0)$ set in conjunction with $\mu(T) = 0$.)
- We know $\mu(t), H(t)$ and $I(t)$ continuous.
- $\dot{\mu}(t)$ need not be continuous at t^*.

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