Notes on "Differential Rents and the Distribution of Earnings"

from Sattinger, *Oxford Economic Papers* 1979, 31(1)

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- This is a version of an hedonic model.
- It features 1-1 matches.
- Assume that we can rank workers and firms by a skill scale: ℓ is amount of labor skill, c is amount of capital owned by firm.
- $F(\ell, c)$ is output. Assume a common production technology. One worker – one firm match $F_\ell > 0$, $F_c > 0$, $F_{\ell\ell} < 0$, $F_{cc} < 0$, no need to make scale restrictions.

- Can be increasing returns to scale technologies.
- Homogeneous output of firms, identical technologies.
- Let $G(\ell)$ be cdf of ℓ in population. Let $K(c)$ be cdf of c in population. Assume both monotone strictly increasing, density has positive support – no mass points.
- Let $W(\ell)$ be wage for worker of type ℓ.
- Let $\pi(c)$ denote "profit" for a firm of type c.

- Assume $\dfrac{\partial^2 F}{\partial \ell \partial c} > 0$ (opposite sign produces negative sorting).
- Assume wage function exists.
- This is something to be proved.
- Firm indexed by c.
- Profit maximization requires that

$$\max_{\ell}(F(\ell, c) - W(\ell))$$

$$\text{FOC: } \frac{\partial F}{\partial \ell} = W'(\ell) \qquad \text{SOC: } \frac{\partial^2 F}{\partial \ell^2} - W''(\ell) < 0$$

- Defines demand for worker of type ℓ for firm type c.

- Differentiate FOC totally with respect to ℓ:

$$W''(\ell) - \frac{\partial^2 F(\ell, c)}{\partial \ell^2} - \frac{\partial^2 F}{\partial \ell \partial c} \frac{dc}{d\ell} = 0$$

$$\underbrace{\left(W''(\ell) - \frac{\partial^2 F(\ell, c)}{\partial \ell^2} \right)}_{>0, \text{ from SOC}} = \underbrace{\left(\frac{\partial^2 F}{\partial \ell \partial c} \right)}_{+} \frac{dc}{d\ell} \tag{1}$$

- $\therefore \frac{dc}{d\ell} > 0$ ("best firms match with best workers")

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- Opposite true if we have $\dfrac{\partial^2 F}{\partial \ell \partial c} < 0$ $(dc/dl < 0)$.
- Retain $\dfrac{\partial^2 F}{\partial \ell \partial c} > 0$ for specificity.
- Profits residually determined:

$$\pi(c) = F(\ell(c), c) - W(\ell(c)).$$

- Observe that the roles of ℓ and c can be reversed (labor hires capital) and labor incomes could be residually determined.

- The continuum hypothesis for skills $\Longrightarrow$ local returns to scale

$$dF = F_\ell d\ell + F_c dc$$

- $\therefore$ we get product exhaustion locally.
- Residual claimant gets marginal product, no matter who is claimant.
- Now suppose number of workers (N_ℓ).
- Number of capitalists (N_c).

- Let W_R be the reserve price of workers (what they could get not working in the sector being studied). Let π_R be reserve price of capitalist. Let ℓ^* be the least productive worker (employed). We need $W(\ell^*) \geq W_R$.
- If all capital employed, and $c \in [\underline{c}, \bar{c}]$, ℓ^* works with $\underbrace{\underline{c},}_{\text{least productive capitalist}}$ assuming that $\pi(\underline{c}) \geq \pi_R$.

- How to establish that decentralized wage setting is optimal and a wage function exists?
- Solve Social Planner's Problem.

$$\frac{\partial^2 F(\ell, c)}{\partial \ell \partial c} > 0 \Rightarrow$$

maximize total output by matching the best with the best.

Proof: trivial based on proof by contradiction

Take a discrete example

$$\begin{aligned} \text{two workers} \qquad & \ell_1 < \ell_2 \\ \text{two firms} \qquad & c_1 < c_2 \end{aligned}$$

From complementarity (or supermodularity)

$$F(\ell_2, c_2) + F(\ell_1, c_1) > F(\ell_2, c_1) + F(\ell_1, c_2)$$

because

$$F(\ell_2, c_2) - F(\ell_1, c_2) > F(\ell_2, c_1) - F(\ell_1, c_1)$$

due to

$$\frac{\partial^2 F(\ell, c)}{\partial \ell \, \partial c} > 0.$$

- Using the fact that the best matches with the best, sort top-down.
- Assume densities "continuous" (absolutely continuous).

$$
\begin{aligned}
N_\ell \int_{\ell(c)}^{\infty} g(\ell)\, d\ell &= N_c \int_{c}^{\infty} k(c)\, dc \\
N_\ell \left(1 - G(\ell(c))\right) &= N_c \left(1 - K(c)\right) \\
\left(1 - G(\ell(c))\right) &= \left(\frac{N_c}{N_\ell}\right)\left(1 - K(c)\right) \\
G^{-1}\left[1 - \left(\frac{N_c}{N_\ell}\right)\left(1 - K(c)\right)\right] &= \ell(c)
\end{aligned}
$$

- This defines the optimal sorting function.

Use survivor function:

$$S(x) = \Pr[X \geq x]$$

$$S_G(\ell) = 1 - G(\ell)$$
$$S_K(c) = 1 - K(c)$$

$$S_G(\ell(c)) = \left(\frac{N_c}{N_\ell}\right) S_K(c)$$
$$\ell(c) = S_G^{-1}\left(\frac{N_c}{N_\ell} S_K(c)\right)$$

- Defines a relationship:

$$\ell = \varphi(c) \quad \text{(most productive match with each order)}$$

This function has an inverse from strictly decreasing survivor function assumption (density has no mass points or holes).

- Feasibility requires, using $\varphi^{-1}(\ell) = c$, that the lowest quality capitalist cover his/her reserve income outside the sector

$$\pi(\underline{c}) = F(\ell(\underline{c}), \underline{c}) - W(\ell^*) \geq \pi_R.$$

- If not satisfied we have unemployed capital.
- Jack up $c^* > \underline{c}$ until constraint satisfied.

- From the allocation derived from the social planner's problem, we can derive the hedonic equation (instead of assuming it).
- The slope of the wage function is given by FOC (using $c = \varphi^{-1}|l$)

$$W'(\ell) = \frac{\partial F}{\partial \ell}(\ell, \varphi^{-1}(\ell))$$

  (the right-hand side determined by the equilibrium sorting).
- This defines the slope of hedonic line with a continuum of labor.

- Note that if we totally differentiate the right-hand side,

$$W''(\ell) = \underset{<0}{F_{\ell\ell}} + \underset{+}{F_{\ell c}} \underset{+}{\frac{dc}{d\ell}}$$

$\therefore$ SOC satisfied, because $W''(\ell) - F_{\ell\ell} \geq 0$ as required.

- The marginal wage at minimum quality ℓ^* satisfies

$$W'(\ell_*) = \frac{\partial F}{\partial \ell}(\ell^*, \varphi^{-1}(\ell^*)).$$

- Competitive labor market forces $W(\ell_*) = W_R$.
- You cannot pay any less than reserve wage.
- If you pay more, all workers from the "reserve" will want to work in the sector being studied and hence it forces wages down.

$$W(\ell) = \int_{\ell^*}^{\ell} \frac{\partial F}{\partial x}(x, \varphi^{-1}(x))dx + W_R.$$

"hedonic function"

- Similarly

$$\pi(c) = \int_{c_*}^{c} \frac{dF}{dz}(\varphi(z), z)dz + \pi_R.$$

(Reserve value of capital is nonnegative; $\pi_R \geq 0$.)

- Under our assumptions (more workers than firms and unemployed worker, $N_c > N_\ell$), rents are assigned to firms.
- Density of earnings is obtained from inverting wage function

$$w(\ell) = \eta(\ell) \qquad \eta^{-1}(w) = \ell \text{ (exists under our assumptions)}$$

- Density of earnings is

$$g(\eta^{-1}(w))\frac{d\eta^{-1}(w)}{dw}$$

Density of profits obtained in a similar way.

Cobb Douglas Example

- $F(\ell, c) = \ell^{\alpha} c^{\beta}$, $\alpha > 0,\ \beta > 0$.
- Assume Pareto distribution of endowments:

$$\begin{aligned} g(\ell) &= j\ell^{-\gamma} \qquad \gamma > 2, \qquad \ell \geq 1 \\ k(c) &= hc^{-\sigma} \qquad \sigma > 2, \qquad c \geq 1. \end{aligned}$$

- This ensures finite variances. Obviously $F_{\ell c} > 0$.
- The higher γ, the more equal is the distribution of ℓ.
- The higher σ, the more equal is the distribution of c.

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- Equilibrium:

$$N_c \int_{c(\ell)}^{\infty} hx^{-\sigma} dx = N_\ell \int_{\ell}^{\infty} j\eta^{-\gamma} d\eta$$

$$c(\ell) = \left[\frac{N_\ell j}{N_c h} \frac{(\sigma - 1)}{(\gamma - 1)} \right]^{\frac{1}{1-\sigma}} (\ell)^{\frac{1-\gamma}{1-\sigma}}.$$

- FOC (for wages) $\alpha \ell^{\alpha-1} c^{\beta} = W'(\ell)$.
- Substitute for $c(\ell)$ to reach

$$\therefore \; W'(\ell) = \alpha \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{\beta}{1-\sigma}} \ell^{P}$$

$$P = \frac{(\alpha - 1)(1 - \sigma) \; + \beta(1 - \gamma)}{1 - \sigma} \gtrless 0$$

$$W(\ell) = \underbrace{\left[\frac{\alpha \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)} \right]^{\frac{\beta}{1-\sigma}}}{\left(\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma} \right)} \right]}_{g_1} \cdot (\ell)^{\left(\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{(1-\sigma)} \right)} + k_1,$$

and where k_1 is a constant of integration, determined by $W_R : \; W(\ell^*) \geq W_R$.

- Obviously $W(\ell) \uparrow$ as $\ell \uparrow$. Convexity or concavity in labor quality hinges on whether

$$P \lesseqgtr 0$$
$$P = (\alpha - 1) + \beta \frac{(1-\gamma)}{1-\sigma}.$$

- If $\alpha + \beta = 1$ (CRS)

$$P = \beta \left[-1 + \frac{1-\gamma}{1-\sigma} \right]$$
$$= \beta \left[\frac{\sigma - \gamma}{1-\sigma} \right] = \beta \left[\frac{\gamma - \sigma}{\sigma - 1} \right]$$

- Convexity or concavity of wage function depends on P.
- If $\gamma > \sigma$, $W(\ell)$ is convex in ℓ. (More firms out in tail than workers – workers get scarcity payment).
- Firms less equally distributed (more "productive" firms out in tail).
- If $\beta \uparrow$ (from CRS) reinforces effect (Renders capital relatively more productive).

- If $\gamma = \sigma$ and $\beta + \alpha > 1$ (β big enough), $P > 0$ and hence produces convexity.
- Increasing returns to scale gives rise to convexity (**scale of productivity of resources effect**).

- Profits can be written as

$$\pi(c) = \ell^{\alpha} c^{\beta} - w(\ell)$$

- From the equilibrium matching condition we obtain

$$\ell = g_0(c)^{\frac{1-\sigma}{1-\gamma}} \qquad g_0 = \left[\frac{N_c h(\gamma - 1)}{N_\ell j(\sigma - 1)}\right]^{\frac{1}{1-\gamma}}$$

$$\pi(c) = \left[g_0(c)^{\frac{(1-\sigma)}{(1-\gamma)}}\right]^{\alpha} c^{\beta} - g_1 \left(g_0(c)^{\frac{(1-\sigma)}{(1-\gamma)}}\right)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}} - k_1$$

$$\frac{\alpha(1-\sigma)}{1-\gamma} + \beta = \frac{\alpha(1-\sigma) + \beta(1-\gamma)}{1-\gamma}$$

$$\pi(c) = \left[g_0^{\alpha} - g_1(g_0)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}}\right] \cdot c^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}} - k_1$$

- For positive marginal productivity of capital, this requires that

$$\alpha + \frac{\beta(\gamma-1)}{\sigma-1} > \left[\frac{N_c h(\gamma-1)}{N_\ell j(\sigma-1)}\right]^{\frac{\gamma(\beta-1)}{(\sigma-1)(\gamma-1)}}$$

- Otherwise, coefficient on $c^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}}$ is negative.

$$\pi(c) = ac^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}} - k_2$$

$$a = (g_0)^{\alpha} - g_1(g_0)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}} > 0$$

(True if $N_c \gg N_\ell$, for example.)

- $\therefore$ convexity of $\pi(c)$ is determined by sign of

$$
\begin{aligned}
&\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\gamma}-1 \\
&=\frac{\alpha(1-\sigma)+(\beta-1)(1-\gamma)-1+\gamma}{1-\gamma} \\
&=\frac{(\gamma-1)(\beta-1)+(\sigma-1)\alpha}{\gamma-1} \\
&=(\beta-1)+\left(\frac{\sigma-1}{\gamma-1}\right)\alpha .
\end{aligned}
$$

- Observe if $\alpha+\beta>1$ then both $\pi(c)$ and $W(\ell)$ can be convex in their arguments. With CRS one must be concave, the other convex.
- Linearity arises when we have $\gamma=\sigma$ and $\alpha+\beta=1$.

- γ big relative to σ (scarcity of labor at top firms (high c firms)).
- α, β big – scale effects – we get convexity at top of distribution.
- Suppose we invoke full employment conditions for capital:

$$N_\ell > N_c \qquad \pi(1) \geq \pi_R$$

- We need to determine the constants for the wage equation.
- Minimum quality labor earns its opportunity cost outside of the sector.
- Rents accrue to other workers.

At lowest level of employment, we have (from matching function $c(\ell)$)

$$1 = \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)}\right]^{\frac{1}{1-\sigma}} (\ell^*)^{\frac{1-\gamma}{1-\sigma}}$$

$$\therefore\ \ell^* = \left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)}\right]^{\frac{1}{\gamma-1}}$$

$$W(\ell^*) = W_R$$

$\therefore k_1 =$

$$W_R - \frac{\alpha(1-\sigma)}{\alpha(1-\sigma)+\beta(1-\gamma)}\left[\frac{N_\ell j(\sigma - 1)}{N_c h(\gamma - 1)}\right]^{\frac{\beta}{1-\sigma}} (\ell^*)^{\frac{\alpha(1-\sigma)+\beta(1-\gamma)}{1-\sigma}}.$$

$\pi(c)$ defined residually. (Need to check $\pi(1) > \pi_R$).

- Pigou's Problem: Why doesn't the distribution of earnings resemble the distribution of ability?
- Distribution of earnings: (generated from distribution of endowments by the pricing function).
- Look at distribution of translated earnings (translated around the constant k_1).

$$(W(\ell) - k_1) \sim (W - k_1)^{-\left[1+\frac{(\gamma-1)(\sigma-1)}{\alpha(\sigma-1)+\beta(\gamma-1)}\right]}$$

Distribution of raw skills $\sim \ell^{-\gamma}$.

- Higher γ is associated with more equality in the distribution of labor skills.

- One way to measure the market-induced change in inequality is the change in the wage distribution from γ.
- Example:

$$1 + \frac{(\gamma - 1)(\sigma - 1)}{\alpha(\sigma - 1) + \beta(\gamma - 1)} < \gamma$$

(wage inequality $>$ inequality in ℓ)

- For this to happen,

$$\frac{1}{\alpha + \beta \dfrac{(\gamma - 1)}{(\sigma - 1)}} < 1$$

- The higher $\alpha + \beta$, the more unequal the distribution of wages.
- Higher $\gamma > \sigma$ (capital more unequally distributed) the greater the wage inequality.

- If $\gamma = \sigma$, $\alpha + \beta = 1$, no induced change in inequality.
- If $\gamma = \sigma$, $\alpha + \beta > 1$, *more* inequality in wages than skills.
- If $\sigma \ll \gamma$, then more inequality in wages than skills (Demand for top talent).
- It is not “superstars” but “superfirms”.

- The wage equation is an hedonic function.
- Hedonic Functions (Tinbergen, 1951, 1956; Rosen, 1974). What can you estimate when you regress W on ℓ? Obviously we can estimate k_1,

$$\frac{\alpha(\sigma-1)+\beta(\gamma-1)}{(\sigma-1)}$$

and slope coefficient (g_1).
- Do not recover any single parameter of interest. We get lowest ℓ in market and from distribution of ℓ and c, we can get γ, σ, h (if c fully employed).
- If we assume $\alpha+\beta=1$ (CRS) and we observe distributions of the factors, we get σ, γ and hence α, β.

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- If we know ℓ^*, we can get j.
- If we know N_ℓ and N_c, we can identify γ, σ but α, β are unknown.
- $\alpha + \beta$ is known.
- CRS $\Rightarrow \alpha$, β known.

Identify the Technology

- Idea (Rosen, 1974). Two-stage estimation procedure. Assume perfect data.
- Assume $\alpha \neq 1$.
- No error term in model, no omitted variables.
- Use FOC for firm,

$$\ln \alpha + (\alpha - 1) \ln \ell + \beta \ln c = \ln W'(\ell)$$

i.e.,

$$\ln \ell = -\frac{\ln \alpha}{\alpha - 1} + \frac{\ln W'(\ell)}{\alpha - 1} - \frac{\beta \ln c}{\alpha - 1}.$$

- Apparently, we can regress $\ln \ell$ on $\ln W'(\ell)$.
- Notice however that from the sorting condition,

$$\ln \ell = \ln g_0 + \left(\frac{\sigma - 1}{\gamma - 1} \right) \ln c.$$

- We get no independent variation. $\ln W'(\ell)$ is redundant.
- Alternatively, $\ln W'(\ell)$ and $\ln c$ are perfectly collinear.

- More general principle:

$$\text{FOC:} \quad \frac{\partial^2 F}{\partial \ell^2} d\ell + \frac{\partial^2 F}{\partial \ell \partial c} dc = dW'(\ell)$$

$$d\ell = \frac{1}{\left(\frac{\partial^2 F}{\partial \ell^2}\right)} d[W'(\ell)] - \frac{\frac{\partial^2 F}{\partial \ell \partial c}}{\frac{\partial^2 F}{\partial \ell^2}} dc.$$

- Functional dependence between c and $W'(\ell)$ does not necessarily imply linear dependence.

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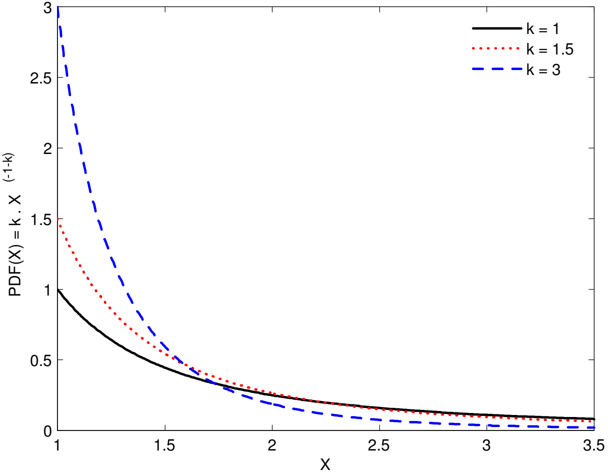
- $\therefore$ we might be able to identify the model.
- Need shifter in regression.
- Functional dependence $\not\Rightarrow$ linear independence

$$y = \alpha_0 + \alpha_1 X + \alpha_2 X^2.$$

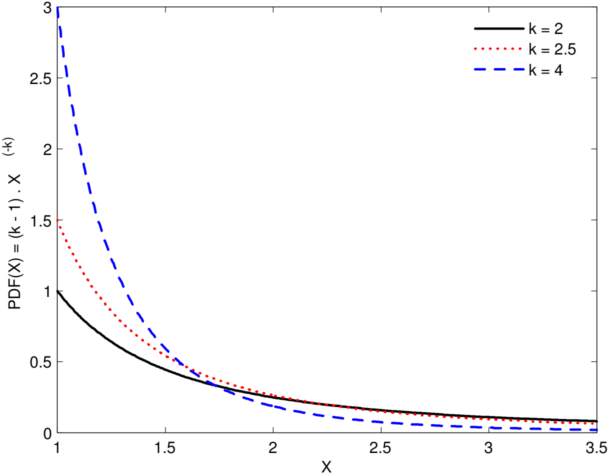
- Obviously X and X^2 only dependent but not linearly dependent.
- We return to this in a bit.

Pareto Distribution



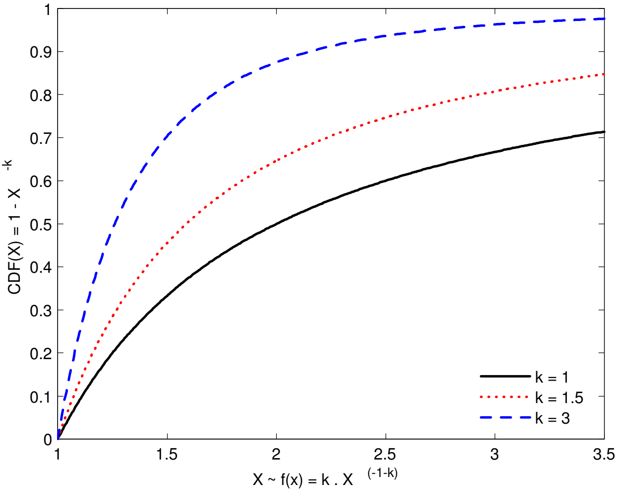
$$X \sim \text{Pareto}(k) \rightarrow f_X(x) = k \cdot x^{-(1+k)}$$

Pareto Distribution



$$X \sim \text{Pareto}(k) \rightarrow f_X(x) = (k-1) \cdot x^{-k}$$

Pareto Distribution



$$X \sim \text{Pareto}(k) \rightarrow F_X(x) = 1 - x^{-k}$$

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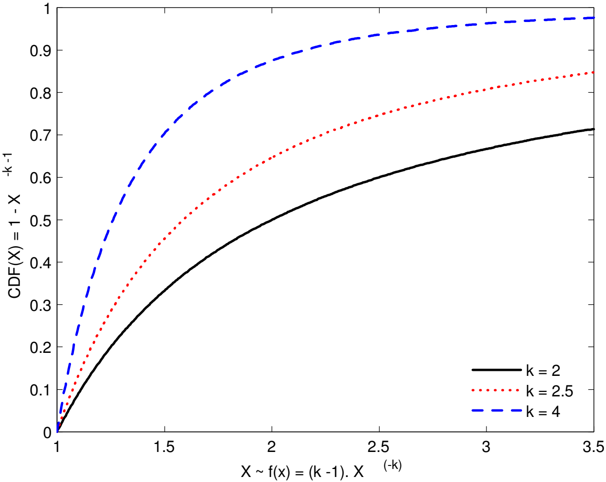
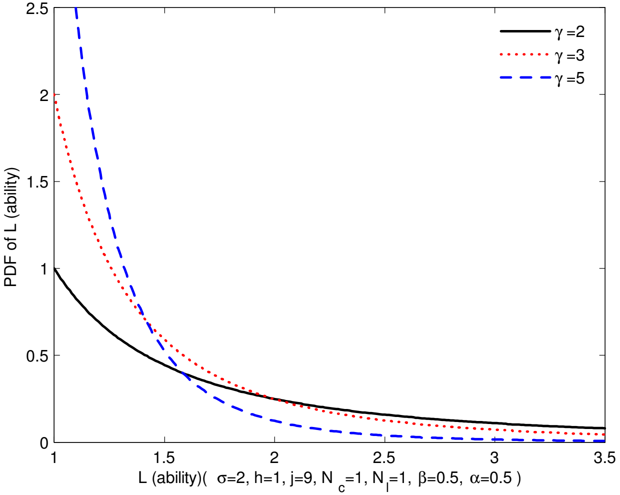
Pareto Distribution

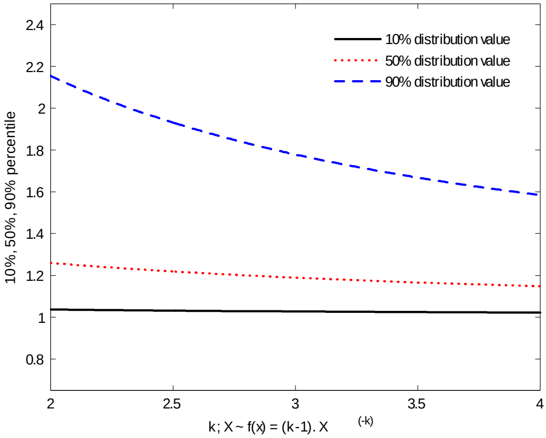


$$X \sim \text{Pareto}(k) \rightarrow F_X(x) = 1 - x^{-(k+1)}$$

Ability Distributions

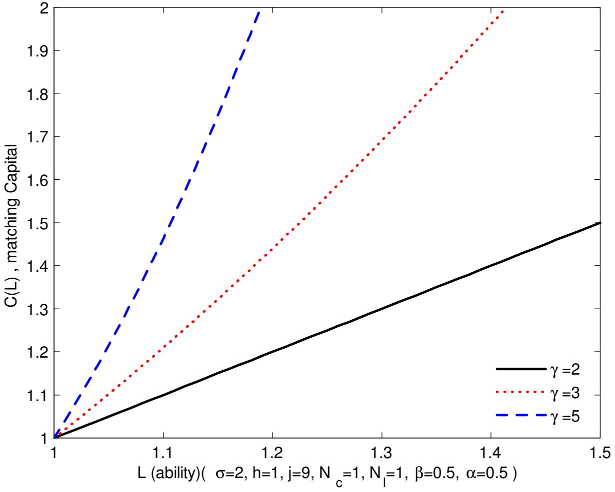


Pareto Percentiles

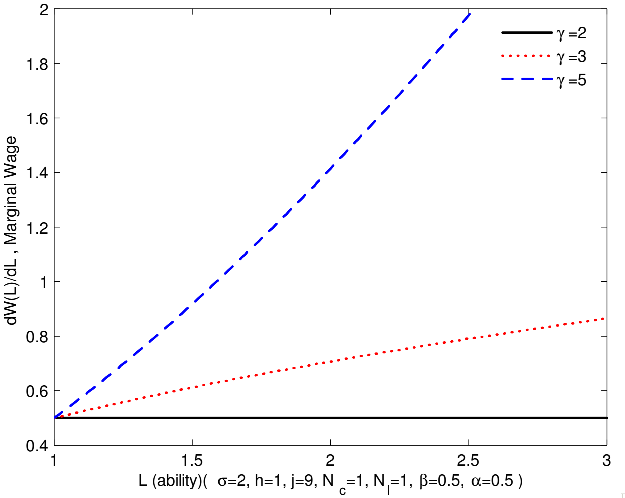


Pareto 10%, 50%, 90% Percentiles for $k \in [2, 4]$

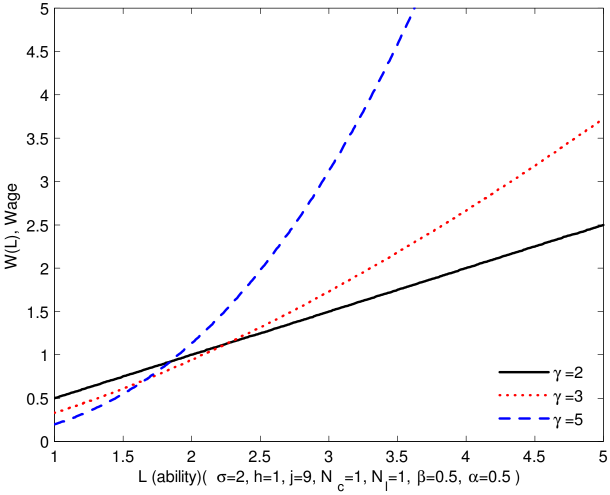
Capital/ability relation



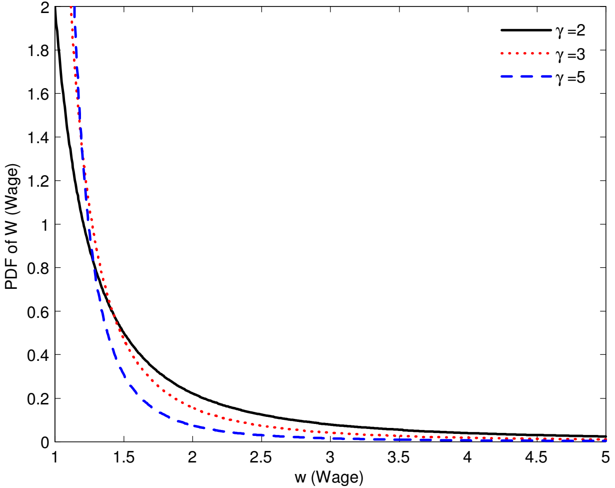
Wage derivative with respect to ability $\dfrac{\partial W(L)}{\partial L}$



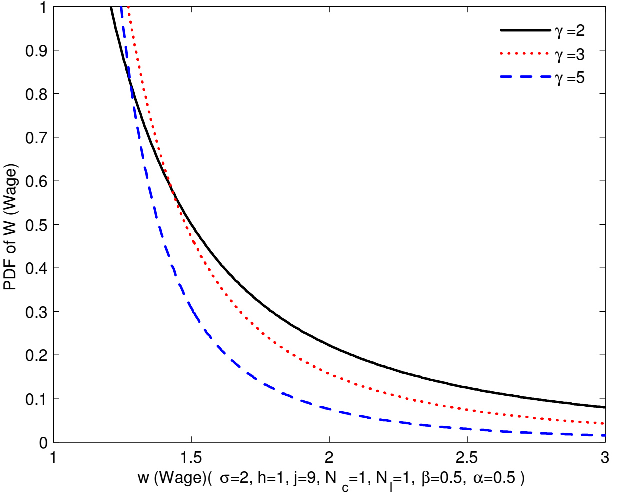
Wage as a function of ability



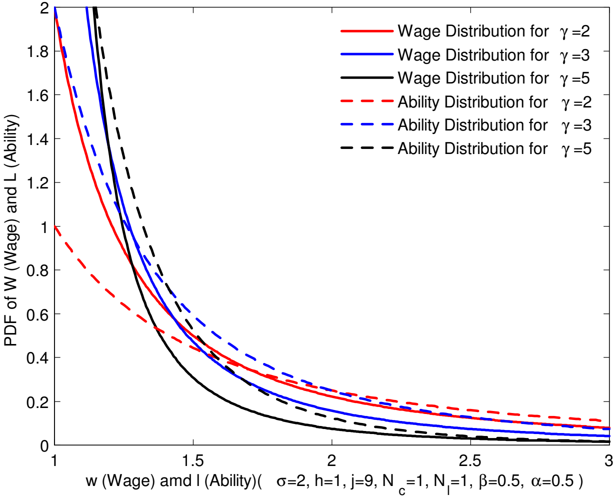
Wage distribution



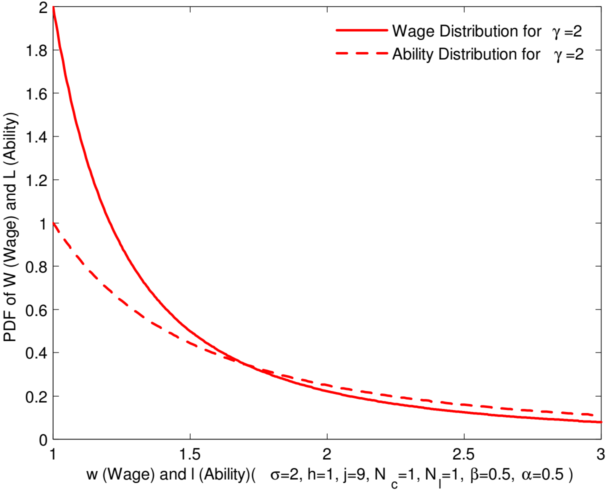
Wage distribution



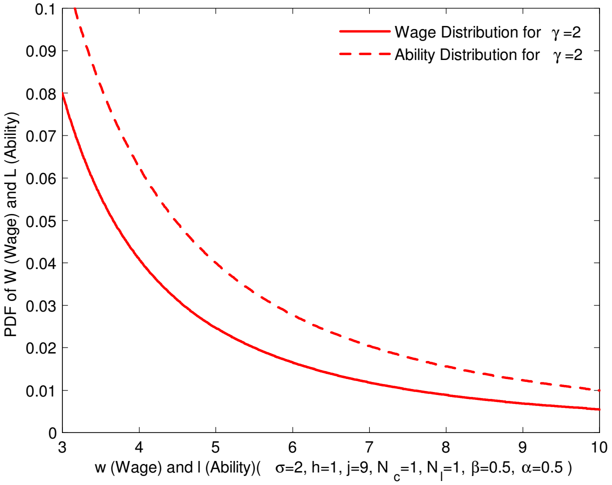
Wage and ability distribution



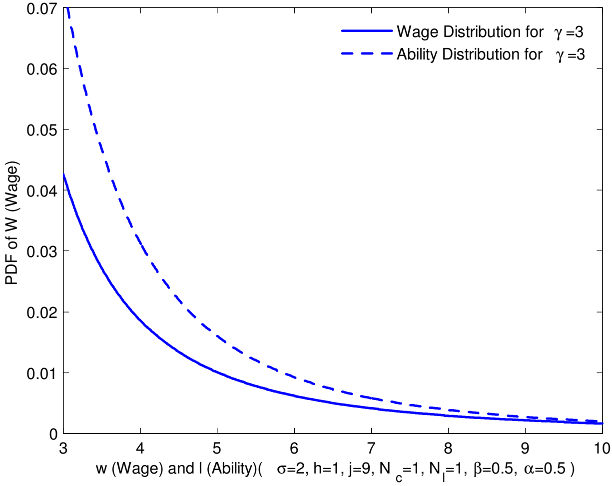
Wage and ability distribution



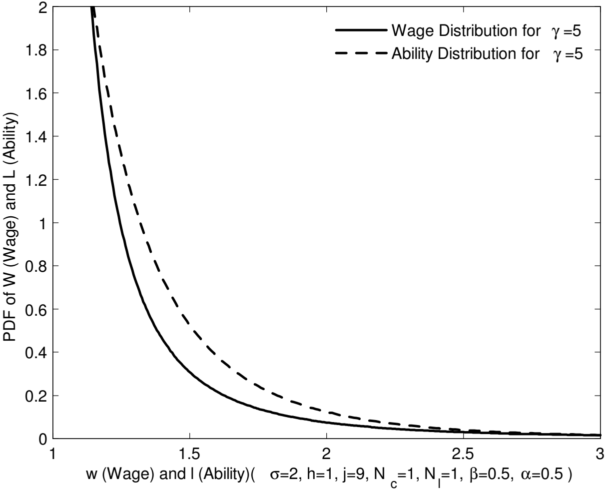
Wage and ability distribution



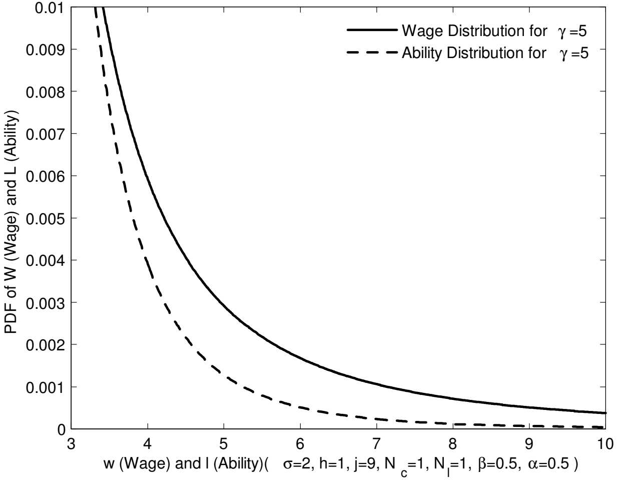
Wage and ability distribution



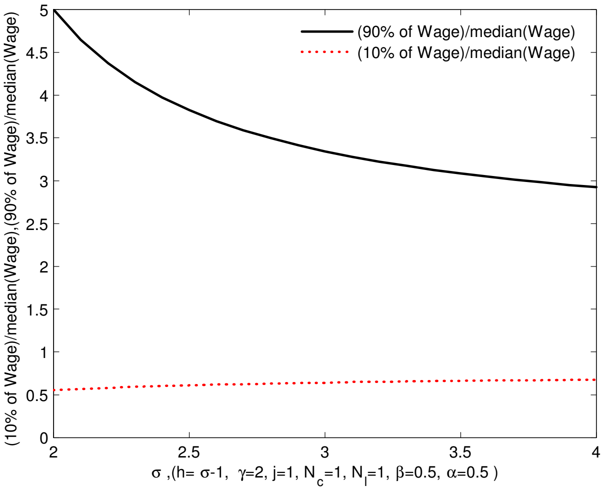
Wage and ability distribution



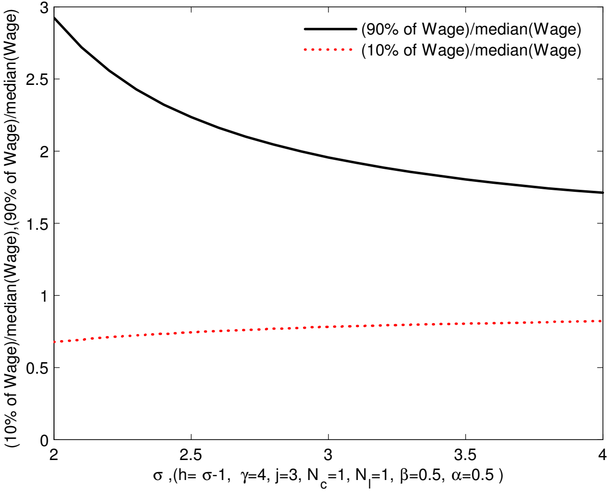
Wage and ability distribution



Wage percentile ratios



Wage percentile ratios



Wage percentile ratios