Ordering Marshallian, Hicks, Frisch Responses to a Price Change

James J. Heckman and David Malison
University of Chicago

Econ 350, Winter 2023

Consider the canonical consumer problem

$$\max_{\mathbf{x}\in\mathbb{R}_+^N} U(\mathbf{x}) \quad \text{subject to } A \geq \mathbf{p}\cdot\mathbf{x} \tag{1}$$

where U is the consumer's utility function defined over consumption bundles $\mathbf{x} = (x_1, \cdots, x_N)'$, A is the consumer's income, $\mathbf{p} = (p_1, \cdots, p_N)$ is a vector of prices, and $\mathbf{p}\cdot\mathbf{x} = \sum_{i=1}^N p_i x_i$ is the consumer's total spending on goods.

- The Lagrangian for this problem is given by

$$\mathcal{L} = U(\mathbf{x}) + \lambda \left(A - \mathbf{p} \cdot \mathbf{x} \right)$$

- The first order-conditions are

$$U_i \leq \lambda p_i \;\text{ with equality if } x_i > 0, \quad i = 1, \cdots, N \tag{2}$$

$$A \geq \mathbf{p} \cdot \mathbf{x}, \quad \lambda \geq 0, \quad (A - \mathbf{p} \cdot \mathbf{x})\lambda = 0 \tag{3}$$

where $U_i = \frac{\partial U}{\partial X_i}$. If $U_i \geq 0$ for all i with at least one strict inequality, then $\lambda > 0$ and equation (3) reduces to

$$A = \mathbf{p} \cdot \mathbf{x} \tag{3'}$$

- If in addition U is continuously differentiable and quasiconcave, these conditions are necessary and sufficient for a solution by the Kuhn-Tucker Theorem.
- That is, any solution to (1) must solve (2) and (3′) and any solution to (2) and (3′) must be as solution to (1).
- Suppose we are at an interior solution so that all the equations in (2) hold with equality.
- Furthermore, assume that U is strictly quasiconcave so that the solution is unique.
- Then we can totally differentiate the system in (2) and (3′):

$$\sum_{j=1}^{N} U_{ij} dx_j = \lambda dp_i + p_i d\lambda$$

$$dA = \sum^{N} (p_i dx_i + x_i dp_i)$$

Place the $N+1$ equations into a matrix:

$$\underbrace{\begin{pmatrix} U_{11} & \cdots & U_{1N} & -p_1 \\ U_{21} & \cdots & U_{2N} & -p_2 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1} & \cdots & U_{NN} & -p_N \\ -p_1 & \cdots & -p_N & 0 \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_N \\ d\lambda \end{pmatrix} = \begin{pmatrix} \lambda dp_1 \\ \vdots \\ \lambda dp_N \\ -dA + \sum_i x_i dp_i \end{pmatrix} \tag{4}$$

Using (2), the matrix $\mathbf{J}$ can be rewritten as

$$\mathbf{J} = \begin{pmatrix} U_{11} & \cdots & U_{1N} & -\frac{U_1}{\lambda} \\ U_{21} & \cdots & U_{2N} & -\frac{U_2}{\lambda} \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1} & \cdots & U_{NN} & -\frac{U_N}{\lambda} \\ -\frac{U_1}{\lambda} & \cdots & -\frac{U_N}{\lambda} & 0 \end{pmatrix}$$

The determinant of $\mathbf{J}$ is given by

$$|\mathbf{J}| = \frac{1}{\lambda^2} \underbrace{\begin{vmatrix} U_{11} & \cdots & U_{1N} & U_1 \\ U_{21} & \cdots & U_{2N} & U_2 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1} & \cdots & U_{NN} & U_N \\ U_1 & \cdots & U_N & 0 \end{vmatrix}}_{|\mathbf{BH}|}$$

- **BH** is the bordered Hessian of the utility function.
- A sufficient condition for strict quasiconcavity of U is that the border-preserving principle minors of **BH** alternate in sign, starting with a negative value.
- This condition also guarantees that the system of equations in (4) has a unique solution. We will maintain this assumption for the remainder of this note.

Large Comparative Statics

Change in income

- Suppose that $dA \neq 0$ and $dp_i = 0$ for all i.
- That is, we first consider the consumer's response to a marginal change in income holding prices fixed.
- In this case, the system in (4) reduces to

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_N \\ d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -dA \end{pmatrix}$$

- Using Cramer's Rule, we can solve for dx_j as

$$dx_j = \frac{\begin{vmatrix} U_{11} & \dots & \overbrace{0}^{\text{Column } j} & \dots & U_{N1} & -p_1 \\ U_{21} & \dots & 0 & \dots & U_{N2} & -p_2 \\ \vdots & & & & & \\ U_{N1} & \dots & 0 & \dots & U_{NN} & -p_N \\ -p_1 & \dots & -dA & \dots & p_N & 0 \end{vmatrix}}{|\mathbf{J}|} = \frac{J_{N+1,j}}{|\mathbf{J}|}(-dA)$$

where $J_{N+1,j}$ is the $(N+1, j)^{th}$ cofactor of the matrix $\mathbf{J}$.

- This implies

$$\frac{dx_j}{dA} = -\frac{J_{N+1,j}}{|\mathbf{J}|} \tag{5}$$

- If this term is positive (negative), x_j is a normal (inferior) good at the given prices and income level.

Uncompensated (Marshallian) change in price

- Now suppose that $dp_i \neq 0$, $dp_j = 0$ for $j \neq i$, and $dA = 0$.
- That is, we want to consider the consumer's response to a *ceteris paribus* change in the price of good i.
- The system in(4) reduces to

$$\mathbf{J} \begin{pmatrix} dx_1^M \\ \vdots \\ dx_N^M \\ d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda dp_i \\ 0 \\ \vdots \\ 0 \\ x_i dp_i \end{pmatrix}$$

where the superscript M is used to denote that the price change is uncompensated (Marshallian).

- Again appealing to Cramer's Rule, we have

$$dx_j^M = \frac{\begin{vmatrix} U_{11} & \cdots & \overbrace{0}^{\text{Column } j} & \cdots & U_{1N} & -p_1 \\ & & \vdots & & & \\ \vdots & & \lambda dp_i & & \vdots & \vdots \\ & & \vdots & & & \\ U_{N1} & \cdots & 0 & \cdots & U_{NN} & -p_N \\ -p_1 & \cdots & x_i dp_i & \cdots & -p_N & 0 \end{vmatrix}}{|\mathbf{J}|} = \frac{J_{ij}}{|\mathbf{J}|}\lambda dp_i + \frac{J_{N+1,j}}{|\mathbf{J}|}x_i$$

which, upon rearrangement, becomes

$$\frac{dx_j^M}{dp_i} = \frac{J_{ij}}{|\mathbf{J}|}\lambda + \frac{J_{N+1,j}}{|\mathbf{J}|}x_i = \frac{J_{ij}}{|\mathbf{J}|}\lambda - \frac{dx_j}{dA}x_i \tag{6}$$

where the last equation follows from (5).

Slutsky-compensated change in price

- Consider again a change in a single price p_i.
- Slutsky-compensation adjusts the consumer's income by $dA^C = x_i dp_i$ so that the original bundle remains affordable at the new prices.
- Proceeding as before, the system in (4) reduces to

$$\mathbf{J}\begin{pmatrix} dx_1^C \\ \vdots \\ dx_N^C \\ d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda dp_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the superscript C is used to denote that the price change is compensated.

- Cramer's Rule now yields the expression

$$dx_j^C = \frac{\begin{vmatrix} U_{11} & \cdots & \overbrace{0}^{\text{Column } j} & \cdots & U_{1N} & -p_1 \\ & & \vdots & & & \\ \vdots & & \lambda dp_i & & \vdots & \vdots \\ & & \vdots & & & \\ U_{N1} & \cdots & 0 & \cdots & U_{NN} & -p_N \\ -p_1 & \cdots & 0 & \cdots & -p_N & 0 \end{vmatrix}}{|\mathbf{J}|} = \frac{J_{ij}}{|\mathbf{J}|}\lambda dp_i$$

or

$$\frac{dx_j^C}{dp_i} = \frac{J_{ij}}{|\mathbf{J}|}\lambda \tag{7}$$

- Our assumption about the principle minors of the bordered Hessian of U ensure that J_{ii} and $|\mathbf{J}|$ have the opposite sign.
- This implies

- TWe can substitute equation (7) into (6) to obtain

$$\frac{dx_j^M}{dp_i} = \frac{dx_j^C}{dp_i} - \frac{dx_j}{dA}x_i \tag{9}$$

- This is the famous Slutsky equation, which says that the effect of an uncompensated price change can be decomposed into a compensated price (substitution) effect and an income effect.
- If X_i is a normal good, then equation (8) implies

$$\frac{dx_i^M}{dp_i} < 0$$

- This result is called the law of demand.

Hicks-compensated change in price

- Suppose we again consider a change in prices, but we adjust the consumer's income in such a way that he is indifferent between the old and new price vectors.
- This is called Hicks compensation.
- To calculate how much utility changes, totally differentiate the utility function:

$$dU = \sum_i U_i dx_i$$

- By the first order conditions in (2), we can rewrite this as

$$dU = \lambda \sum_i p_i dx_i \tag{10}$$

- We can also totally differentiate the budget constraint:

$$dA = \sum_i (p_i dx_i + x_i dp_i)$$

- Substituting into equation (10), we have

$$dU = \lambda \left(dA - \sum_i x_i dp_i \right)$$

- This equation implies that the consumer is indifferent to a small change in prices after receiving Slutsky-compensation.
- The equivalence of Slutsky and Hicks compensation breaks down for non-marginal changes, a problem that can lead to "substitution bias" in price indices.

Frisch-compensated change in price

- Up until now, λ has played a minor role in the analysis.
- Recall the Envelope Theorem result that λ can be interpreted as the marginal utility of income, the additional amount of utility the consumer can attain from a marginal increase in income.
- Using Cramer's Rule on equation(4), we can calculate the change in λ as we change income as

$$d\lambda = \frac{\begin{vmatrix} U_{11} & \cdots & U_{1N} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1} & \cdots & U_{NN} & 0 \\ -p_1 & \cdots & -p_N & -dA \end{vmatrix}}{|\mathbf{J}|} = \frac{|\mathbf{H}|}{|\mathbf{J}|}(-dA)$$

where $\mathbf{H}$ is the Hessian (not the bordered Hessian) of the utility function.

- This implies
$$\frac{d\lambda}{dA} = -\frac{|\mathbf{H}|}{|\mathbf{J}|} \tag{11}$$
- A sufficient condition for the strict concavity of U is that the principle minors of its Hessian alternate in sign, starting with a negative value.
- We will assume this condition holds, so that $(-1)^N|\mathbf{H}| > 0$.
- From our assumption about the determinants of the bordered Hessian, we also know that $(-1)^N|\mathbf{J}| > 0$.
- This implies marginal utility of income is falling.

- The assumption that the utility function is concave is not without loss of generality, since not all convex preference orderings are "concavifiable".
- Restrictions on preferences which guarantee a concave utility representation, as well as what those restrictions imply about consumer demand behavior, are explored in the mathematical economics literature (see **????**).

- The effect of a Slutsky-compensated change in price on λ is given by

$$d\lambda^C = \frac{\begin{vmatrix} U_{11} & \cdots & U_{1N} & 0 \\ & & & \vdots \\ \vdots & \ddots & \vdots & \lambda dP_i \\ & & & \vdots \\ U_{N1} & \cdots & U_{NN} & 0 \\ -p_1 & \cdots & -p_N & 0 \end{vmatrix}}{|\mathbf{J}|} = \frac{J_{i,N+1}}{|\mathbf{J}|}\lambda dp_i = -\frac{dx_i}{dA}\lambda dp_i$$

$$\Rightarrow \qquad \frac{d\lambda^C}{dp_i} = -\lambda\frac{dx_i}{dA}$$

where the last equality comes from the symmetry of **J** and equation (5).

- If x_i is a normal good, the compensated marginal utility of

- Now consider the effect of a general price change:

$$d\lambda = \frac{\begin{vmatrix} U_{11} & \cdots & U_{1N} & \lambda dp_1 \\ U_{21} & \cdots & U_{2N} & \lambda dp_2 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1} & \cdots & U_{NN} & \lambda dp_N \\ -p_1 & \cdots & -p_N & -dA + \sum_i x_i dp_i \end{vmatrix}}{|\mathbf{J}|}$$

$$= \sum_i \frac{J_{iN}}{|\mathbf{J}|}\lambda dp_i + (-dA + \sum_i x_i dp_i)\frac{|\mathbf{H}|}{|\mathbf{J}|}$$

$$= -\sum_i \frac{dx_i}{dA}\lambda dp_i - (-dA + \sum_i x_i dp_i)\frac{d\lambda}{dA}$$

- Frisch compensation keeps the marginal utility of income constant ($d\lambda = 0$):

$$0 = -\sum_i \frac{dx_i}{dA}\lambda dp_i - (-dA^F + \sum_i x_i dp_i)\frac{d\lambda}{dA}$$

- If all goods are normal and utility is strictly concave, dA^F will be smaller than Slutsky-compensation.
- A compensated increase in price will make the marginal utility of income fall, since the consumer will be able to buy less with a marginal dollar.
- To keep marginal utility constant, the consumer must be given less compensation.
- The Frisch-compensated response to a *ceteris paribus* change in p_i will therefore exceed the Hicks-compensated response:

$$0 > \frac{dx_i^C}{dp_i} > \frac{dx_i^F}{dp_i} = \frac{dx_i^C}{dp_i} + \frac{\lambda}{\frac{d\lambda}{dA}} \left(\frac{dx_i}{dA} \right)^2 \tag{12}$$

- The difference between the uncompensated and Frisch-compensated response is given by

$$\frac{dx_i^U}{} - \frac{dx_i^F}{} = -\left(x_i + \frac{\lambda}{} \frac{dx_i}{} \right) \frac{dx_i}{}$$

- Whether or not the uncompensated response exceeds the Frisch-compensated response depend on the sign of the term in parentheses.
- Using equations (5) and (7), we can also write the Frisch response in terms of determinants as

$$\frac{dx_i^F}{dp_i} = \lambda \left(\frac{J_{ii}|\mathbf{H}| - J_{N+1,i}^2}{|\mathbf{J}||\mathbf{H}|} \right) \tag{13}$$

- Suppose without loss of generality that $i = N$ (we can always rearrange the labelling of the goods to make this the case).
- We can partition the adjutant matrix of $\mathbf{J}$ as

$$\operatorname{adj}\mathbf{J} = \left(\begin{array}{cccc|cc} J_{11} & J_{21} & \cdots & J_{N-1,1} & J_{N,1} & J_{N+1,1} \\ J_{12} & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ J_{1,N-1} & & \cdots & J_{N-1,N-1} & J_{N,N-1} & J_{N+1,N-1} \\ \hline J_{1,N} & & \cdots & J_{N-1,N} & J_{N,N} & J_{N+1,N} \\ J_{1,N+1} & & \cdots & J_{N-1,N+1} & J_{N,N+1} & |\mathbf{H}| \end{array}\right) = \Big(\,-$$

- Using the partitioned determinant formula, we get

$$|\operatorname{adj}\mathbf{J}| = |\mathbf{C}||\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T| = |\mathbf{J}|^N$$

where the last equality comes from the fact that

$$\frac{1}{|\mathbf{J}|}\operatorname{adj}\mathbf{J} = \mathbf{J}^{-1} \tag{14}$$

- But $|\mathbf{C}|$ is just the numerator of equation (12). Thus we can rewrite (12) as

$$\frac{dx_N^F}{dp_N} = \lambda\frac{|\mathbf{J}|^{N-1}|(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T))^{-1}|}{|\mathbf{H}|} \tag{15}$$

- From the partitioned inverse formula, we know that

$$(\text{adj}\,\mathbf{J})^{-1} = \left(\begin{array}{c|c} (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T)^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{C} \\ \hline -\mathbf{C}^{-1}\mathbf{B}^T(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T)^{-1} & (\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})^{-1} \end{array}\right.$$

- Equation (14) then implies

$$\frac{1}{|\mathbf{J}|}\mathbf{H}_{NN} = (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T)^{-1}$$

so that equation (15) simplifies to

$$\frac{dx_N^F}{dp_N} = \lambda \frac{|\mathbf{H}_{NN}|}{|\mathbf{H}|} \tag{16}$$

- A simpler way to obtain the expression in (16) is to consider the system of equations in (2) as defining $\mathbf{x}$ as a function of $\mathbf{p}$ and λ:

$$U_i(\mathbf{x}(\mathbf{p}, \lambda)) = \lambda p_i \qquad \text{for } i = 1, \cdots, N$$

- At a particular value of λ, these functions will satisfy the original budget constraint. Differentiating this system yields

$$\mathbf{H} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_N \end{pmatrix} = \begin{pmatrix} \lambda dp_1 \\ \lambda dp_2 \\ \vdots \\ \lambda dp_N \end{pmatrix} \tag{17}$$

- Cramer's Rule applied to the system in (17) then yields equation (16).

Monotonic Transformations

- Suppose we consider the utility function

$$V(\mathbf{x}) = f(U(\mathbf{x}))$$

where f is a differentiable, strictly increasing function.

- V represents the same underlying preferences as U, so we should hope that our predictions remain unchanged if we perform the above calculations using V.

- The first-order conditions in equation (2) become

$$V_i = U_i f' = \lambda^V p_i$$

- Since $(\mathbf{x}, \lambda)$ was a solution to the original system of equations, we know that $(\mathbf{x}, f'\lambda)$ is a solution to the new system of equations.

- Thus we must have

$$\lambda = \frac{\lambda^V}{f'} \tag{18}$$

Replacing V with U causes the $\mathbf{J}$ matrix in equation (4) to become

$$\mathbf{J}^V = \begin{pmatrix} U_{11}f' + U_1^2 f'' & \cdots & U_{1N}f' + U_1U_N f'' & -p_1 \\ U_{21}f' + U_2U_1 f'' & \cdots & U_{2N}f' + U_2U_N f'' & -p_1 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1}f' + U_NU_1 f'' & \cdots & U_{NN}f' + U_N^2 f'' & -p_1 \\ -p_1 & \cdots & -p_N & 0 \end{pmatrix}$$

$$= \begin{pmatrix} U_{11}f' & \cdots & U_{1N}f' & -p_1 \\ U_{21}f' & \cdots & U_{2N}f' & -p_1 \\ \vdots & \ddots & \vdots & \vdots \\ U_{N1}f' & \cdots & U_{NN}f' & -p_1 \\ -p_1 & \cdots & -p_N & 0 \end{pmatrix} + f'' \begin{pmatrix} U_1 \\ \vdots \\ U_N \\ 0 \end{pmatrix} \begin{pmatrix} U_1 & \cdots & U_N & 0 \end{pmatrix}$$

$$= \mathbf{C}_1\mathbf{J}\mathbf{C}_2 + f''\mathbf{z}\mathbf{z}'$$

where

$$\mathbf{C}_1 = \begin{pmatrix} 1 & & \\ & \ddots & \end{pmatrix} \quad \mathbf{C}_2 = \begin{pmatrix} f' & & \\ & \ddots & \end{pmatrix} \quad \begin{pmatrix} U_1 \\ \vdots \end{pmatrix}$$

- Using the matrix determinant lemma, we have

$$\begin{aligned} |\mathbf{J}^V| &= |\mathbf{C}_1\mathbf{J}\mathbf{C}_2|(1 + f''\mathbf{z}'\mathbf{C}_2^{-1}\mathbf{J}^{-1}\mathbf{C}_1^{-1}\mathbf{z}) \\ &= |\mathbf{J}|\left(1 + \frac{f''}{f'}\mathbf{z}'\mathbf{J}^{-1}\mathbf{z}\right)(f')^{N-1} \end{aligned} \tag{19}$$

- Recalling the adjoint formula for a matrix inverse given in (14), $\mathbf{J}^{-1}$ can be written as

$$\mathbf{J}^{-1} = \frac{1}{|\mathbf{J}|}\begin{pmatrix} J_{11} & J_{21} & \cdots & J_{N+1,1} \\ J_{12} & J_{22} & \cdots & J_{N+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ J_{1,N+1} & J_{2,N+1} & \cdots & J_{N+1,N+1} \end{pmatrix}$$

- Thus we have

$$\mathbf{z}'\mathbf{J}^{-1}\mathbf{z} = \frac{1}{|\mathbf{J}|}\sum_i\sum_j U_iU_jJ_{ji}$$

- Notice that for $i = 1, \cdots, N$, the term $\sum_j U_j J_{ji}$ is the cofactor expansion of the determinant of the matrix which replaces the i^{th} column $\mathbf{J}$ with the vector $\mathbf{z}$.

- By the first order conditions, we know that $\mathbf{z}$ is proportional to the $(N+1)^{th}$-column of $\mathbf{J}$, so the determinant of this matrix must thus be zero.

- This implies $\sum_j U_j J_{ji} = 0$ for all i, so that the expression in (19) simplifies to

$$|\mathbf{J}^V| = |\mathbf{J}|(f')^{N-1} \tag{20}$$

- Using similar methods, it can be shown that

$$J^V_{N+1,j} = J_{N+1,j}(f')^{N-1} \quad \text{for } j = 1, \cdots, N \tag{21}$$

$$J^V_{ij} = J_{ij}(f')^{N-2} \quad \text{for } i = 1, \cdots, N, \quad j = 1, \cdots, N \tag{22}$$

- Equations (18)-(22) then imply

$$-\frac{J^V_{N+1,j}}{|\mathbf{J}^V|} = -\frac{J_{N+1,j}}{|\mathbf{J}|} \quad \text{for } j = 1, \cdots, N \tag{23}$$

$$\frac{J^V_{ij}}{|\mathbf{J}^V|}\lambda^V = \frac{J_{ij}}{|\mathbf{J}|}\lambda \quad \text{for } i = 1, \cdots, N, \quad j = 1, \cdots, N \tag{24}$$

- Equations (23) and (24) imply that the income and Slutsky-Hicks compensated price effects will be the same when V is used instead of U.
- Equation (9) then implies that the uncompensated effects will also be the same.

What about the Frisch response? Proceeding as before, we find that

$$|\mathbf{H}^V| = |\mathbf{H}| \left(1 + \underbrace{\frac{f''}{f'} \begin{pmatrix} U_1 & \cdots & U_N \end{pmatrix} \mathbf{H}^{-1} \begin{pmatrix} U_1 \\ \vdots \\ U_N \end{pmatrix}}_{\kappa} \right) (f')^N \tag{25}$$

and

$$|\mathbf{H}^V_{ii}| = |\mathbf{H}_{ii}| \left(1 + \underbrace{\frac{f''}{f'} \begin{pmatrix} U_1 & \cdots & U_{i-1} & U_{i+1} & \cdots & U_N \end{pmatrix} \mathbf{H}^{-1} \begin{pmatrix} U_1 \\ \vdots \\ U_{i-1} \\ U_{i+1} \\ \vdots \\ U_N \end{pmatrix}}_{\kappa_i} \right)$$

- This time, the terms in the parentheses does not simplify to unity unless $f'' = 0$.
- We thus have

$$\lambda^V \frac{|\mathbf{H}_{ii}^V|}{|\mathbf{H}^V|} = \lambda \frac{|\mathbf{H}_{ii}|}{|\mathbf{H}|} \frac{\left(1 + \frac{f''}{f'}\kappa_i\right)}{\left(1 + \frac{f''}{f'}\kappa\right)}$$

  where κ and κ_i refer to the quadratic forms in equation (25) and (26), respectively.
- This result implies the Frisch response is not invariant to arbitrary monotonic transformations.
- Some transformations may reverse the ordering in equation (12).

Adding Endowments

- Now suppose we assume the consumer's income comes in the form of a vector of endowments of the goods $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_N)'$.
- The consumer's budget constraint is

$$\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \bar{\mathbf{x}}$$

- The first-order conditions in (2) are unchanged. The system of equations in (4) is now

$$\mathbf{J} \begin{pmatrix} dx_1 \\ \vdots \\ dx_N \\ d\lambda \end{pmatrix} = \begin{pmatrix} \lambda dp_1 \\ \vdots \\ \lambda dp_N \\ \sum_i (x_i - \bar{x}_i) dp_i \end{pmatrix}$$

- The Slutsky equation in (9) becomes

$$\frac{dx_j^U}{dp_i} = \frac{dx_j^C}{dp_i} - \frac{dx_j}{dA}(x_i - \bar{x}_i) \tag{27}$$

- If the consumer is a net seller of x_i and x_i is a normal good, the income effect is positive. Slutsky compensation is given by

$$dA^C = \sum_i (x_i - \bar{x}_i) dp_i$$

- In the case where only p_i changes and the consumer is a net seller of x_i, Slutsky compensation is negative.

- Frisch compensation can be written as

$$dA^F = \sum_i \left(\frac{\lambda}{\frac{d\lambda}{dA}} \frac{dx_i}{dA} + x_i - \bar{x}_i \right) dp_i$$

- In the case of normal goods and strictly concave utility, Frisch compensation remains more negative than Slutsky compensation.
- We can order the responses to an own-price change as

$$\frac{dx_i^U}{dp_i} > \frac{dx_i^C}{dp_i} > \frac{dx_i^F}{dp_i}$$

where $\frac{dx_i^C}{dp_i}$ and $\frac{dx_i^F}{dp_i}$ are both negative but $\frac{dx_i^U}{dp_i}$ may be positive.

THE UNIVERSITY OF CHICAGO

The Labor-Leisure Choice

- Suppose we consider the case where there are two goods, consumption c and leisure l.
- The consumer is endowed with 1 unit of time and A units of consumption.
- The consumer works for $1-l$ hours and earns a wage w.
- The consumer's budget constraint is now

$$pc \leq a + w(1-l)$$

which can be arranged as

$$pc + wl \leq A + w$$

- The system of equations in (4) is then given by

$$\begin{pmatrix} U_{cc} & U_{cl} & -p \\ U_{lc} & U_{ll} & -w \\ -p & -w & 0 \end{pmatrix} \begin{pmatrix} dc \\ dl \\ d\lambda \end{pmatrix} = \begin{pmatrix} \lambda dp \\ \lambda dw \\ (c-A)dp + (l-1)dw \end{pmatrix} \tag{28}$$

- The income effect for leisure can be calculated from equation (5) as

$$\frac{dl}{dA} = \frac{U_{lc}p - U_{cc}w}{-p^2U_{ll} + 2pwU_{lc} - U_{cc}w^2}$$

- The compensated effect of an increase in w can be calculated from equation (7):

$$\frac{dl^C}{dw} = \frac{-\lambda p^2}{-p^2U_{ll} + 2pwU_{lc} - U_{cc}w^2}$$

  and the uncompensated price effect can be calculated using equation (27):

$$\frac{dl^U}{dw} = \frac{-\lambda p^2 + (U_{lc}p - U_{cc}w)(1 - l)}{-p^2U_{ll} + 2pwU_{lc} - U_{cc}w^2}$$

- All these effects are invariant to monotonic transformations of the utility function.

- The Frisch effect can be calculated using equation (16):

$$\frac{dL^F}{dW} = \frac{\lambda U_{cc}}{U_{cc}U_{ll} - U_{cl}^2}$$

- Define labor supply as $n = 1 - l$. Substituting for λ using the first-order condition $U_l = \lambda w$, we can write the *Frisch elasticity of labor supply* as

$$\epsilon^F = \frac{dn^F}{dw}\frac{w}{n} = -\frac{1}{n}\frac{U_{cc}U_l}{U_{cc}U_{ll} - U_{cl}^2}$$

- If consumption and leisure are additively separable in the utility function, this expression simplifies to

$$\epsilon^F = -\frac{1}{n}\frac{U_l}{U_{ll}} \tag{29}$$

- Suppose that the utility function is given by

$$U(c,l) = \phi(c) - \gamma \frac{\epsilon}{1+\epsilon}(1-l)^{\frac{1+\epsilon}{\epsilon}}$$

- Then equation (29) simplifies to

$$\epsilon^F = \epsilon$$

- This utility function has a constant Frisch elasticity of labor supply. As an exercise, determine the Frisch elasticity of labor supply for the utility function

$$U(c,l) = \frac{\left(c^{\gamma} l^{1-\gamma}\right)^{1-\sigma}}{1-\sigma}$$

- Both these specifications are commonly used in the literature.

Application to a Life-Cycle Model

- Suppose a consumer lives for $T+1$ periods and that the consumer's preferences over consumption and leisure streams $\{c_t, l_t\}_{t=0}^{T}$ are ordered by the utility function

$$\sum_{t=0}^{T} \beta^t U(c_t, l_t) \tag{30}$$

- The consumer earns a real wage w_t in period t and is endowed with one unit of time each period.
- The consumer is also endowed with some initial assets A_0 at $t=0$ and can borrow and lend freely at constant interest rate r.
- The consumer's lifetime budget constraint is given by

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \sum_{t=0}^{T} \frac{w_t l_t}{(1+r)^t} \leq A_0 + \sum_{t=0}^{T} \frac{w_t}{(1+r)^t} = \bar{A} \tag{31}$$

where $\bar{A}$ is the present value of the consumer's lifetime income

- Suppose instead of considering the lifetime problem all at once, the consumer first solves the period t problem first:

$$\max_{c_t, l_t} \; U(c_t, l_t) \text{ subject to } c_t + w_t l_t \leq E_t \tag{33}$$

where E_t are the resources allocated to period t. Let $V(w_t, E_t)$ denote the value function of this problem. The consumer then solves the problem

$$\max_{\{E_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t V(w_t, E_t) \quad \text{subject to } \sum_{t=0}^{T} \frac{E_t}{(1+r)^t} \leq \bar{A} \tag{34}$$

- The first-order conditions to the problem in (34) is

$$\beta^t \frac{\partial V^t}{\partial E} = \lambda_t = \frac{\lambda}{(1+r)^t} \tag{35}$$

where $V^t = V(w_t, E_t)$ and $\lambda_t = \beta^t \frac{\partial V^t}{\partial E}$.

- Totally differentiating this expression yields

$$\frac{\partial \lambda_t}{\partial E_t} dE_t + \frac{\partial \lambda_t}{\partial w_t} dw_t = \frac{d\lambda}{(1+r)^t} \qquad \text{for } t = 0, 1, \cdots, T \qquad (36)$$

- Differentiating the budget constraint yields

$$\sum_{t=0}^{T} \frac{dE_t}{(1+r)^t} = \sum_{t=0}^{T} \frac{dw_t}{(1+r)^t} + dA_0 \qquad (37)$$

Placing these equations into a matrix gives us

$$\underbrace{\begin{pmatrix} \frac{\partial \lambda_0}{\partial E} & & & & -1 \\ & \frac{\partial \lambda_1}{\partial E} & & & -\frac{1}{1+r} \\ & & \ddots & & \vdots \\ & & & \frac{\partial \lambda_T}{\partial E} & -\frac{1}{(1+r)^T} \\ -1 & -\frac{1}{1+r} & \cdots & -\frac{1}{(1+r)^T} & \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} dE_0 \\ dE_1 \\ \vdots \\ dE_T \\ d\lambda \end{pmatrix} = \begin{pmatrix} -\frac{\partial \lambda_0}{\partial w_0} dw_0 \\ -\frac{\partial \lambda_1}{\partial w_1} dw_1 \\ \vdots \\ -\frac{\partial \lambda_T}{\partial w_T} dw_T \\ -\sum_{t=0}^{T} \frac{dw_t}{(1+r)^t} - \end{pmatrix}$$

- Due to its diagonal structure, the determinant of **J** can easily be found using the partitioned determinant formula:

$$|\mathbf{J}| = -\left(\prod_{t=0}^{T} \frac{\partial \lambda_t}{\partial E}\right)\left(\sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^{2t}\left(\frac{\partial \lambda_t}{\partial E}\right)^{-1}\right) \tag{38}$$

- If utility within each period is strictly concave, then we already showed above that $\frac{\partial \lambda_t}{\partial E} < 0$ for all t.
- It is easy to see that the border-preserving principle minors of **J** alternate in sign, so that the utility function in equation (34) is strictly quasiconcave.

- We first solve for the effect of a small change in initial assets dA_0. Proceeding as before, we have

$$\frac{dE_t}{dA_0} = -\frac{J_{T+2,t+1}}{|\mathbf{J}|} \tag{39}$$

- To find $J_{T+2,t+1}$, expand along row $t+1$.
- The only non-zero element in this row occurs in the final column.
- Once this row and column are removed, only the final row contains more than a single element.

- Expanding along column $t+1$ then yields a diagonal matrix.
- To illustrate how this works, suppose $T=3$ and $t=1$.
- We seek to find the determinant

$$J_{5,2} = \begin{vmatrix} \frac{\partial \lambda_0}{\partial E} & & & & -1 \\ & & & & -\frac{1}{1+r} \\ & & \frac{\partial \lambda_2}{\partial E} & & -\frac{1}{(1+r)^2} \\ & & & \frac{\partial \lambda_3}{\partial E} & -\frac{1}{(1+r)^3} \\ -1 & -1 & -\frac{1}{(1+r)^2} & -\frac{1}{(1+r)^3} & \end{vmatrix}$$

Expanding on the second row yields

$$J_{5,2} = \frac{1}{1+r} \begin{vmatrix} \frac{\partial \lambda_0}{\partial E} & & & \\ & & \frac{\partial \lambda_2}{\partial E} & \\ & & & \frac{\partial \lambda_3}{\partial E} \\ -1 & -1 & -\frac{1}{(1+r)^2} & -\frac{1}{(1+r)^3} \end{vmatrix}$$

Expanding along the second column yields

$$J_{5,2} = -\frac{1}{1+r}\begin{vmatrix} \frac{\partial \lambda_0}{\partial E} & & \\ & \frac{\partial \lambda_2}{\partial E} & \\ & & \frac{\partial \lambda_3}{\partial E} \end{vmatrix} = -\frac{1}{1+r}\prod_{t \neq 1} \frac{\partial \lambda_t}{\partial E}$$

- In general, this procedure will yield the expression

$$J_{T+2,t+1} = -\frac{1}{(1+r)^t} \prod_{t' \neq t} \frac{\partial \lambda_{t'}}{\partial E}$$

which implies from equation (38) and (39) that

$$\frac{dE_t}{dA_0} = \left(\frac{1}{1+r}\right)^t \left(\frac{\partial \lambda_t}{\partial E}\right)^{-1} \left(\sum_{t'=0}^{T} \left(\frac{1}{1+r}\right)^{2t'} \left(\frac{\partial \lambda_{t'}}{\partial E}\right)^{-1}\right)^{-1} > 0$$

Interpretation:

- The Frisch elasticity estimates the effect of evolutionary wage change.
- In a perfect foresight model, it is the effect of changing the relative price of time in different periods.
- These price changes are perfectly anticipated.
- Marshallian effects refer to wage changes with income effects (Blundell and MaCurdy, 1999; MaCurdy, 1978, 1981).

Questions to address:

1. Add uncertainty about future wages and incomes (see e.g., MaCurdy, 1978). Does the interpretation survive?
2. Are the Frisch elasticities useless for interpreting data?

Explain better utility constant (this is lifetime utility) from utility constant within a period.

Show the Hessian for the lifetime utility problem. Interpret within a utility tree model

Life cycle model:

$E_t =$ expenditure in t

U_t is utility in t

Agent: Max $U_t + \lambda_t(E_t - P_t' \cdot X_t)$

This solves the expenditure constant (for t period problem)

Stage 2: Allocate $\sum_{t=0}^{T} \frac{E_t}{(1+r)^t} = A$ over the life cycle so that the $\lambda_t = \lambda$ all t (with $r > 0, \lambda_t = \frac{\lambda}{(1+r)^t}$)

Question: In this representation, what is the Frisch effect of a change in P_{1t} on X_{1t}?

Blundell, R. and T. E. MaCurdy (1999). Labor supply: A review of alternative approaches. In O. C. Ashenfelter and D. E. Card (Eds.), *Handbook of Labor Economics*, Volume 3A, pp. 1559–1695. Amsterdam: Elsevier.

MaCurdy, T. E. (1978). *Two Essays on the Life Cycle*. Ph.D. dissertation, University of Chicago, Chicago.

MaCurdy, T. E. (1981, December). An empirical model of labor supply in a life-cycle setting. *Journal of Political Economy 89*(6), 1059–1085.