Notes on Frisch Demands*

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Frisch Demands

Simple Two Good Model to Interpret Frisch Demands

This is a model for each period of a general life cycle model. Abstract from interest rates and time preference.

$$max \ U(C, L)$$

s.t. $PC + WL = E$

E is the resources available to be spent within the period. Assume fixed for the moment.

 $U_C = \lambda P$ $U_L = \lambda W$

C = Consumption

L = Leisure

 $\lambda = {\sf LaGrange}$ Multiplier associated with the budget constraint

Treat leisure as an ordinary good for the moment.

Totally Differentiate

$$\begin{bmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{bmatrix} \begin{bmatrix} dC \\ dL \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dP \\ \lambda dW \\ -dE + CdP + LdW \end{bmatrix}$$

Let bars denote determinants

$$\frac{\partial L}{\partial E} = \frac{\begin{vmatrix} U_{CC} & 0 & -P \\ U_{CL} & 0 & -W \\ -P & -1 & 0 \end{vmatrix}}{|\cdot|}$$

Notation $|\cdot| = \begin{vmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{vmatrix} > 0$ (second order conditions)

$$\frac{dL}{dE} = \left\{ \underbrace{\overbrace{(-U_{CC}W + PU_{CL})}^{>0}}_{\substack{|\cdot|\\ >0}} \right\} > 0 \text{ (From Assumed Normality)}$$

Now this can be related to displacements for λ :

$$\frac{\partial \lambda}{\partial W} = \frac{\begin{vmatrix} U_{CC} & U_{CL} & 0 \\ U_{CL} & U_{LL} & \lambda \\ -P & -W & 0 \end{vmatrix}}{|\cdot|}$$
$$= \frac{-\lambda}{|\cdot|} \begin{vmatrix} U_{CC} & U_{CL} \\ -P & -W \end{vmatrix}$$
$$= (-\lambda) \frac{(-U_{CC}W + U_{CL}P)}{|\cdot|}$$
$$= \lambda \frac{(U_{CC}W - U_{CL}P)}{|\cdot|}$$

< 0 from normality of leisure

Thus

$$\frac{\partial \lambda}{\partial W} = -\frac{\partial L}{\partial E}$$

(Intuition: The cost of a component of utility rises, the marginal utility of income declines.)

Observe

$$\frac{\partial \lambda}{\partial E} = \frac{-\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}}{|\cdot|}$$
$$\frac{d\lambda}{dE} < 0 \text{ under strict concavity}$$
(Diminishing Marginal Utility of Income)

Take total differential:

$$d\lambda = \frac{\partial \lambda}{\partial P} dP + \frac{\partial \lambda}{\partial W} dW + \frac{\partial \lambda}{\partial E} (dE - CdP - LdW)$$

(This sign pattern assumes that all goods are normal.)

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Consider

 $\left. \frac{\partial L}{\partial W} \right|_{\lambda}$

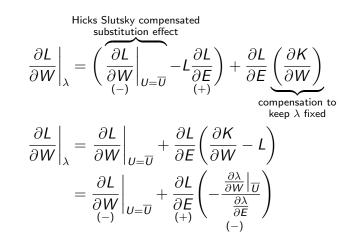
Compensate the agent to keep λ fixed. Let dK be the compensation.

$$d\lambda = 0 = \left(\frac{\partial\lambda}{\partial W}\right) dW + \left(\frac{\partial\lambda}{\partial E}\right) (dK - LdW)$$
$$0 = \frac{\partial\lambda}{\partial W} dW - LdW + dK$$
$$dK = \left(L - \frac{\partial\lambda}{\partial W} - \frac{\partial\lambda}{\partial E}\right) dW$$

 \therefore The compensation required to keep λ constant is smaller than what is required to keep utility constant for the same change in the wage.

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Clearly $\frac{\partial L}{\partial W}\Big|_{\lambda} \le \frac{\partial L}{\partial W}\Big|_{U=\overline{U}} \le 0$

Digression

Direct derivation (assume λ fixed).

 $U_C = \lambda P$ $U_L = \lambda W$

 $U_{CC}dC + U_{CL}dL = \lambda dP$ $U_{CL}dC + U_{LL}dL = \lambda dW$

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Take Cramer's Rule:

$$\frac{\partial L}{\partial W}\Big|_{\lambda} = \frac{\begin{vmatrix} U_{CC} & 0 \\ U_{CL} & \lambda \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}} = \frac{(U_{CC}\lambda)}{U_{CC}U_{LL} - U_{CL}^2} < 0$$

Claimed result follows because

$$U_{CC} < 0$$

 $U_{CC}U_{LL} - U_{CL}^2 > 0$
(concavity)

Frisch Demands

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$$\frac{\partial C}{\partial W}\Big|_{\lambda} = \frac{\begin{vmatrix} 0 & U_{CL} \\ \lambda & U_{LL} \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}}$$
$$= \frac{-\lambda U_{CL}}{U_{CC} U_{LL} - U_{CL}^2}$$

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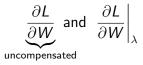
Back to the main thread:

$$\mathbf{D} \geq \frac{\partial L}{\partial W} \bigg|_{U = \overline{U}} \geq \frac{\partial L}{\partial W} \bigg|_{\lambda}$$

Intuition: we have to pay less compensation to keep λ fixed than U fixed and leisure is a normal good, so we get less leisure with λ fixed and hence the λ -constant wage response is more negative. Moreover,

$$\frac{\partial L}{\partial W}\Big|_{U=\overline{U}} \ge \frac{\partial L}{\partial W} \qquad \text{(From normality of leisure.)}$$

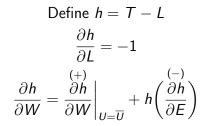
How to order



We have thus far abstracted from the fact that people have a time endowment, T. (Implicitly, we set T = 0). The Hicks-Slutsky uncompensated wage effect for leisure is (for $T \ge 0$)

$$\frac{\partial L}{\partial W} = \frac{\partial L}{\partial W} \bigg|_{U = \overline{U}} + (T - L) \frac{\partial L}{\partial E}$$

This makes the wage effect on labor supply ambiguous.



K is compensation required to keep λ constant.

$$\frac{\partial h}{\partial W}\Big|_{\lambda} = \frac{\partial h}{\partial W}\Big|_{U=\overline{U}} + \frac{\partial h}{\partial E}h + \frac{\partial h}{\partial E}\left(\frac{\partial K}{\partial W}\right) = F + F = OQC$$

What is *dK*?

$$d\lambda = 0 = \left(\frac{\partial \lambda}{\partial W}\right) \Big|_{\overline{U}} dW + \left(\frac{\partial \lambda}{\partial E}\right) (dK + hdW)$$
$$0 = \frac{\frac{\partial \lambda}{\partial W} \Big|_{\overline{U}}}{\frac{\partial \lambda}{\partial \overline{E}}} dW + hdW + dK$$
$$\frac{\partial K}{\partial W} = -\underbrace{\left(\frac{\partial \lambda}{\partial W} \Big|_{\overline{U}}}_{+} + h\right)}_{+} < 0$$

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$$\frac{\partial h}{\partial W}\Big|_{\lambda} = \frac{\frac{\partial h}{\partial h}}{\partial W}\Big|_{U=\overline{U}} - \frac{\frac{\partial h}{\partial E}\left(\frac{\frac{\partial \lambda}{\partial W}}{\frac{\partial \lambda}{\partial E}}\right)}{(+)}$$
$$\therefore 0 \le \frac{\partial h}{\partial W}\Big|_{U=\overline{U}} \le \frac{\partial h}{\partial W}\Big|_{\lambda}$$

and with respect to the uncompensated Marshallian elasticity:

$$\left. \frac{\partial h}{\partial W} < \frac{\partial h}{\partial W} \right|_{U = \overline{U}} \le \left. \frac{\partial h}{\partial W} \right|_{\lambda}$$

In the case of labor supply more income is transferred out periods where wages increase. dK is larger (more negative) in this case than the previous one considered.

(There is a more negative wage effect on borrowing, *i.e.* a force toward saving. This is intuitively so because wage increase generates income in the period which is likely transferred to other periods.)

- This is for a single period.
- We connect back to the multiperiod model by using the intertemporal arbitrage condition

$$\lambda_{t+1} = \frac{1+\beta}{1+r}\lambda_t$$

(see Browning, Heckman, and Hansen).

- This determines the allocations of the *E_t* across the periods in a two stage budgeting procedure.
- In those periods where λ_t is high ship resources in (borrow).
- Where it is low, ship out (save).
- As we showed, holding $U = \overline{U}$, $W \uparrow \lambda \downarrow$.
- Tends to create a force for saving in periods of wage increases.
- Possibly offsetting it, is substitution toward consumption when leisure goes down.