Notes on Frisch Demands*

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Econ 350, Winter 2023

*I thank David Malison and Mohsen Mirtaher for helpful comments.

Simple Two Good Model to Interpret Frisch Demands

This is a model for each period of a general life cycle model. Abstract from interest rates and time preference.

$$\max U(C, L)$$
$$\text{s.t. } PC + WL = E$$

E is the resources available to be spent within the period. Assume fixed for the moment.

$$U_C = \lambda P$$
$$U_L = \lambda W$$

$C =$ Consumption

$L =$ Leisure

$\lambda =$ LaGrange Multiplier associated with the budget constraint

Treat leisure as an ordinary good for the moment.

Comparative Statics of Model

Totally Differentiate

$$\begin{bmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{bmatrix} \begin{bmatrix} dC \\ dL \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dP \\ \lambda dW \\ -dE + CdP + LdW \end{bmatrix}$$

Let bars denote determinants

$$\frac{\partial L}{\partial E} = \frac{\begin{vmatrix} U_{CC} & 0 & -P \\ U_{CL} & 0 & -W \\ -P & -1 & 0 \end{vmatrix}}{|\cdot|}$$

Notation $|\cdot| = \begin{vmatrix} U_{CC} & U_{CL} & -P \\ U_{CL} & U_{LL} & -W \\ -P & -W & 0 \end{vmatrix} > 0$ (second order conditions)

$$\frac{dL}{dE} = \left\{ \frac{\overbrace{(-U_{CC}W + PU_{CL})}^{>0}}{\underbrace{|\cdot|}_{>0}} \right\} > 0 \text{ (From Assumed Normality)}$$

Now this can be related to displacements for λ:

$$\begin{aligned} \frac{\partial \lambda}{\partial W} &= \frac{\begin{vmatrix} U_{CC} & U_{CL} & 0 \\ U_{CL} & U_{LL} & \lambda \\ -P & -W & 0 \end{vmatrix}}{|\cdot|} \\ &= \frac{-\lambda}{|\cdot|} \begin{vmatrix} U_{CC} & U_{CL} \\ -P & -W \end{vmatrix} \\ &= (-\lambda) \frac{(-U_{CC}W + U_{CL}P)}{|\cdot|} \\ &= \lambda \frac{(U_{CC}W - U_{CL}P)}{|\cdot|} \\ &\quad < 0 \text{ from normality of leisure} \end{aligned}$$

Thus

$$\frac{\partial \lambda}{\partial W} = -\frac{\partial L}{\partial E}$$

(Intuition: The cost of a component of utility rises, the marginal utility of income declines.)

Observe

$$\frac{\partial \lambda}{\partial E} = \frac{-\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}}{|\cdot|}$$

$$\frac{d\lambda}{dE} < 0 \text{ under strict concavity}$$

(Diminishing Marginal Utility of Income)

Take total differential:

$$d\lambda = \overset{(-)}{\frac{\partial \lambda}{\partial P}} dP + \overset{(-)}{\frac{\partial \lambda}{\partial W}} dW + \overset{(-)}{\frac{\partial \lambda}{\partial E}} (dE - CdP - LdW)$$

(This sign pattern assumes that all goods are normal.)

Consider

$$\left.\frac{\partial L}{\partial W}\right|_{\lambda}$$

Compensate the agent to keep λ fixed. Let dK be the compensation.

$$d\lambda = 0 = \left(\overset{(-)}{\frac{\partial \lambda}{\partial W}}\right) dW + \left(\overset{(-)}{\frac{\partial \lambda}{\partial E}}\right)(dK - LdW)$$

$$0 = \frac{\frac{\partial \lambda}{\partial W}}{\frac{\partial \lambda}{\partial E}} dW - LdW + dK$$

$$dK = \left(L - \frac{\overset{(+)}{\frac{\partial \lambda}{\partial W}}}{\frac{\partial \lambda}{\partial E}}\right) dW$$

$\therefore$ The compensation required to keep λ constant is smaller than what is required to keep utility constant for the same change in the wage.

So

$$\left.\frac{\partial L}{\partial W}\right|_{\lambda} = \Bigg(\overbrace{\underset{(-)}{\left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}}}}^{\text{Hicks Slutsky compensated substitution effect}} - L \underset{(+)}{\frac{\partial L}{\partial E}} \Bigg) + \frac{\partial L}{\partial E} \underbrace{\left(\frac{\partial K}{\partial W}\right)}_{\text{compensation to keep } \lambda \text{ fixed}}$$

$$\left.\frac{\partial L}{\partial W}\right|_{\lambda} = \left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}} + \frac{\partial L}{\partial E}\left(\frac{\partial K}{\partial W} - L\right)$$

$$= \underset{(-)}{\left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}}} + \underset{(+)}{\frac{\partial L}{\partial E}} \Bigg(\underset{(-)}{-\frac{\left.\frac{\partial \lambda}{\partial W}\right|_{\overline{U}}}{\frac{\partial \lambda}{\partial E}}} \Bigg)$$

Clearly $\left.\frac{\partial L}{\partial W}\right|_{\lambda} \leq \left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}} \leq 0$

Digression

Direct derivation (assume λ fixed).

$$U_C = \lambda P$$
$$U_L = \lambda W$$

$$U_{CC} dC + U_{CL} dL = \lambda dP$$
$$U_{CL} dC + U_{LL} dL = \lambda dW$$

Take Cramer's Rule:

$$\left.\frac{\partial L}{\partial W}\right|_{\lambda} = \frac{\begin{vmatrix} U_{CC} & 0 \\ U_{CL} & \lambda \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}}$$
$$= \frac{(U_{CC}\lambda)}{U_{CC}U_{LL} - U_{CL}^2} < 0$$

Claimed result follows because

$$U_{CC} < 0$$
$$U_{CC}U_{LL} - U_{CL}^2 > 0$$
$$\text{(concavity)}$$

$$\left.\frac{\partial C}{\partial W}\right|_{\lambda} = \frac{\begin{vmatrix} 0 & U_{CL} \\ \lambda & U_{LL} \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}} = \frac{-\lambda U_{CL}}{U_{CC}U_{LL} - U_{CL}^2}$$

$$U_{CL} > 0 \implies \left.\frac{\partial C}{\partial W}\right|_{\lambda} < 0$$

$$U_{CL} < 0 \implies \left.\frac{\partial C}{\partial W}\right|_{\lambda} > 0$$

(See e.g. Heckman, 1974, AER)

Back to the main thread:

$$0 \geq \left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}} \geq \left.\frac{\partial L}{\partial W}\right|_{\lambda}$$

Intuition: we have to pay less compensation to keep λ fixed than U fixed and leisure is a normal good, so we get less leisure with λ fixed and hence the λ-constant wage response is more negative.
Moreover,

$$\left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}} \geq \frac{\partial L}{\partial W} \qquad \text{(From normality of leisure.)}$$

How to order

$$\underbrace{\frac{\partial L}{\partial W}}_{\text{uncompensated}} \text{ and } \left.\frac{\partial L}{\partial W}\right|_{\lambda}$$

?

We have thus far abstracted from the fact that people have a time endowment, T. (Implicitly, we set $T = 0$). The Hicks-Slutsky uncompensated wage effect for leisure is (for $T \geq 0$)

$$\frac{\partial L}{\partial W} = \left.\frac{\partial L}{\partial W}\right|_{U=\overline{U}} + (T - L)\frac{\partial L}{\partial E}$$

This makes the wage effect on labor supply ambiguous.

$$\text{Define } h = T - L$$

$$\frac{\partial h}{\partial L} = -1$$

$$\frac{\partial h}{\partial W} = \left.\overset{(+)}{\frac{\partial h}{\partial W}}\right|_{U=\overline{U}} + h\left(\overset{(-)}{\frac{\partial h}{\partial E}}\right)$$

K is compensation required to keep λ constant.

$$\left.\frac{\partial h}{\partial W}\right|_{\lambda} = \left.\overset{(+)}{\frac{\partial h}{\partial W}}\right|_{U=\overline{U}} + \overset{(-)}{\frac{\partial h}{\partial E}}h + \overset{(-)}{\frac{\partial h}{\partial E}}\left(\frac{\partial K}{\partial W}\right)$$

What is dK?

$$d\lambda = 0 = \overset{(-)}{\left(\frac{\partial \lambda}{\partial W}\right)}\Bigg|_{\overline{U}} dW + \overset{(-)}{\left(\frac{\partial \lambda}{\partial E}\right)} (dK + hdW)$$

$$0 = \frac{\frac{\partial \lambda}{\partial W}\Big|_{\overline{U}}}{\frac{\partial \lambda}{\partial E}} dW + hdW + dK$$

$$\frac{\partial K}{\partial W} = -\underbrace{\left(\frac{\frac{\partial \lambda}{\partial W}\Big|_{\overline{U}}}{\frac{\partial \lambda}{\partial E}} + h \right)}_{+} < 0$$

So

$$\left.\frac{\partial h}{\partial W}\right|_{\lambda} = \overset{(+)}{\left.\frac{\partial h}{\partial W}\right|_{U=\overline{U}}} \underbrace{-\overset{(-)}{\frac{\partial h}{\partial E}} \left(\overset{(+)}{\frac{\left.\frac{\partial \lambda}{\partial W}\right|_{\overline{U}}}{\frac{\partial \lambda}{\partial E}}} \right)}_{(+)}$$

$$\therefore 0 \leq \left.\frac{\partial h}{\partial W}\right|_{U=\overline{U}} \leq \left.\frac{\partial h}{\partial W}\right|_{\lambda}$$

and with respect to the uncompensated Marshallian elasticity:

$$\frac{\partial h}{\partial W} < \left.\frac{\partial h}{\partial W}\right|_{U=\overline{U}} \leq \left.\frac{\partial h}{\partial W}\right|_{\lambda}$$

In the case of labor supply more income is transferred out periods where wages increase. dK is larger (more negative) in this case than the previous one considered.

(There is a more negative wage effect on borrowing, *i.e.* a force toward saving. This is intuitively so because wage increase generates income in the period which is likely transferred to other periods.)

- This is for a single period.
- We connect back to the multiperiod model by using the intertemporal arbitrage condition

$$\lambda_{t+1} = \frac{1+\beta}{1+r}\lambda_t$$

(see Browning, Heckman, and Hansen).
- This determines the allocations of the E_t across the periods in a two stage budgeting procedure.
- In those periods where λ_t is high ship resources in (borrow).
- Where it is low, ship out (save).
- As we showed, holding $U = \overline{U}$, $W \uparrow \lambda \downarrow$.
- Tends to create a force for saving in periods of wage increases.
- Possibly offsetting it, is substitution toward consumption when leisure goes down.