Notes on "Assignment Problems and the Location of Economic Activities"

Koopmans and Beckmann, *Econometrica* 25(1), 1957

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Assignment Problem:

- No measure of "skill" or "capital" required. Just output of matches.

- **Perfect Certainty:**
  No transactions costs
  n **workers** n **firms** (not strictly required)
  a homogeneous transferrable output.
  Can match at most one worker to one firm.

- a_{ij} : output of Worker i at Firm j.
- $A = (a_{ij})$ matrix of all possible assignments.
- We solve social planner's problem first to maximize output – then ask if it can be supported by a decentralized pricing function. Any assignment can be written as a **permutation matrix** $P = (P_{ij})$. Each row and column has $n - 1$ zeros and 1 "1."
- Example of $P = (P_{ij})$:

	Firm		
	1	0	0
Worker	0	0	1
	0	1	0

- Worker 1 - Firm 1; Worker 2 - Firm 3; Worker 3 - Firm 2

Value of Total Output in Society

$$V = \sum_i \sum_j P_{ij} a_{ij}$$

- Problem: Find a P_{ij} that maximizes total output.
- Assume $a_{ij} \geq 0$.

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- First: Consider fractional assignment problem.
- We split up fractions of workers and fractions of firms and allocate fractions.

$$\max_{\text{w.r.t. } X_{ij}} \quad \sum_{i,j} a_{ij} X_{ij}$$

- X_{ij} fraction of i assignment to j such that

$$\sum_{j} X_{ij} = 1 \qquad i = 1, \ldots, n$$
$$\sum_{i} X_{ij} = 1 \qquad j = 1, \ldots, n.$$

- $X_{ij} \geq 0$.

- Solution can always be depicted on an "edge".
- Example: Take $n = 2$.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Output is

$$a_{11}X_{11} + a_{12}X_{12} + a_{21}X_{21} + a_{22}X_{22}.$$

- Now

$$X_{11} + X_{12} = 1$$
$$X_{21} + X_{22} = 1$$
$$X_{11} + X_{21} = 1$$
$$X_{12} + X_{22} = 1$$

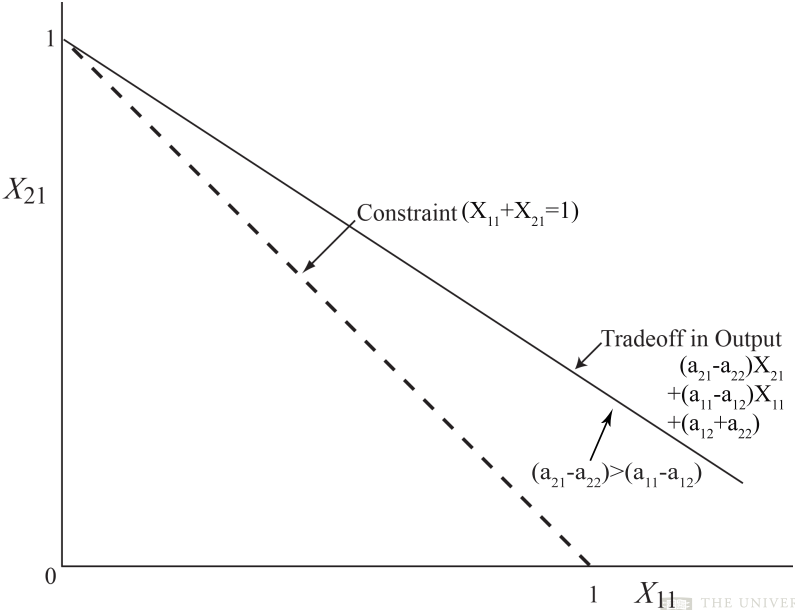
- $\therefore$ output can be written

$$= a_{11}X_{11} + a_{12}(1 - X_{11}) + a_{21}X_{21} + a_{22}(1 - X_{21})$$

$$= \underbrace{(a_{11} - a_{12})}_{\substack{\text{“gain” in output of Worker 1} \\ \text{in Firm 1 relative to working at Firm 2}}} X_{11} + \overbrace{(a_{21} - a_{22})}^{\substack{\text{gain in output of Worker 2} \\ \text{in Firm 1 relative to Firm 2}}} X_{21} + (a_{12} + a_{22})$$

Assume $(a_{21} - a_{22})$

- We obtain the following figure, assuming $a_{21} - a_{22} > a_{11} - a_{12}$.


1
X_{21}
Constraint (X_{11}+X_{21}=1)
Tradeoff in Output
(a_{21}-a_{22})X_{21}
+(a_{11}-a_{12})X_{11}
+(a_{12}+a_{22})
(a_{21}-a_{22})>(a_{11}-a_{12})
0
1
X_{11}

- Constraints $\Rightarrow$ that when $(a_{11} - a_{12}) = -(a_{21} - a_{22})$, and the slope $= 1$, the solution is indeterminate but lies along 45° line $\therefore$ we have solution at extremes as one case.

- Solutions are given at corners generically.
  $\therefore$ Use Linear Programming to solve problem and we get P as generic solution.
- Permute original subscripts so that in equilibrium new labels have Worker i matched with Firm i. From standard duality theory in Linear Programming, we can derive dual prices. (Koopmans and Beckmann).

Theorem:

- There exists a system of wages w_i, $i = 1, \ldots, n$ a system of profits π_j, $j = 1, \ldots, n$, $a_{kk} = \pi_k + w_k$, $k = 1, \ldots, n$ (recall that we have relabeled original subscripts — if necessary — so that firm k is matched with worker k in equilibrium).
- Worker k and Firm i must be able to get less with other assignments than they do in an optimal assignment. In particular,

$$\pi_i \geq a_{ki} - w_k$$

(profits greater for Firm i with Worker i than with other workers given equilibrium assignments).

- On the worker side, we have

$$w_i \geq a_{ik} - \pi_k$$

(wages greater with Firm i than with any other firm).

Theorem:

- Social optimum is decentralizable.
  Competition in the market leads to an optimum.
  Further: Given a set of π^s and w^s that satisfy (a) and (b), **output is maximized**

Proof:

$$V = \sum_{i,k} a_{ik} P_{ik}$$

By hypothesis $\pi_i + w_k \geq a_{ik}$

$$\begin{aligned} V &= \sum_{i,k} P_{ik} a_{ik} \leq \sum_{i,k} P_{ik} (w_i + \pi_k) \\ &= \left(\sum_k P_{ik} \right) \sum_i w_i + \left(\sum_i P_{ik} \right) \sum \pi_k \\ &= \sum w_i + \sum \pi_k = \sum a_{ii} \end{aligned}$$

(using bistochastic nature of permutation matrices, i.e., that rows and columns sum to one).

- **Observe:** Optimization for society does not necessarily imply picking best absolute matches in society.
- **Moreover:** Underlying principle of the problem is *not* comparative advantage as in the Roy Model.
- It is *opportunity cost*, more generally, although comparative advantage can be consistent with opportunity cost.
- **Example 1:**

		Firm 1	**Firm** 2
Worker	1	5	1
Worker	2	6	4

Max Output $= 9$

- Here: Optimum is $1 \rightarrow 1, 2 \longrightarrow 2$
- **Worker 1**: Has comparative advantage in Firm 1 (5/1 vs 6/4)
- **Worker 2**: Has comparative advantage in Firm 2 (1/5 vs 4/6)

Example 2:

- Suppose instead that

		Firm 1	Firm 2
Worker	1	1	9
	2	2	11

Here: Optimum: $1 \to 1, 2 \to 2$

- Worker 1 has a comparative advantage in Firm 2:

$$\frac{9}{1} > \frac{11}{2}$$

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- If both workers could work at the same firm (2) total output = 20.
- This change rules of 1-1 assignment – lets there be unlimited supply of firms. Ruled out in this case (but consistent with the Roy model).
- Worker 1 assigned to Firm 2, gain 8 units, Worker 2 assigned to Firm 1, lose 9 units.
- In second assignment problem, Worker 2 has absolute advantage over Worker 1.

- **Absolute Advantage:** $a_{11} > a_{21}$ Worker 1 better than Worker 2 at each firm.

$$a_{12} > a_{22}.$$

- **Comparative Advantage**: $a_{11}/a_{12} > a_{21}/a_{22}$.
- Worker is more productive in sector 1.
- In the assignment problem, we have that for an optimum allocation

$$a_{11} + a_{22} > a_{12} + a_{21}$$

$$\text{i.e.,} \quad a_{11} - a_{12} > a_{21} - a_{22}.$$

- Neither absolute nor comparative advantage is the controlling principle.
- Basic principle is always **opportunity cost** in economics.

Properties of Equilibrium

- Consider out of equilibrium matches and their associated prices

$$a_{ik} - a_{kk} \leq w_i + \pi_k - (w_k + \pi_k)$$

- The system is supported by wages *alone* and if wages satisfy above

$$(a_{ik} - a_{kk} \leqq w_i - w_k)$$

  can define $\tilde{\pi}_i = a_{ii} - w_i$ that support equilibria.

- Each firm need only know all wages and its net output with all workers – not output of other firms (informational decentralization).

- Observe also: Nonuniqueness of the price equilibrium

$$\pi_k^* = \pi_k - \lambda \qquad w_i^* = w_i + \lambda$$

supports optimum as well for any λ ($\lambda \geq 0; \leq 0$).

- Moreover can tamper (a bit) with individual wages.
- Select λ_k and η_i so that $\pi_k^* = \pi_k - \lambda_k$,

$$i = 1, \ldots, n, \qquad k = 1, \ldots, n$$

- Leads to rent division problem.
- Solved in Sattinger (1979) and classical hedonic models by continuum assumption.

- Observe also that if

$$a_{ik} > a_{kk} \Rightarrow w_i > w_k$$

(Worker i more productive at k than Worker k his wage is higher).

- Obviously if $a_{jm} \geq a_{\ell m} \qquad m = 1, \ldots, k$

$$w_j \geq w_\ell$$

Pairwise j is better than ℓ. If it is true for all pairs then each pair can be ordered. We have an ordinal efficiency scale for workers based on ranks.

- No notion yet of efficiency units: complete ordering defines a kind of ordinal efficiency unit.
- Not a *scale* which entails a sense of cardinality.
- Suppose we postulate a scale for workers:

$$\ell_1 > \ell_2 > \cdots > \ell_n$$

- Another scale for firms:

$$c_1 > c_2 > \cdots > c_n$$

Firm

Worker

$$
\begin{array}{|cccc|}
\hline
a_{11} & a_{12} & \dots & a_{1n} \\
\vdots & & & \vdots \\
a_{n1} & \dots & \dots & a_{nn} \\
\hline
\end{array}
$$

- We can define a function

$$a_{11} = g(\ell_1, c_1), \quad a_{12} = g(\ell_1, c_2)$$

- We might get monotonicity in both arguments. If optimum has best worker with best firm $\Rightarrow$ complementarity in the sense that $\ell_j > \ell_k$, $c_j > c_k$ implies

$$g(\ell_j, c_j) + g(\ell_k, c_k) > g(\ell_j, c_k) + g(\ell_k, c_j) \qquad (*)$$

$$\text{i.e.,} \quad g(\ell_j, c_j) - g(\ell_k, c_j) \geq g(\ell_j, c_k) - g(\ell_k, c_k)$$

- Increments in output between ℓ_j and ℓ_k higher the bigger c.
- **Note, however, nothing in problem defines an order or even requires complementarity or any sorting condition.**

- Conversely, if we have complementarity in this sense, then best worker must be matched with best firm. Then, given complementarity we can meaningfully talk about absolute advantage and it is the controlling principle.
- Recall that, for such a model, *comparative* advantage is not relevant
- Why? Because we have (assuming $g > 0$)

$$\frac{g(\ell_j, c_j)}{g(\ell_j, c_k)} > \frac{g(\ell_k, c_j)}{g(\ell_k, c_k)} \qquad (**)$$

(comparative advantage).
- $(**) \not\Rightarrow (*)$

- Now, if we have that

$$g(\ell_j, c_j) - g(\ell_k, c_j) \leq g(\ell_j, c_k) - g(\ell_k, c_k),$$

  the factors are substitutes. Solution is to match best worker with worst firm.
- To see why, take a 2-person problem:

$$\ell_1 > \ell_2, c_1 > c_2,$$

  but $g(\ell_1, c_1) - g(\ell_2, c_1) \leq g(\ell_1, c_2) - g(\ell_2, c_2)$, i.e., we have $g(\ell_1, c_1) + g(\ell_2, c_2) < g(\ell_1, c_2) + g(\ell_2, c_1)$.
- We get an inverse ordering.

- Example: Cobb Douglas

$$g = \ell c \Rightarrow + \text{ sorting}$$
$$g = \ell / c \Rightarrow - \text{ sorting.}$$

- Comparability of workers not required to define an equilibrium but we have that we get notions of "best" and "worst" – really of only heuristic value.

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- Suppose that the number of workers and number of firms is not equal? Who is unemployed? Assume capital fully employed. Which type of labor is unemployed?
  Take our ordinal efficiency units assumption.

$$\ell_1 > \ell_2 > \cdots > \ell_N > \ell_{N+1} > \ell_{N+2} > etc.$$

$$c_1 > c_2 > \cdots > c_N.$$

Two Cases: They are (A) All workers have the same reservation wage or (B) Reservation wage determined by the match technology with a zero argument for missing partner. They produce the same sorting outcome.

(A) All workers have same reservation wage w_R (What they earn if not working) (Ricardian notion).

- Assume worst worker is laid off. **We show that this is optimal.**
- Assume worst worker paired with worst employed c (complementarity). Then replace worst worker with someone *below* him, e.g., ℓ_{N+1}.
- Total output loss is

$$\begin{aligned} & -\left[g(\ell_N, c_N) - w_R\right] + \left[g(\ell_{N+1}, c_N) - w_R\right] \\ = {} & g(\ell_{N+1}, c_N) - g(\ell_N, c_N) < 0 \end{aligned}$$

- Least productive are the unemployed. (Obviously true if best ℓ work with worst c, i.e., substitute case). What governs this case is the greater productivity of Worker ℓ_N.

(B) Now suppose that the reservation wage comes from *same technology*, i.e., $g(\ell_{N+1}, 0) \neq 0$, $g(0, c_j) \neq 0$ all $N+1$ all c_j.

Then test the previous equilibrium: gain in moving in ℓ_{N+1} in place of ℓ_N:

$$[g(\ell_{N+1}, c_N) - g(\ell_{N+1}, 0)] - [g(\ell_N, c_N) - g(\ell_N, 0)] \leq 0.$$

$\therefore$ lay off worst.

1. Substitutability implies opposite.
2. In that case, you do not employ *best* workers. (Replace c_N with c_1 above to make proof rigorous). ■

Refine Bounds on Wages

- We can refine bounds on wages. (See Sattinger, Factor Pricing in Assignment Problem). See also Shapley and Shubik.

Bounds

- Bounds on wages and an implicit technology: Set $(a_{ij}) = A$ assignment matrix (non-negative elements). Assume it is of rank r. Use the singular value decomposition (spectral decomposition).
- Let A be square (not really needed). See C. R. Rao (1971) $\underset{\sim}{\lambda}$ is matrix of eigenvalues of A. Then we know from linear algebra that there exists

  $$\begin{array}{lll} P & (M \times r) & \lambda(r \times r) \\ Q & M \times r & A = P\lambda Q' \end{array}$$ where columns of P are orthonormal (mutually orthogonal L unit length) true even if

  A is $m \times n$ $\quad m \neq n$

  P is $m \times r$ $\quad \lambda$ is $r \times r$ $\quad Q$ is $n \times r$.
- Note this P is not the permutation matrix previously introduced.

- Unique if all $\lambda_i > 0$. Then

$$a_{ij} = \sum_{k=1}^{r} \lambda_k p_{ik} q_{jk}.$$

- Rank one case is Cobb-Douglas (assuming there exists a cardinal scale)

$$a_{ij} = \ell_i c_j$$

$$A = \left(\overset{n \times 1}{\underset{\sim}{\ell}} \right) \left(\overset{l \times n}{\underset{\sim}{c}} \right)'$$

$$A = (\underset{\sim}{\ell})(\underset{\sim}{c})'.$$

- The spectral decomposition assigns a Cobb-Douglas interaction to each component:

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$$p_{ik} \text{ is quality } k \text{ of Worker } i$$
$$q_{jk} \text{ is quality } k \text{ of Firm } j.$$

- $\therefore$ implicitly we have a sum of Cobb-Douglas technologies in qualities

$$\begin{pmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nr} \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$$
$$\begin{pmatrix} q_{11} & \cdots & q_{n1} \\ \vdots & & \vdots \\ q_{1r} & \cdots & q_{nr} \end{pmatrix}.$$

- Observe $a_{ii} = w_i + \pi_i$. Now $a_{ji} \leq w_j + \pi_i$

$$a_{ii} = w_i + \pi_i$$
$$a_{ji} - a_{ii} \leq w_j - w_i.$$

- $\therefore w_i - w_j \leq a_{ii} - a_{ji}$.

- Similarly we have that

$$a_{ij} - a_{jj} \leq w_i - w_j$$

$$\therefore\ a_{ij} - a_{ii} \leq w_i - w_j \leq a_{ii} - a_{ji}$$

- Now use spectral decomposition

$$a_{ij} - a_{jj} = \sum_{k=1}^{r} \lambda_k \overbrace{(p_{ik} - p_{jk})}^{\substack{\text{difference in traits between} \\ \text{worker at Firm } i \text{ and} \\ \text{worker at Firm } j}} \underset{\uparrow}{q_{jk}}$$

trait k at Firm j

like marginal product

$$\therefore \sum_{k=1}^{r} \lambda_k (p_{ik} - p_{jk}) q_{jk} \leq w_i - w_j \leq \sum_{k=1}^{r} \lambda_k (p_{ik} - p_{jk}) q_{ik}$$

- Left and right hand sides are differential marginal product using Firm j and Firm i^s attributes, respectively. Suppose all firms alike: $q_{jk} = q_{ik}$ (firms possess no identity). Then we get Gorman-Lancaster form of the model

$$w_i - w_j = \sum_{k=1}^{r} \lambda_k (p_{ik} - p_{jk}) q_k$$

(pure factor structure model).
- Otherwise, we get the notion that workers have different productivities depending on properties of firms). (Then characteristics payment will depend on distributions of firms.