Interpreting IV
More On Roy Model

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Identifying Policy Parameters

- Commonly used specifications

$$Y_1 = \mu_1(X) + U_1, \qquad Y_0 = \mu_0(X) + U_0, \qquad C = \mu_C(Z) + U_C, \tag{1}$$

where (X, Z) are observed by the analyst, and U_0, U_1, U_C are unobserved.

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- Define Z to include all of X.
- Variables in Z not in X are instruments.
- $Z \perp\!\!\!\perp (U_0, U_1, U_C) | X$
- $I_D = E(Y_1 - Y_0 - C \mid \mathcal{I}) = \mu_D(Z) - V$
  $\mu_D(Z) = E(\mu_1(X) - \mu_0(X) - \mu_C(Z) \mid \mathcal{I})$
  $V = -E(U_1 - U_0 - U_C \mid \mathcal{I})$.
- Choice equation:
  $$D = 1(\mu_D(Z) > V). \tag{2}$$
- In the early literature that implemented this approach $\mu_0(X)$, $\mu_1(X)$, and $\mu_C(Z)$ were assumed to be linear in the parameters, and the unobservables were assumed to be normal and distributed independently of X and Z.

Useful fact (previously discussed):

$$\text{Choice Probability}: \quad P(z) = \Pr(D = 1 \mid Z = z) = \Pr(\mu_D(z) \geq V) = \Pr\left(\frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V}\right)$$

$$P(z) = F_{\left(\frac{V}{\sigma_V}\right)}\left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$U_D = F_{\left(\frac{V}{\sigma_V}\right)}\left(\frac{V}{\sigma_V}\right); \quad \text{Uniform}(0,1)$$

$$
\begin{aligned}
P(z) &= \Pr\left(F_{\frac{V}{\sigma_V}}\left(\frac{\mu_D(z)}{\sigma_V}\right) \geq F_{\left(\frac{V}{\sigma_V}\right)}\left(\frac{V}{\sigma_V}\right)\right) \\
&= \Pr\left(P(z) \geq U_D\right)
\end{aligned}
$$

$P(z)$ is the $p(z)^{\text{th}}$ quantile of U_D.

- It is also a monotonic transformation of the mean utility $\frac{\mu_D(z)}{\sigma_V}$
- So $P(z)$ is a monotonic transformation of utility

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Recall

$$\begin{aligned} Y &= DY_1 + (1 - D)Y_0 \\ &= Y_0 + D(Y_1 - Y_0) \end{aligned}$$

Keep X implicit (condition on $X = x$)

$$\begin{aligned} E(Y \mid Z = z) &= E(Y_0) + \underbrace{E(Y_1 - Y_0 \mid D = 1, Z = z)P(z)}_{\text{from law of iterated expectations}} \\ &= E(Y_0) + E(Y_1 - Y_0 \mid P(z) \geq U_D)P(z) \end{aligned}$$

$\therefore$ It depends on Z only through $P(Z)$.

$$E(Y \mid Z = z') = E(Y_0) + E(Y_1 - Y_0 \mid P(z') \geq U_D)P(z')$$

Index Sufficiency

- **Question:** Why? Under what conditions?

- What is $E(Y_1 - Y_0 \mid P(z) \geq U_D)$? (Treatment on the treated)

Derivation

- Let the joint density of $(Y_1 - Y_0, U_D)$ be

$$f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D).$$

- It does not depend on Z.
- It may, in general, depend on X.
- $$E(Y_1 - Y_0 \mid P(z) \geq U_D) = \frac{\int_{-\infty}^{\infty} \int_{0}^{P(z)} (y_1 - y_0) f_{y_1 - y_0, u_D}(y_1 - y_0, u_D)\, du_D d(y_1 - y_0)}{\Pr(P(z) \geq U_D)}$$

- Recall that

$$U_D = F_{\left(\frac{V}{\sigma_V}\right)}\left(\frac{V}{\sigma_V}\right).$$

- U_D is a quantile of the V/σ_V distribution.

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- By construction, U_D is Uniform$(0,1)$ (this is the definition of a quantile).
- $\therefore f_{U_D}(u_D) = 1$.
- Also, $\Pr(P(z) \geq U_D) = P(z)$.
- Notice, by law of conditional probability,

$$f_{Y_1-Y_0,U_D}(y_1 - y_0, u_D) = f_{Y_1-Y_0,U_D}(y_1 - y_0 \mid U_D = u_D) \underbrace{f_{U_D}(u_D)}_{=1}.$$

$$E(Y_1 - Y_0 \mid P(z) \geq U_D)$$

$$= \frac{\int_0^{P(z)} \int_{-\infty}^{\infty} (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D)\, d(y_1 - y_0)\, du_D}{P(z)}$$

$$E(Y_1 - Y_0 \mid P(z) \geq U_D)$$

$$= \frac{\int_0^{P(z)} \int_{-\infty}^{\infty} (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0 \mid U_D = u_D)\, d(y_1 - y_0)\, du_D}{P(z)}$$

$$= \frac{\int_0^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D)\, du_D}{P(z)}$$

- Definition: $E(Y_1 - Y_0 \mid U_D = u_d)$ is marginal treatment effect (MTE)
- If $P(z) = U_d$, agent with $Z = z$ is indifferent between "0" and "1."

$$\therefore E(Y \mid Z = z) = E(Y_0) + \int_0^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

$$\frac{\partial E(Y \mid Z = z)}{\partial P(z)} = \underbrace{E(Y_1 - Y_0 \mid U_D = P(z))}_{\substack{\text{EOTM or marginal gains for} \\ \text{people with } U_D = P(z)}}$$

$$E(Y \mid Z = z') = E(Y_0) + \int_0^{P(z')} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

- Consider mean of Y for two different values of Z
- Suppose $P(z) > P(z')$

$$\therefore E(Y \mid Z = z) - E(Y \mid Z = z') =$$

$$= \int_{P(z')}^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D) du_D$$

$$= E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z')) \Pr(P(z) \geq U_D \geq P(z'))$$

Notice

$$\Pr(P(z) \geq U_D \geq P(z')) = \int_{P(z')}^{P(z)} du_D$$

$$= P(z) - P(z')$$

$$E(Y \mid Z = z) - E(Y \mid Z = z')$$

$$= E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z'))(P(z) - P(z'))$$

- This is LATE: will see why in next slides

$$\frac{E(Y \mid Z = z) - E(Y \mid Z = z')}{P(z) - P(z')} = \text{LATE}(z, z')$$

$$= \frac{\int_{P(z')}^{P(z)} \text{MTE}(u_D) du_D}{P(z) - P(z')}$$