Ability Bias, Errors in Variables and Sibling Methods: Background

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Ability Bias

- Consider the model:

$$\log y_{it} = \alpha_0 + \alpha_1 S_i + U_{it}$$

  where y_{it} = income, S_i = schooling, and α_0 and α_1 are parameters of interest.
- Omitted from the above specification is unobserved ability, which is captured in the residual term U_{it}.

- We thus re-write the above as:

$$\log y_{it} = \alpha_0 + \alpha_1 S_i + a_i + \varepsilon_{it}$$

  where a_i is ability, $(\varepsilon_{it}, \varepsilon_{i't}) \perp\!\!\!\perp (S_i, S_{i'})$, and we believe that $Cov(a_i, S_i) \neq 0$.
- S_i is schooling of a sibling (could be a twin).
- Thus, $E(U_{it} \mid S_i) \neq 0$, so that OLS on our original specification gives biased and inconsistent estimates.

Strategies for Estimation

1. *Use proxies for ability*: Find proxies for ability and include them as regressors. Examples may include: height, weight, etc. The problem with this approach is that proxies may measure ability with error and thus introduce additional bias (see Section 9).
2. *Fixed Effect Method*: Find a paired comparison. Examples may include a genetic twin or sibling with similar or identical ability. Consider two individuals i and i':

$$\begin{aligned}\log y_{it} - \log y_{i't} &= (\alpha_0 + \alpha_1 S_i + U_{it}) - (\alpha_0 + \alpha_1 S_{i'} + U_{i't}) \\ &= \alpha_1 (S_i - S_{i'}) + (a_i - a_{i'}) + (\varepsilon_{it} - \varepsilon_{i't})\end{aligned}$$

Note: if $a_i = a_{i'}$, then OLS performed on our fixed effect estimator is unbiased and consistent.

- If $a_i \neq a_{i'}$, then we just get a different bias (see Section 7). Further, if S_i is measured with error, we may exacerbate the bias in our fixed effect estimator (see Section 9).

OLS vs. Fixed Effect (FE)

- In the *OLS* case with ability bias, we have:

$$\text{plim } (\alpha_1^{OLS}) = \alpha_1 + \frac{Cov(a,S)}{Var(S)}$$

- We also impose:

$$\begin{aligned} Var(S) &= Var(S^{'}) \\ Cov(a,S) &= Cov(a',S') \\ Cov(a',S) &= Cov(a,S') \end{aligned}$$

- With these assumptions, our fixed effect estimator is given by:

$$\begin{aligned}\text{plim } \alpha_1^{FE} &= \alpha_1 + \frac{Cov\left(S - S', (a - a') + (\varepsilon - \varepsilon')\right)}{Var(S - S')} \\ &= \alpha_1 + \frac{Cov(a, S) - Cov(a', S)}{Var(S) - Cov(S, S')}.\end{aligned}$$

Note that if $Cov(a', S) = 0$, and ability is positively correlated with schooling, then the fixed effect estimator is upward biased. From the preceding, we see that the fixed effect estimator has more asymptotic bias if:

$$\frac{Cov(a, S) - Cov(a', S)}{Var(S) - Cov(S, S')} > \frac{Cov(a, S)}{Var(S)}$$

$$\Rightarrow \frac{Cov(a, S)}{Var(S)} > \frac{Cov(a', S)}{Cov(S, S')}.$$

Prove.

Measurement Error

- Say $S^* = S + \nu$, where S^* is observed schooling.
- Our model now becomes:

$$\log y = \alpha_0 + \alpha_1 S + U = \alpha_0 + \alpha_1 S^* + (a + \varepsilon - \alpha_1 \nu)$$

and the fixed effect estimator gives:

$$\begin{aligned} \log y - \log y' &= (\alpha_0 + \alpha_1 S + U) - (\alpha_0 + \alpha_1 S' + U') \\ &= \alpha_1 (S^* - S^{*'}) + (U - U') + \alpha_1 (\nu' - \nu) \end{aligned}$$

- Now we wish to examine which estimator (*OLS* or fixed effect), has more asymptotic bias given our measurement error problem.
- For the remaining arguments of this section, we assume:

$$E(\nu \mid S) = E(\nu' \mid S) = E(\nu \mid \nu') = 0$$

so that the *OLS* estimator gives:

$$\begin{aligned} \text{plim } \alpha_1^{OLS} &= \alpha_1 + \frac{Cov(S^*, a + \varepsilon - \alpha_1 \nu)}{Var(S^*)} \\ &= \alpha_1 + \frac{Cov(a, S) - \alpha_1 Var(\nu)}{Var(S) + Var(\nu)}. \end{aligned}$$

- The fixed effect estimator gives:

$$
\begin{aligned}
\text{plim } \alpha_1^{FE} &= \alpha_1 + \frac{Cov\left(S^* - S^{*'}, (U - U') + \alpha_1(\nu' - \nu)\right)}{Var(S^* - S^{*'})} \\
&= \alpha_1 + \frac{Cov\left((S - S'), (a - a')\right) - \alpha_1 Var(\nu' - \nu)}{Var(S - S') + Var(\nu' - \nu)} \\
&= \alpha_1 + \frac{Cov(a, S) - Cov(a, S') - \alpha_1 Var(\nu)}{Var(S) + Var(\nu) - Cov(S', S)}.
\end{aligned}
$$

- Under what conditions will the fixed effect bias be greater?
- From the above, we know that this will be true if and only if:

$$\frac{Cov(a,S) - Cov(a,S') - \alpha_1 Var(\nu)}{Var(S) + Var(\nu) - Cov(S',S)} > \frac{Cov(a,S) - \alpha_1 Var(\nu)}{Var(S) + Var(\nu)}$$

$$\Rightarrow Cov(a,S')\left(Var(S) + Var(\nu)\right) >$$

$$\left(\alpha_1 Var(\nu) - Cov(a,S)\right) Cov(S',S)$$

$$\Rightarrow \frac{Cov(a,S) - \alpha_1 Var(\nu)}{Var(S) + Var(\nu)} > \frac{Cov(a,S')}{Cov(S',S)}.$$

- If this inequality holds, taking differences can actually worsen the fit over *OLS* alone.
- Intuitively, we see that we have differenced out the true component, S, and compounded our measurement error problem with the fixed effect estimator.

- In the special case $a = a'$, the condition is

$$\frac{-\alpha_1 Var(\nu)}{Var(S) + Var(\nu) - Cov(S', S)} > \frac{Cov(a, S) - \alpha_1 Var(\nu)}{Var(S) + Var(\nu)}.$$

Errors in Variables

The Model

- Suppose that the equation for earnings is given by:

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + U_t$$

where $E\left(U_t \mid X_{1t}, X_{2t}\right) = 0 \;\forall\; t, t'$.

- Also define:

$$X^*_{1t} = X_{1t} + \varepsilon_{1t} \quad \text{and} \quad X^*_{2t} = X_{2t} + \varepsilon_{2t}.$$

- Here, X_{1t}^* and X_{2t}^* are observed and measure X_{1t} and X_{2t} with error.
- We also impose that $X_i \perp\!\!\!\perp \varepsilon_j \; \forall \; i, j$.
- So, our initial model can be equivalently re-written as:

$$Y_t = X_{1t}^* \beta_1 + X_{2t}^* \beta_2 + (U_t - \varepsilon_{1t}\beta_1 - \varepsilon_{2t}\beta_2).$$

- Finally, by assumed independence of X and ε, we write:

$$\Sigma_{x^*} = \Sigma_x + \Sigma_\epsilon.$$

McCallum's Problem

- **Question:** Is it better for estimation of β_1 to include other variables measured with error? Suppose that X_{1t} is not measured with error, in the sense that $\varepsilon_{1t} = 0$, while X_{2t} is measured with error. Below, we consider both excluding and including X_{2t}, and investigate the asymptotic properties of both cases.

Excluded X_{2t}

- The equation for earnings with omitted X_2 is:

$$y = X_1\beta_1 + (U + X_2\beta_2)$$

- Therefore, by arguments similar to those in the appendix, we know:

$$\text{plim } \tilde{\beta}_1 = \beta_1 + \frac{\sigma_{12}}{\sigma_{11}}\beta_2. \tag{1}$$

- Here, σ_{12} is the covariance between the regressors, and σ_{11} is the variance of X_1.
- Before moving on to a more general model for the inclusion of X_{2t}, let us first consider the classical case for including both variables.

- Suppose

$$\Sigma_{\epsilon} = \begin{bmatrix} \sigma_{11}^{*} & 0 \\ 0 & \sigma_{22}^{*} \end{bmatrix}, \Sigma_{x} = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}.$$

- We know that:

$$\text{plim } \hat{\beta} = \left[I - (\Sigma_{x^*})^{-1} (\Sigma_{\epsilon}) \right] \beta \qquad (2)$$

where the coefficient and regressor vectors have been stacked appropriately (see Appendix for derivation).

- Note that Σ_{ϵ} represents the variance-covariance matrix of the measurement errors, and Σ_{x} is the variance-covariance matrix of the regressors.

Derivation of Equation (2)

- We can write

$$y_t = x^* \beta + (U_t - \epsilon_{1t}\beta_1 - \epsilon_{2t}\beta_2),$$

where:

$$x^* = \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

and x_1^*, x_2^*, are $T \times 1$.

- So:

$$
\begin{aligned}
\hat{\beta}^{OLS} &= \left(x^{*'}x^{*}\right)^{-1}(x^{*'}y) \\
&= \beta + \left(x^{*'}x^{*}\right)^{-1}\left(x^{*'}(U - \epsilon_1\beta_1 - \epsilon_2\beta_2)\right) \\
&= \beta + \left(\frac{\left(x^{*'}x^{*}\right)}{T}\right)^{-1} \\
&\qquad \times \left(\left(\frac{x^{*'}U}{T}\right) - \left(\frac{x^{*'}\epsilon_1\beta_1}{T}\right) - \left(\frac{x^{*'}\epsilon_2\beta_2}{T}\right)\right) \\
&\rightarrow \beta + \left(E\left(x^{*'}x^{*}\right)\right)^{-1} \\
&\qquad \times \left(E\left(x^{*'}U\right) - E\left(x^{*'}\epsilon_1\right)\beta_1 - E\left(x^{*'}\epsilon_2\right)\beta_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \beta - \begin{bmatrix} E\left(x_1^{*'}x_1^*\right) & E\left(x_1^{*'}x_2^*\right) \\ E\left(x_2^{*'}x_1^*\right) & E\left(x_2^{*'}x_2^*\right) \end{bmatrix}^{-1} \\
&\qquad \times \left(E\begin{bmatrix} x_1^{*'}\epsilon_1 \\ x_2^{*'}\epsilon_1 \end{bmatrix}\beta_1 + E\begin{bmatrix} x_1^{*'}\epsilon_2 \\ x_2^{*'}\epsilon_2 \end{bmatrix}\beta_2 \right) \\
&= \beta - \begin{bmatrix} E\left(x_1^{*'}x_1^*\right) & E\left(x_1^{*'}x_2^*\right) \\ E\left(x_2^{*'}x_1^*\right) & E\left(x_2^{*'}x_2^*\right) \end{bmatrix}^{-1} \\
&\qquad \times \begin{bmatrix} E\left(x_1^{*'}\epsilon_1\right) & E\left(x_1^{*'}\epsilon_2\right) \\ E\left(x_2^{*'}\epsilon_1\right) & E\left(x_2^{*'}\epsilon_2\right) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(I - (\Sigma_{x^*})^{-1} \left[\begin{array}{cc} E\left(\left(\varepsilon_1^{'} + x_1^{'}\right)\epsilon_1\right) & E\left(\left(\varepsilon_1^{'} + x_1^{'}\right)\epsilon_2\right) \\ E\left(\left(\varepsilon_2^{'} + x_2^{'}\right)\epsilon_1\right) & E\left(\left(\varepsilon_2^{'} + x_2^{'}\right)\epsilon_2\right) \end{array} \right] \right) \\
&\qquad \times \left[\begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] \\
&= \left(I - (\Sigma_{x^*})^{-1} (\Sigma_{\varepsilon}) \right) \beta,
\end{aligned}
$$

where the second-to-last step follows from the independence of x and ε. This type of argument is also used to derive the probability limit of the β's in section 2.

- Straightforward computations thus give:

$$
\begin{aligned}
&\text{plim } \hat{\beta} \\
&= \left[I - \begin{bmatrix} \sigma_{11} + \sigma_{11}^* & 0 \\ 0 & \sigma_{22} + \sigma_{22}^* \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11}^* & 0 \\ 0 & \sigma_{22}^* \end{bmatrix} \right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\sigma_{11}}{\sigma_{11} + \sigma_{11}^*} & 0 \\ 0 & \dfrac{\sigma_{22}}{\sigma_{22} + \sigma_{22}^*} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.
\end{aligned}
$$

Included X_{2t}

- In McCallum's problem we suppose that $\sigma_{12}^* = 0$.
- Further, as X_{1t} is not measured with error, $\sigma_{11}^* = 0$.
- Substituting this into equation 2 yields:

$$\text{plim } \hat{\beta} = \beta - \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} + \sigma_{22}^* \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22}^* \end{bmatrix} \beta$$

- With a little algebra, the above gives:

$$\begin{aligned}
\text{plim } \hat{\beta}_1 &= \beta_1 + \beta_2 \left(\frac{\sigma_{12}}{\sigma_{11}} \right) \left(\frac{\sigma_{22}^*}{\sigma_{22} + \sigma_{22}^* - \frac{\sigma_{12}^2}{\sigma_{11}}} \right) \\
&= \beta_1 + \beta_2 \left(\frac{\sigma_{12}}{\sigma_{11}} \right) \left(\frac{\sigma_{22}^*}{\sigma_{22} \left(1 - \rho_{12}^2\right) + \sigma_{22}^*} \right)
\end{aligned}$$

  where ρ_{12}^2 is simply the correlation coefficient, $\frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}$. Further, we know that:

$$0 < \rho_{12}^2 < 1$$

  so including X_{2t} results in less asymptotic bias (inconsistency).
- (We get this result by comparing the above with the bias from excluding X_{2t} in section 18, the result captured in equation (1)).
- So, we have justified the kitchen sink approach. This result

General Case

- In the most general case, we have:

$$\text{plim } \hat{\beta} = \beta - \left(\Sigma_{x^*}\right)^{-1} \Sigma_{\varepsilon}\beta$$

$$= \beta - \begin{bmatrix} \sigma_{11} + \sigma_{11}^* & \sigma_{12} + \sigma_{12}^* \\ \sigma_{12} + \sigma_{12}^* & \sigma_{22} + \sigma_{22}^* \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

- With a little algebra we find:

$$\det(\Sigma_{x^*}) = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{22}^* + \sigma_{11}^*\sigma_{22} + \sigma_{11}^*\sigma_{22}^* - \sigma_{12}^{*^2} - 2\sigma_{12}\sigma_{12}^* - \sigma_{12}^2$$

- Therefore:

$$\text{plim } \hat{\beta} = \beta - \frac{1}{\det(\Sigma_{x^*})} \begin{bmatrix} \sigma_{22} + \sigma_{22}^* & -(\sigma_{12} + \sigma_{12}^*) \\ -(\sigma_{12} + \sigma_{12}^*) & \sigma_{11} + \sigma_{11}^* \end{bmatrix} \times \begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

- Supposing $\sigma_{12}^* = 0$, we get:

$$\det(\tilde{\Sigma}_{x^*}) = \det(\Sigma_{x^*}) \,|_{\sigma_{12}^* = 0}$$
$$= \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{22}^* + \sigma_{11}^*\sigma_{22} + \sigma_{11}^*\sigma_{22}^* - \sigma_{12}^2$$

- And thus:

$$\text{plim } \hat{\beta} = \beta - \begin{bmatrix} \frac{(\sigma_{22}+\sigma_{22}^*)\sigma_{11}^*}{\det(\tilde{\Sigma}_{x^*})} & \frac{-\sigma_{12}\sigma_{22}^*}{\det(\tilde{\Sigma}_{x^*})} \\ \frac{-\sigma_{11}^*\sigma_{12}}{\det(\tilde{\Sigma}_{x^*})} & \frac{(\sigma_{11}+\sigma_{11}^*)\sigma_{22}^*}{\det(\tilde{\Sigma}_{x^*})} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

- Note that if $\beta_2\sigma_{12} < 0$, *OLS* may not be downward biased for β_1.
- If $\beta_2 = 0$, we get:

$$\text{plim } \hat{\beta}_2 = \frac{\beta_1\sigma_{12}\sigma_{11}^*}{\det(\tilde{\Sigma}_{x^*})}$$

  so, if X_2 were a race variable and blacks get lower quality schooling, (where schooling is measured by X_{1t},) then $\sigma_{12} < 0$, and hence $\hat{\beta}_2 < 0$.
- This would be a finding in support of labor market discrimination.

The Kitchen Sink Revisited

- McCallum's analysis suggests that one should toss in a variable measured with error if there is no measurement error in X_{1t}.
- But suppose that there is measurement error in X_{1t}.
- Is it still better to include the additional variable measured with error as a regressor?
- We proceed by imposing $\beta_2 = 0$.
- (i) **Excluded $\boldsymbol{X}_{2t}$.**
- The equation for earnings with measurement error in X_1 and excluded X_2 is:

$$
\begin{aligned}
y &= \left(X_1^* + \varepsilon_1\right)\beta_1 + \left(U + X_2\beta_2\right) \\
&= X_1^*\beta_1 + \left(U + X_2\beta_2 + \beta_1\varepsilon_1\right)
\end{aligned}
$$

- Therefore:

$$\operatorname{plim}\ \tilde{\beta}_1 = \beta_1 - \beta_1\left(\frac{\sigma_{11}^*}{\sigma_{11}+\sigma_{11}^*}\right) = \beta_1\left(\frac{\sigma_{11}}{\sigma_{11}+\sigma_{11}^*}\right) = \beta_1\left(\frac{1}{1+\frac{\sigma_{11}^*}{\sigma_{11}}}\right) \tag{3}$$

- (ii) **Included $\boldsymbol{X}_{2t}$.**

- From our analysis in the General Case, we know that:

$$\text{plim } \hat{\beta}_1 = \beta_1 \left(\frac{(\sigma_{22} + \sigma_{22}^{*})\, \sigma_{11} - \sigma_{12}^{2}}{\det(\tilde{\Sigma}_{x^*})} \right). \tag{4}$$

- If $\sigma^*_{22} = 0$, so that X_{2t} is not measured with error:

$$\begin{aligned} \text{plim } \hat{\beta}_1 &= \beta_1 \left(\frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2 + \sigma^*_{11}\sigma_{22}} \right) \\ &= \beta_1 \left(\frac{1 - \rho_{12}^2}{1 - \rho_{12}^2 + \frac{\sigma^*_{11}}{\sigma_{11}}} \right). \end{aligned} \tag{5}$$

- Comparing eqn (4) and eqn (5), we see that adding the variable measured without error always exacerbates the bias.

- For, the bias in the excluded case will be smaller if:

$$\beta_1 \left(\frac{1}{1 + \frac{\sigma_{11}^*}{\sigma_{11}}} \right) > \beta_1 \left(\frac{1 - \rho_{12}^2}{1 - \rho_{12}^2 + \frac{\sigma_{11}^*}{\sigma_{11}}} \right)$$

$$\iff \left(1 - \rho_{12}^2 + \frac{\sigma_{11}^*}{\sigma_{11}} \right) > \left(1 + \frac{\sigma_{11}^*}{\sigma_{11}} \right) (1 - \rho_{12}^2)$$

$$\iff 0 > -\rho_{12}^2 \frac{\sigma_{11}^*}{\sigma_{11}}.$$

which is always the case, provided $\rho_{12}^2 > 0$.

- (Note that the coefficients on β_1 for both the excluded and included case are less than one.
- So, the larger coefficient is the one with less bias, as stated above.)

- Now suppose that $\sigma_{22}^* > 0$, so that both variables are measured with error.
- Then:

$$\begin{aligned}
\text{plim } \hat{\beta}_1 &= \beta_1 \left(\frac{(\sigma_{22} + \sigma_{22}^*)\,\sigma_{11} - \sigma_{12}^2}{\det(\tilde{\Sigma}_{x^*})} \right) \\
&= \beta_1 \left(\frac{1 + \dfrac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2}{1 + \dfrac{\sigma_{11}^*}{\sigma_{11}} + \dfrac{\sigma_{11}^*}{\sigma_{11}} \dfrac{\sigma_{22}^*}{\sigma_{22}} + \dfrac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2} \right).
\end{aligned}$$

- Intuitively, adding measurement error in X_{2t} can only worsen the bias, and thus exclusion should again be preferred to inclusion.

- Formally, including X_{2t} gives more bias if and only if:

$$\beta_1 \left(\frac{1 + \frac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2}{1 + \frac{\sigma_{11}^*}{\sigma_{11}} + \frac{\sigma_{11}^*}{\sigma_{11}} \frac{\sigma_{22}^*}{\sigma_{22}} + \frac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2} \right) < \beta_1 \left(\frac{1}{1 + \frac{\sigma_{11}^*}{\sigma_{11}}} \right)$$

$$\iff \left(1 + \frac{\sigma_{11}^*}{\sigma_{11}}\right) \left(1 + \frac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2\right)$$

$$< \left(1 + \frac{\sigma_{11}^*}{\sigma_{11}} + \frac{\sigma_{11}^*}{\sigma_{11}} \frac{\sigma_{22}^*}{\sigma_{22}} + \frac{\sigma_{22}^*}{\sigma_{22}} - \rho_{12}^2\right)$$

$$\iff -\rho_{12}^2 \frac{\sigma_{11}^*}{\sigma_{11}} < 0.$$

- Thus, provided $\rho_{12}^2 > 0$, including X_{2t} results in more bias than excluding it.
- If $\rho_{12}^2 = 0$, the bias from including X_{2t} is obviously seen to be:

$$\beta_1 \left(\frac{1 + \frac{\sigma_{22}^*}{\sigma_{22}}}{1 + \frac{\sigma_{11}^*}{\sigma_{11}} + \frac{\sigma_{11}^*}{\sigma_{11}}\frac{\sigma_{22}^*}{\sigma_{22}} + \frac{\sigma_{22}^*}{\sigma_{22}}} \right) = \beta_1 \left(\frac{1 + \frac{\sigma_{22}^*}{\sigma_{22}}}{\left(1 + \frac{\sigma_{22}^*}{\sigma_{22}}\right)\left(1 + \frac{\sigma_{11}^*}{\sigma_{11}}\right)} \right)$$

$$= \beta_1 \left(\frac{1}{1 + \frac{\sigma_{11}^*}{\sigma_{11}}} \right)$$

so that including and excluding X_{2t} yields the same result.

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- Finally, from the General Case section, we have:

$$\text{plim } \hat{\beta}_1 = \frac{\beta_1 \left(\sigma_{22} + \sigma_{22}^*\right) \sigma_{11} - \sigma_{12}^2 + \beta_2 \left(\sigma_{12}\sigma_{22}^*\right)}{\sigma_{11}\sigma_{22} - \sigma_{12}^2 + \sigma_{11}^*\sigma_{22}^* + \sigma_{11}^*\sigma_{22} + \sigma_{11}\sigma_{22}^*}.$$

- L'Hôpital's rule on the above shows that:

$$\begin{aligned} \sigma_{11}^* \quad &\longrightarrow \quad \infty \lim \left(\text{plim } \hat{\beta}_1\right) = 0, \text{ and} \\ \lim_{\sigma_{22}^* \to \infty} \left(\text{plim } \hat{\beta}_1\right) \quad &= \quad \frac{\beta_1\sigma_{11} + \beta_2\sigma_{12}}{\sigma_{11} + \sigma_{11}^*} \\ &= \quad \frac{\beta_1\sigma_{11}}{\sigma_{11} + \sigma_{11}^*} + \frac{\beta_2\sigma_{12}}{\sigma_{11} + \sigma_{11}^*}. \end{aligned}$$

Appendix

Twin Methods

Basic Principle: Monozygotic or MZ (identical) twins are more similar than Dizygotic or DZ (fraternal) twins. The key assumption is that if environmental factors are the same for both types of twins, then we can estimate genetic components to outcomes.

Univariate Twin Model

- Let $y =$ observed phenotypic variable, $x =$ unobserved genotype, and $u =$ environment.
- Further, suppose that we can write our model additively:

$$y = x + u$$

and assume independence of x and u so that $\sigma_y^2 = \sigma_x^2 + \sigma_u^2$.

- Now suppose that we have data on another individual:

$$y^* = x^* + u^*$$

- Then our phenotypic covariance is:

$$Cov\left(y, y^*\right) = Cov\left(x, x^*\right) + Cov\left(u, u^*\right)$$

where we are imposing the assumption:

$$Cov\left(x, u^*\right) = Cov(x^*, u) = 0.$$

- Defining standardized forms and some simplifying notation, let

$$\tilde{y} \equiv \frac{y}{\sigma_y}, \quad \tilde{x} \equiv \frac{x}{\sigma_x}, \quad \tilde{u} \equiv \frac{u}{\sigma_u}, h^2 \equiv \frac{\sigma_x^2}{\sigma_y^2}, \quad \rho^2 \equiv \frac{\sigma_u^2}{\sigma_y^2}$$

Thus, $\tilde{y}\sigma_y = \tilde{x}\sigma_x + \tilde{u}\sigma_u$ which implies $\tilde{y} = h\tilde{x} + \rho\tilde{u}$. We can also derive the identity:

$$h^2 + \rho^2 = \frac{\sigma_x^2}{\sigma_y^2}. + \frac{\sigma_u^2}{\sigma_y^2} = 1$$

where the last step follows from our assumption of independence.

- Now we wish to consider the correlation between observed phenotypes of our two individuals:

$$
\begin{aligned}
C &= Corr(y, y^*) \\
&= Corr(h\tilde{x} + p\tilde{u}, h\tilde{x}^* + \rho\tilde{u}^*) \\
&= h^2 \frac{Cov(\tilde{x}, \tilde{x}^*)}{Var(\tilde{x})} + \rho^2 \frac{Cov(\tilde{u}, \tilde{u}^*)}{Var(\tilde{u})} \\
&= h^2 g + \rho^2 \nu
\end{aligned}
$$

say, with g and ν defined as above.
- We assume that $g_{MZ} = 1$ and that $g_{DZ} < 1$.
- That is, the genotypic variable is perfectly correlated among identical twins, but less than perfectly correlated among fraternal twins.
- Replacing this result into the above produces:

$$
\begin{aligned}
C_{MZ} &= h^2 + \nu_{MZ}\rho^2 \\
C_{DZ} &= h^2 g_{DZ} + \nu_{DZ}\rho^2
\end{aligned}
$$

- Therefore:

$$\begin{aligned} C_{MZ} - C_{DZ} &= (1 - g_{DZ})h^2 + (\nu_{MZ} - \nu_{DZ})\rho^2 \\ &= (1 - g_{DZ})h^2 + (\nu_{MZ} - \nu_{DZ})(1 - h^2) \end{aligned}$$

where the last equality follows from our established identity.

- Solving for h^2, we find:

$$h^2 = \frac{(C_{MZ} - C_{DZ}) - (\nu_{MZ} - \nu_{DZ})}{(1 - g_{DZ}) - (\nu_{MZ} - \nu_{DZ})}.$$

- The only known in the right hand side of the above equality is the expression $(C_{MZ} - C_{DZ})$, which is simply the correlation coefficient of the observed phenotypic variable.
- The remaining two expressions, $(1 - g_{DZ})$ and $(\nu_{MZ} - \nu_{DZ})$ can not be computed as they represent statistics on variables we don't observe.
- One could impose $\nu_{MZ} = \nu_{DZ}$ so that:

$$h^2 = \frac{C_{MZ} - C_{DZ}}{1 - g_{DZ}}.$$

- The expression g_{DZ} is a measure of how closely the genetic variable is correlated across our two observations.
- One could then guess or estimate a value for this parameter to derive corresponding estimates of h^2, the ratio of how much variance in the phenotypic variable is explained by variance in the genetic component.
- Other studies have attempted to include $Cov(x, u) \neq 0$ but this presents an identification problem.