Notes on Identification of the Roy Model and the Generalized Roy Model

James Heckman
University of Chicago

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Roy Model

(Y_0, Y_1) potential outcomes

$I^* = Y_1 - Y_0$ choice **index**

Observe Y_1 if $Y_1 \geq Y_0$.

Observe Y_0 if $Y_1 < Y_0$.

Cannot simultaneously observe Y_0 and Y_1.

We can conduct an identification analysis assuming we know

$$I = \frac{I^*}{\sigma_{Y_1 - Y_0}} = \frac{Y_1 - Y_0}{\sigma_{Y_1 - Y_0}}$$

for each person where $D = \mathbf{1}(I > 0)$.

Why do we know this? Conditions established in the literature

[Source: Cosslett (1983), Manski (1988), Matzkin (1992)]

We observe (Y_0, D) and (Y_1, D). We never observe the full triple (Y_0, Y_1, D) for anyone.

- Under conditions specified in the literature, $F(Y_0, I|X, Z)$ and $F(Y_1, I|X, Z)$ are identified where:

$$Y_0 = \mu_0(X) + U_0 \quad E(Y_0 \mid X) = \mu_0(X) \tag{1}$$
$$Y_1 = \mu_1(X) + U_1 \quad E(Y_1 \mid X) = \mu_1(X) \tag{2}$$
$$I^* = \mu_I(X, Z) + U_I \tag{3}$$
$$I = \frac{\mu_I(X, Z)}{\sigma_{U_I}} + \frac{U_I}{\sigma_{U_I}} \tag{4}$$

- Assume $(X, Z) \perp\!\!\!\perp (U_0, U_1, U_I)$.
- Source: Heckman (1990), Heckman and Honoré (1990)
- The key idea in these papers is “sufficient” variation in Z holding X fixed.

Identifying the Index Choice Probability

- From the left-hand side of

$$\Pr(D = 1|X, Z) = \Pr(\mu_I(X, Z) + U_I \geq 0|X, Z),$$

we can identify the distribution of $\frac{U_I}{\sigma_{U_I}}$, as well as $\frac{\mu_I(X,Z)}{\sigma_{U_I}}$.

- Just invert known f_{U_I} to establish $\frac{\mu_I(X,Z)}{\sigma_I}$. **Prove**.
- This is true under normality or for assumed functional forms for the distribution of $\frac{U_I}{\sigma_{U_I}}$.
- Also, we do not have to assume the distribution of U_I is known or that the functional form of $\mu_I(X, Z)$ is linear, e.g. $\mu_I(X, Z) = X\beta_I + Z\gamma_I$.
- See the conditions in the Matzkin (1992) paper and the survey in Matzkin, 2007, *Handbook of Econometrics*.

- Suppose U_I is symmetric around zero:

$$\Pr(D = 1|X, Z) = \int_{-\mu_I(X,Z)}^{\infty} f(U_I) dU_I$$
$$= 1 - F_{U_I}\left(\frac{\mu_I(X, Z)}{\sigma_{U_I}}\right)$$
$$\Rightarrow F_{U_I}{}^{-1}[1 - \Pr(D = 1|X, Z)] = \frac{\mu_I(X, Z)}{\sigma_{U_I}}$$

- Can recover $\mu_I(X, Z)$ nonparametrically

- Suppose functional form of distribution unknown?
- To approach this, use the following:

$$\begin{aligned}\Pr(D = 1|X, Z) &= \Pr(U_I \geq -\mu_I(X, Z)) \qquad (**)\\ &= \int_{-\mu_I(X,Z)}^{\infty} f(U_I) dU_I\end{aligned}$$

- Suppose $\mu_I(X,Z)$ differentiable in Z.
- Z has 2 (or more) elements.

$$\frac{\frac{\partial \Pr(D=1|X,Z)}{\partial Z_1}}{\frac{\partial \Pr(D=1|X,Z)}{\partial Z_2}} = \frac{\left(\frac{\partial \mu_I(X,Z)}{\partial Z_1}\right) f_{U_I}(\mu_I(X,Z))}{\left(\frac{\partial \mu_I(X,Z)}{\partial Z_2}\right) f_{U_I}(\mu_I(X,Z))}$$

$$= \frac{\frac{\partial \mu_I(X,Z)}{\partial Z_1}}{\frac{\partial \mu_I(X,Z)}{\partial Z_2}}$$

Example

- Suppose $\mu_I(X, Z) = \gamma Z$

$$\frac{\frac{\partial \mu_I(X,Z)}{\partial Z_1}}{\frac{\partial \mu_I(X,Z)}{\partial Z_2}} = \frac{\gamma_1}{\gamma_2}$$

- Normalize $\gamma_1 = 1$; can identify all the other terms.
- To see what is going on, notice that we can define a set of X, Z such that $P(X, Z)$ is constant, which traces out a P isoquant.

- To identify F_{U_I} non-parametrically requires full support of Z and restrictions on $\mu_I(X,Z)$. See Matzkin (1992).
- A key condition is

$$\text{Support}\left(\frac{\mu_I(X,Z)}{\sigma_{U_I}}\right) \supseteq \text{Support}\left(\frac{U_I}{\sigma_{U_I}}\right)$$

  and other regularity conditions.
- Commonly it is assumed that for a fixed X

$$\text{Support}\left(\frac{\mu_I(X,Z)}{\sigma_{UI}}\right) = (-\infty,\infty).$$

- This is called "identification at infinity." When we vary Z (for each X) we trace out the full support of $\frac{U_I}{\sigma_{U_I}}$.
- **Problem: Prove this using the first line of (**) realizing that you know $\frac{\mu_I}{\sigma_{UI}}$.**

Identifying the Joint Distribution of (Y_0, I)

We know the conditional distribution of Y_0:

$$F(Y_0 \mid D = 0, X, Z) = \Pr(Y_0 \leq y_0 \mid \mu_I(X, Z) + U_I \leq 0, X, Z)$$

Multiply this by $\Pr(D = 0 \mid X, Z)$:

$$F(Y_0 \mid D = 0, X, Z) \Pr(D = 0 \mid X, Z) = \Pr(Y_0 \leq y_0, I^* \leq 0 \mid X, Z) \quad (*)$$

We can follow the analysis of Heckman (1990), Heckman and Smith (1998), and Carneiro, Hansen, and Heckman (2003).

Left hand side of (*) is known from the data.

Right hand side:

$$\Pr\left(Y_0 \leq y_0, \frac{U_I}{\sigma_{U_I}} < -\frac{\mu_I(X,Z)}{\sigma_{U_I}} \mid X, Z\right)$$

Since we know $\frac{\mu_I(X,Z)}{\sigma_{U_I}}$ from the previous analysis, we can vary it for each fixed X.

- If $\mu_I(X,Z)$ gets small ($\mu_I(X,Z) \to -\infty$), recover the marginal distribution Y and in this limit set we can identify the marginal distribution of

$$Y_0 = \mu_0(X) + U_0 \quad \therefore \quad \text{can identify } \mu_0(X) \text{ in limit.}$$

(See Heckman, 1990, and Heckman and Vytlacil, 2007.)

- More generally, we can form:

$$\Pr\left(U_0 \leq y_0 - \mu_0(X), \frac{U_I}{\sigma_{U_I}} \leq \frac{-\mu_I(X,Z)}{\sigma_{U_I}} \mid X, Z\right)$$

- X and Z can be varied and y_0 is a number.
- We can trace out joint distribution of $\left(U_0, \frac{U_I}{\sigma_{U_I}}\right)$ by varying (y_0, Z) for each fixed X (strictly speaking, varying y_0, Z).

∴ Recover joint distribution of

$$(Y_0, I) = \left(\mu_0(X) + U_0, \frac{\mu_I(X,Z) + U_I}{\sigma_{U_I}}\right).$$

Three key ingredients.

1. The independence of (U_0, U_I) and (X, Z).
2. The assumption that we can set $\dfrac{\mu_I(X,Z)}{\sigma_{U_I}}$ to be very small (so we get the marginal distribution of Y_0 and hence $\mu_0(X)$).
3. The assumption that $\dfrac{\mu_I(X,Z)}{\sigma_{U_I}}$ can be varied independently of $\mu_0(X)$.

Trace out the joint distribution of $\left(U_0, \frac{U_I}{\sigma_{U_I}}\right)$. Result generalizes easily to the vector case. (Carneiro, Hansen, and Heckman, 2003, IER)

Another way to see this is to write:

$$F(Y_0 \mid D = 0, X, Z) \Pr(D = 0 \mid X, Z)$$

This is a function of $\mu_0(X)$ and $\dfrac{\mu_I(X, Z)}{\sigma_{U_I}}$ (Index sufficiency)

Varying the $\mu_0(X)$ and $\dfrac{\mu_I(X,Z)}{\sigma_{U_I}}$ traces out the distribution of $\left(U_0, \dfrac{U_I}{\sigma_{U_I}}\right)$.

This means effectively that we observe the pairs $\left(\frac{I}{\sigma_{U_I}}, Y_1\right)$ and $\left(\frac{I}{\sigma_{U_I}}, Y_0\right)$.

We never observe the triple $\left(\frac{I}{\sigma_{U_I}}, Y_0, Y_1\right)$.

- Use the intuition that we “know” I.
- We observe

$$F(Y_0 \mid I < 0, X, Z)$$

and

$$F(Y_1 \mid I \geq 0, X, Z)$$

and

$$\Pr(I \geq 0 \mid X, Z)$$

and can construct the joint distributions $F(Y_0, I \mid X, Z)$ and $F(Y_1, I \mid X, Z)$.

Roy Normal Case

Armed with normality (or the nonparametric assumptions in Heckman and Honoré, 1990), we can estimate

$$\mathrm{Cov}(I, Y_1) = \frac{\sigma^2_{Y_1} - \sigma_{Y_1,Y_0}}{\sigma^2_{Y_1} + \sigma^2_{Y_0} - 2\sigma_{Y_1,Y_0}}$$

$$\mathrm{Cov}(I, Y_0) = -\frac{\sigma^2_{Y_0} - \sigma_{Y_1,Y_0}}{\sigma^2_{Y_1} + \sigma^2_{Y_0} - 2\sigma_{Y_1,Y_0}}$$

We know Var Y_1, Var Y_0 (e.g. normal selection model or use limit sets)

$\therefore$ Cov(Y_0, Y_1) is identified (actually over-identified).

This line of argument does not generalize if we add a cost component (C) that is unobserved (or partly so).

The intuition is clear. In the Roy model the decision rule is generated solely by (Y_1, Y_0). Knowing agent choices we observe the relative order (and magnitude) of Y_1 and Y_0.

Thus we get a second valuable piece of information from agent choices. This information is ignored in statistical approaches to program evaluation.

But does this analysis generalize?

Generalized Roy Model

Add cost

$$I = Y_1 - Y_0 - C$$

and assume that we do not directly observe C.

$$\text{Observe } Y_1 \mid I > 0,$$
$$\text{Observe } Y_0 \mid I < 0,$$

and

$$I = \frac{Y_1 - Y_0 - C}{\sqrt{\text{Var}(Y_1 - Y_0 - C)}}.$$

We can identify Var Y_1 and can identify Var Y_0.

But we cannot directly identify Cov(Y_0, Y_1) which measures comparative advantage.

Notice, however, we can determine if

$$E(Y_1 \mid I > 0) > E(Y_1)$$
$$E(Y_0 \mid I < 0) > E(Y_0)$$

(Are people who work in a sector above average for the sector?)

- Note that if

$$
\begin{aligned}
U_1 &= \lambda_1 \theta + \varepsilon_1 \\
U_0 &= \lambda_0 \theta + \varepsilon_0 \\
U_C &= \lambda_C \theta + \varepsilon_C
\end{aligned}
$$

- $(\varepsilon_0, \varepsilon_1, \varepsilon_C)$ mutually independent and independent of θ.
- $E(\varepsilon_j) = 0 \;\; j \in \{0, 1, C\}$.
- θ scalar.

- Then we can identify joint distributions.
- $I = \mu_1(X) - \mu_0(X) - \mu_C(Z) + U_1 - U_0 - U_C$.
- We can identify the joint distributions of I, Y_1, I_1, Y_0,
- But

$$Cov(I, Y_1) = \frac{\sigma_{11} - \sigma_{10} - \sigma_{1C}}{\sigma_{U_I}}$$

$$Cov(I, Y_0) = \frac{\sigma_{01} - \sigma_{0C} - \sigma_{0C}}{\sigma_{U_I}}$$

- **Problem:** Show how the one factor assumption facilitates identification of joint distribution.
- Suppose instead you have a measure:

$$M = \theta + U_M$$
$$U_M \perp\!\!\!\perp (U_0, U_1, U_C)$$

- How does that aid in identifying the model?

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