Sampling Plans and Initial Condition Problems For Continuous Time Duration Models

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Sampling plans and initial condition problems: Duration Models

Consider a random sample of unemployment spells in progress. For sampled spells, one of the following duration times may be observed:

- time in state up to sampling date (T_b) (recall of time spent)
- time in state after sampling date (T_a) (sampling forward)
- total time in completed spell observed at origin of sample ($T_c = T_a + T_b$)

Duration of spells beginning after the origin date of the sample, denoted T_d, are not subject to initial condition problems. The intake rate at time $-t_b$ (assuming sample occurs at time 0: the proportion of the population entering a spell at $-t_b$.

Assume:

- A time homogenous environment, i.e. constant intake rate, $k(-t_b) = k, \forall b$
- A model without observed or unobserved explanatory variables.
- No right censoring, so $T_c = T_a + T_b$
- Underlying distribution $f(x)$ is nondefective
- $m = \int_0^{\infty}(x)dx < \infty$

The proportion of the population experiencing a spell at $t = 0$, the origin date of the sample, is

$$\begin{aligned} P_0 &= \int_0^{\infty} k(-t_b)(1 - F(t_b))dt_b = k\int_0^{\infty}(1 - F(t_b))dt_b \\ &= k\left[t_b(1 - F(t_b))|_0^{\infty} - \int_0^{\infty} t_b d(1 - F(t_b))\right] \\ &= k\int_0^{\infty} t_b f(t_b)dt_b = km \end{aligned}$$

where $1 - F(t_b)$ is the probability the spell lasts from $-t_b$ to 0 (or equivalently, from 0 to $-t_b$).

So the density of a spell of length t_b interrupted at the beginning of the sample ($t = 0$) is

$$\begin{aligned} g(t_b) &= \frac{\text{proportion surviving til } t = 0 \text{ from batch } t_b}{\text{total surviving til } t = 0} \\ &= \frac{k(-t_b)(1 - F(t_b))}{P_0} = \frac{1 - F(t_b)}{m} \neq f(t_b) \end{aligned}$$

Notice: g is the distribution of T_b in the population constructed by sampling rule of source population.
Distinguish from F : *cdf* of the true population. G : *cdf* of the sampled spells.

The probability that a spell lasts until t_c given that it has lasted from $-t_b$ to 0, is the conditional density of t_c given $0 < t_b < t_c$.

$$f(t_c|t > t_b > 0) = \frac{f(t_c)}{1 - F(t_b)}; t_c \geq t_b \geq 0$$

So the density of a spell **in the sampled population** that lasts, t_c is

$$\begin{aligned} g(t_c) &= \int_0^{t_c} f(t_c|t \geq t_b) f(t \geq t_b) dt_b \\ &= \int_0^{t_c} \frac{f(t_c)}{m} dt_b = \frac{f(t_c) t_c}{m} \end{aligned}$$

Likewise, the density of a sampled spell that lasts until t_a is

$$\begin{aligned} g(t_a) &= \int_0^\infty f(t_a + t_b | t_b) Pr(t \geq t_a \geq 0)) dt_b \\ &= \int_0^\infty \frac{f(t_a + t_b)}{m} dt_b \\ &= \frac{1}{m} \int_{t_a}^\infty f(t_b) dt_b \\ &= \frac{1 - F(t_a)}{m} \end{aligned}$$

(Stationarity, mirror images have same densities). So the functional form of $f(t_b) = f(t_a)$: Consequences of stationarity.

Some useful results that follow from this model:

- If $f(t) = \theta e^{-t\theta}$, then $g(t_b) = \theta e^{-t_b\theta}$ and $g(t_a) = \theta e^{-t_a\theta}$.
  Proof:

$$
\begin{aligned}
f(t) &= \theta e^{-t\theta} \rightarrow m = \frac{1}{\theta}, \\
F(t) &= 1 - e^{-t\theta} \rightarrow g(t_a) = \frac{1 - F(t)}{m} = \theta e^{-t\theta}
\end{aligned}
$$

- $E(T_a) = \frac{m}{2}(1 + \frac{\sigma^2}{m^2})$.

Proof:

$$
\begin{aligned}
E(T_a) &= \int t_a f(t_a) dt_a = \int t_a \frac{1 - G(t_a)}{m} dt_a \\
&= \frac{1}{m}\left[\frac{1}{2} t_a^2 (1 - F(t_a))|_0^\infty - \int \frac{1}{2} t_a^2 d(1 - F(t_a))\right] \\
&= \frac{1}{m} \int \frac{1}{2} t_a^2 F(t_a) dt_a = \frac{1}{2m}[var(t_a) + E^2(t_a)] \\
&= \frac{1}{2m}[\sigma^2 + m^2]
\end{aligned}
$$

- $E(T_b) = \frac{m}{2}(1 + \frac{\sigma^2}{m^2})$. **Proof**: See proof of Proposition 2.
- $E(T_c) = m(1 + \frac{\sigma^2}{m^2})$. **Proof**:

$$E(T_c) = \int \frac{t_c^2 F(t_c)}{m} dt_c = \frac{1}{m}(var(t_c) + E^2(t_c))$$

$$\rightarrow E(T_c) = 2E(T_a) = 2E(T_b), E(T_c) > m \text{ unless } \sigma^2 = 0$$

Some Additional Results:

$$h(t) = \text{ hazard} : h(t) = \frac{f(t)}{1 - F(t)}.$$

- $h'(t) > 0 \rightarrow E(T_a) = E(T_b) < m$. **Proof:** See Barlow and Proschan.
- $h'(t) < 0 \rightarrow E(T_a) = E(T_b) > m$. **Proof**: See Barlow and Proschan.

Examples

Specification of the Distribution

Weibull Distribution

- Parameters: $\lambda > 0, k > 0$
- Probability Density Function (PDF):

$$\frac{\lambda}{k}\left(\frac{t}{\lambda}\right)^{k-1}\exp\left(-\left(\frac{t}{k}\right)^{k}\right)$$

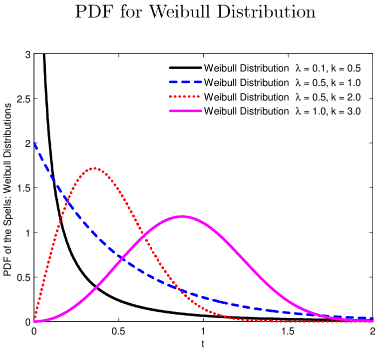
- Cumulative Density Function:

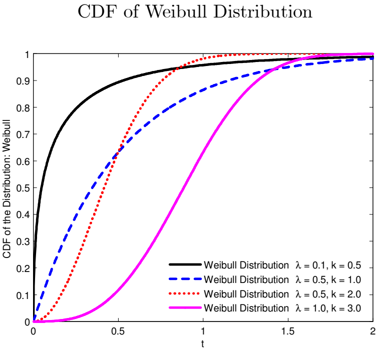
$$1-\exp\left(-\left(\frac{t}{k}\right)^{k}\right)$$

- Set of Parameters:

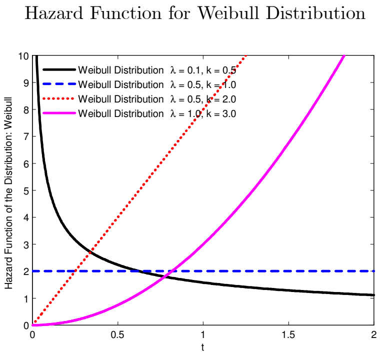
$$\left(\begin{array}{l}\lambda_1, k_1 = 0.5\\ \lambda_2, k_1 = 1.0\\ \lambda_3, k_1 = 2.0\\ \lambda_3, k_1 = 3.0\end{array}\right), \quad \text{respectively}$$

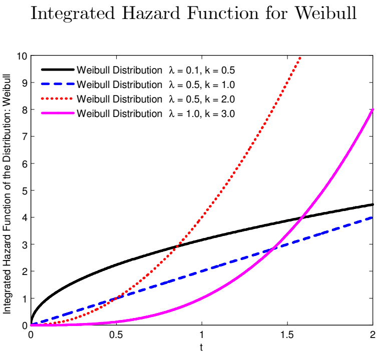
Basic Distribution Graphs



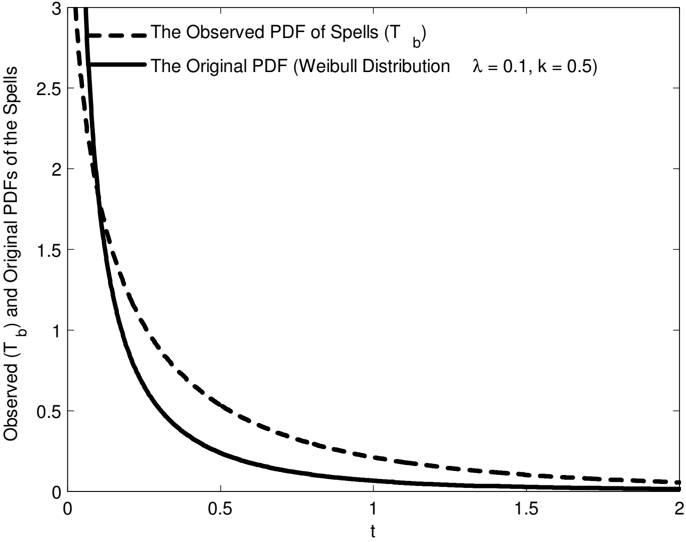


Basic Duration Graphs

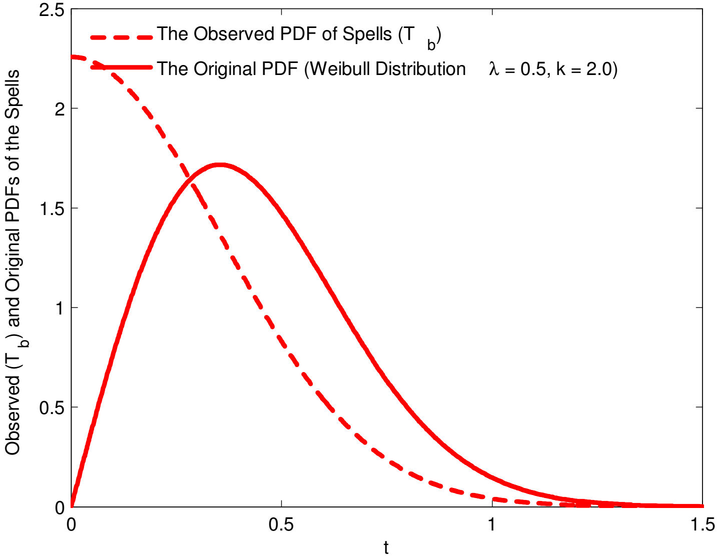




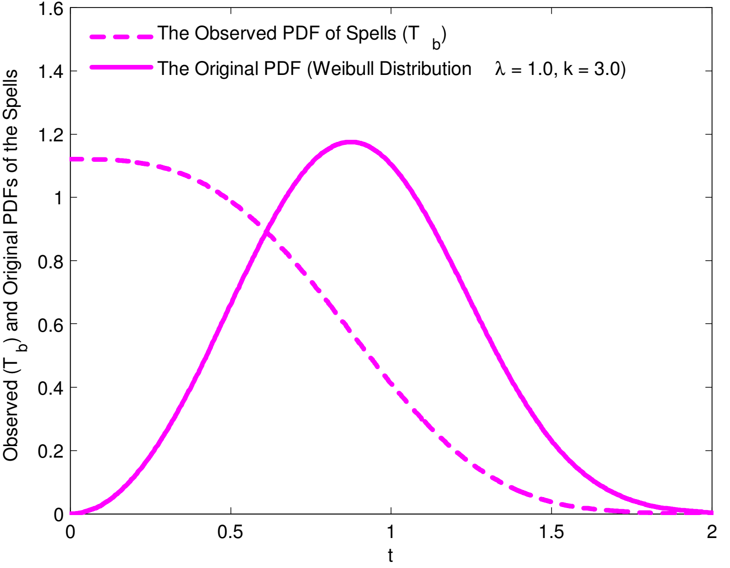
Observed and Original Distribution for T_b (Example 1)



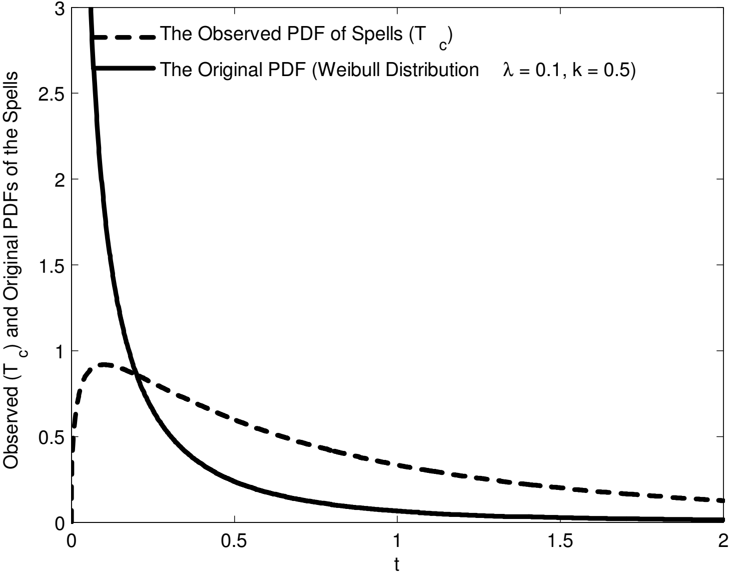
Observed and Original Distribution for T_b (Example 3)



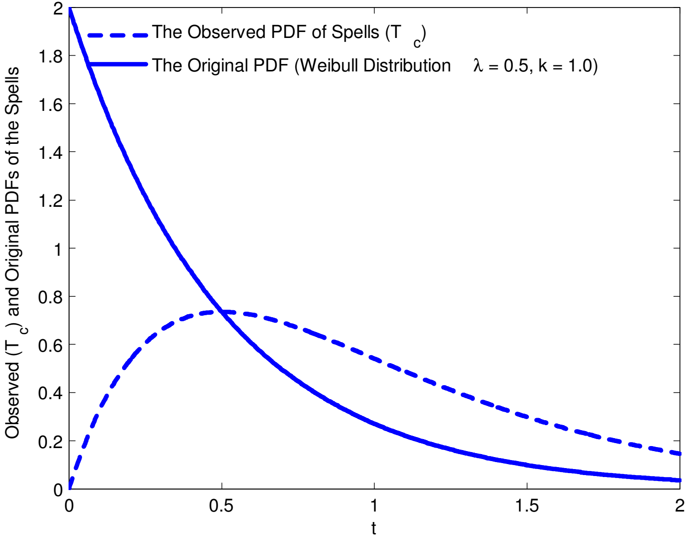
Observed and Original Distribution for T_b (Example 4)



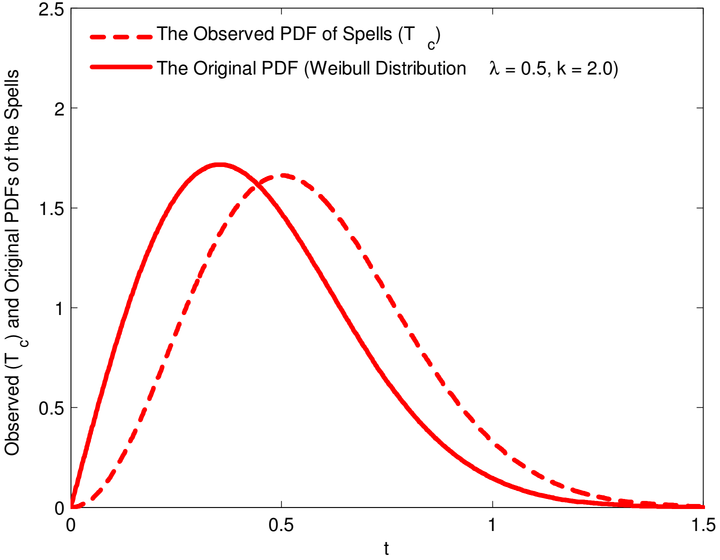
Observed and Original Distribution for T_c (Example 1)



Observed and Original Distribution for T_c (Example 2)



Observed and Original Distribution for T_c (Example 3)



Observed and Original Distribution for T_c (Example 4)